Optimum Resource Allocation and Signalling Schemes in Fading CDMA Channels

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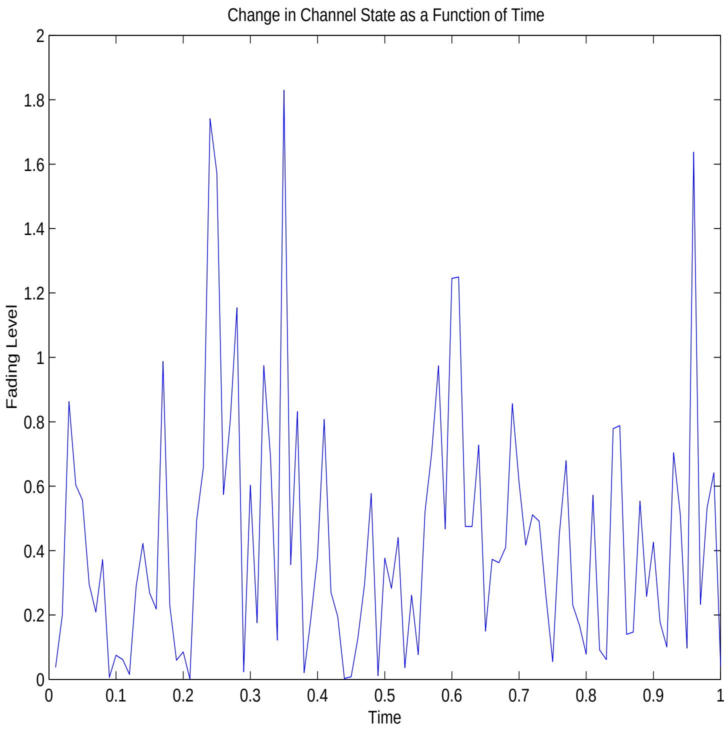
Joint work with Onur Kaya.

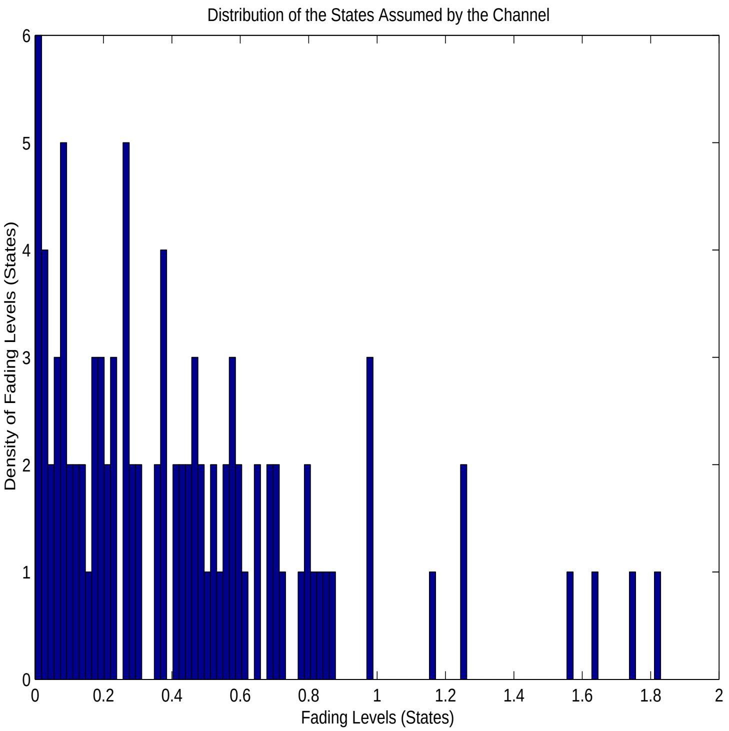
Introduction

- Fading: random fluctuations in channel gains.
- If perfect channel state information (CSI) is available at transmitters
  - Dynamic resource allocation to improve quality-of-service or capacity
- Quality-of-service based
  - Provide all users with desired SIR levels
  - Satisfy SIR requirements with minimum transmit power
  - Compensate for channel fading; more power if bad channel, less if good channel
- Capacity based
  - Maximize information theoretic ergodic capacity subject to average power constraints
  - Exploit variations; more power if good channel, less if bad, no power if very bad

Illustration of the Channel States

$$r = \sqrt{phx} + n$$





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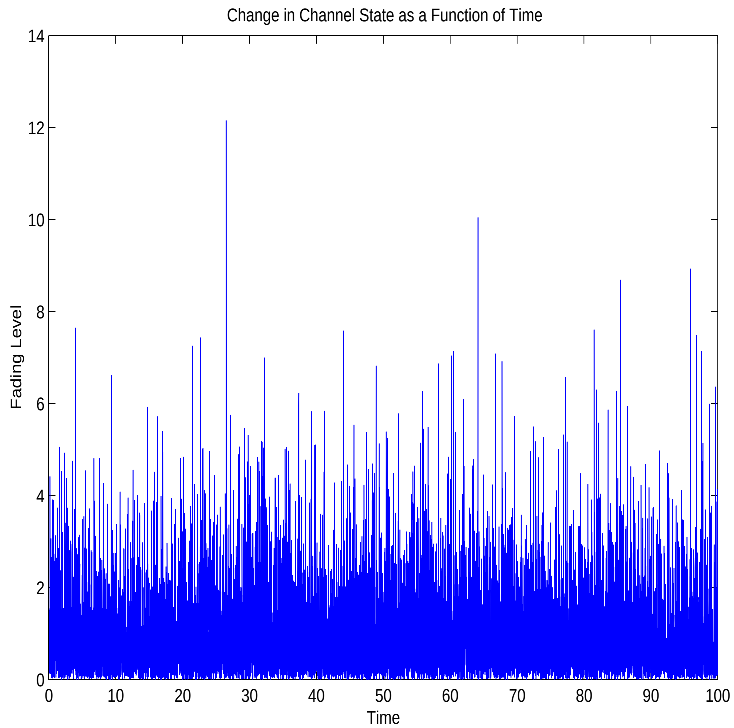
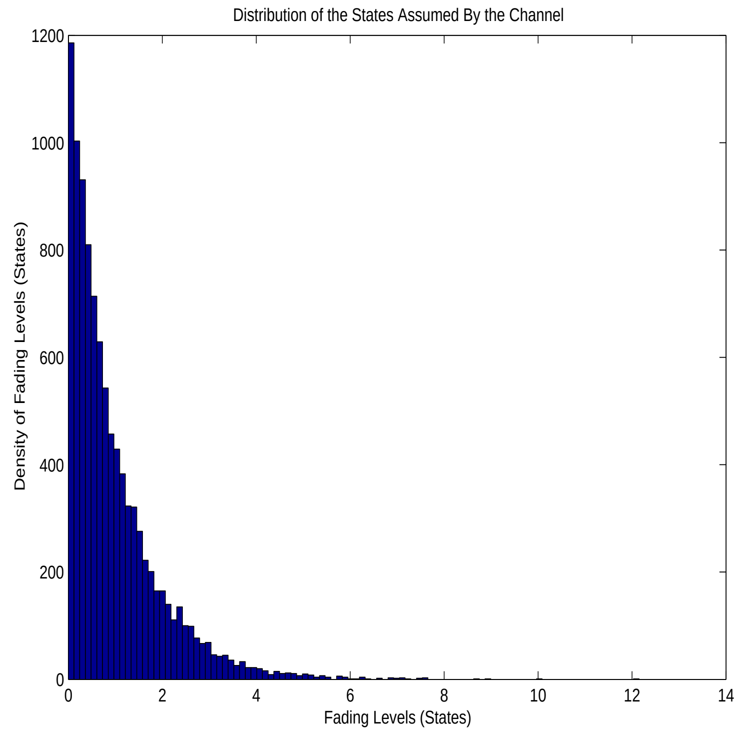




Single User Channel (Goldsmith-Varaiya 1994)

- Channel capacity for single user

$$
\begin{aligned}
C &= \log(1 + SNR) \\
&= \log\left(1 + \frac{p}{\sigma^2}\right)
\end{aligned}
$$

- In the presence of fading, the capacity for a fixed channel state h,

$$
C(h) = \log\left(1 + \frac{p(h)h}{\sigma^2}\right)
$$

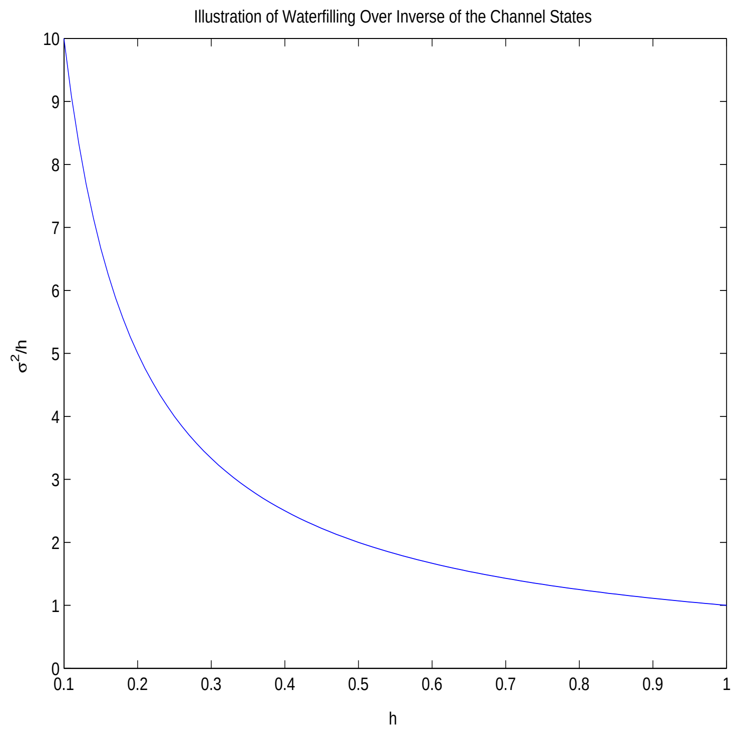
- Maximize the ergodic (expected) capacity, given an average power constraint

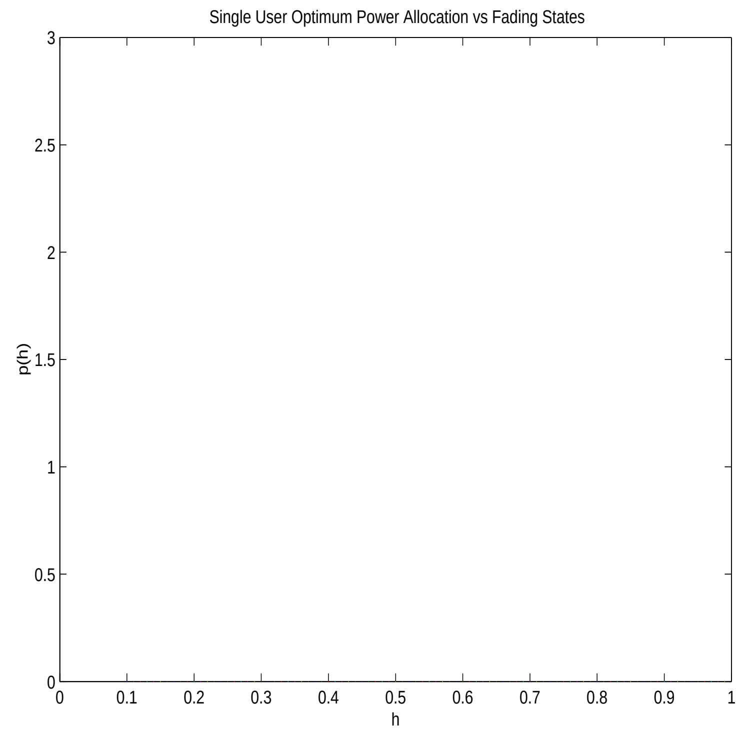
$$
\begin{aligned}
\max_{\{p(h)\}} \quad & E_h\left[\log\left(1 + \frac{p(h)h}{\sigma^2}\right)\right] \\
\text{s.t.} \quad & E_h[p(h)] \le \bar{p}
\end{aligned}
$$

Single User Channel Solution-Waterfilling

- Optimal power allocation: waterfilling of power over time

$$p(h) = \left(\frac{1}{\lambda} - \frac{\sigma^2}{h}\right)^+$$





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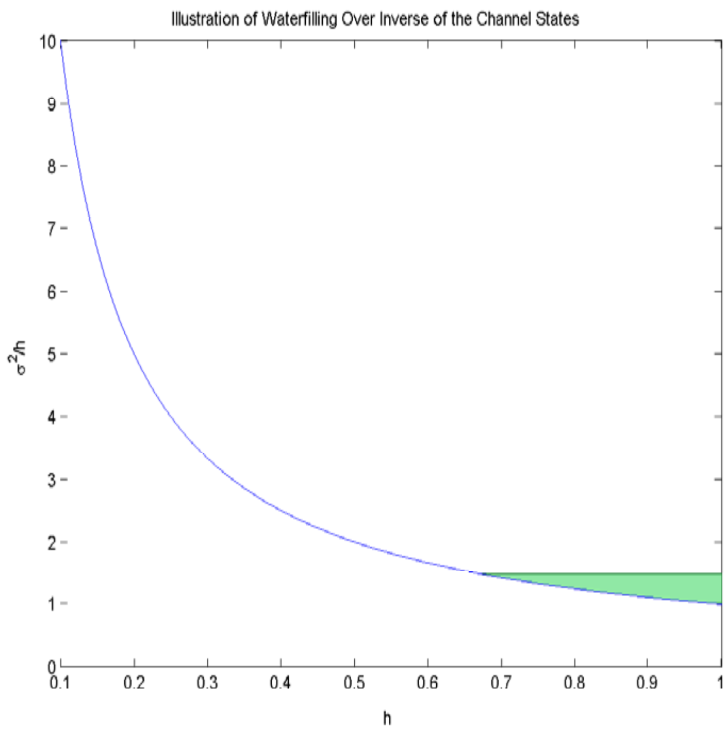

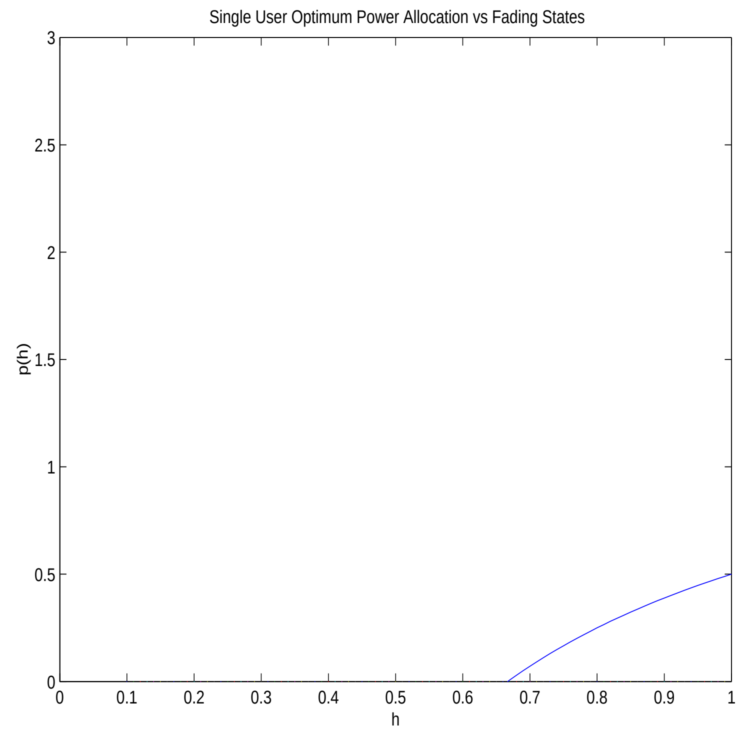

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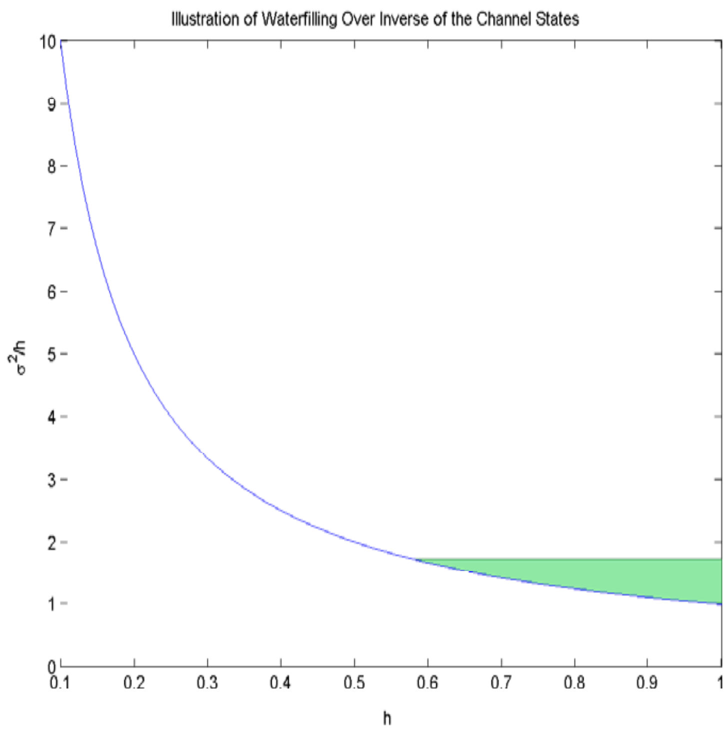

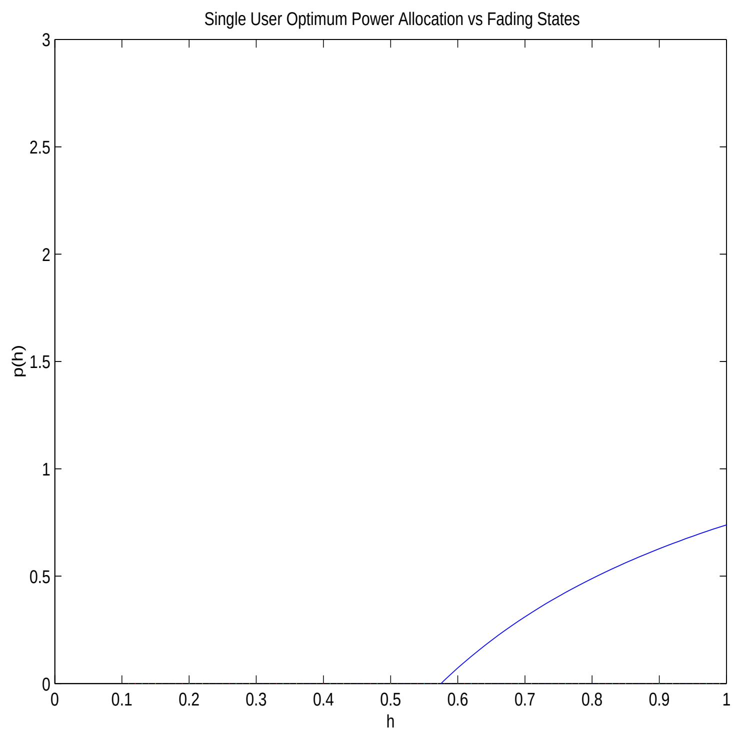

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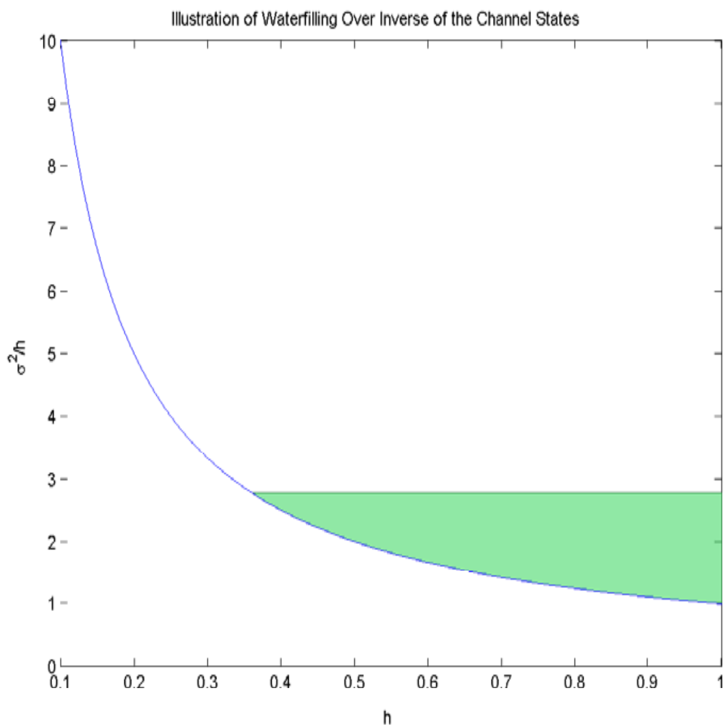

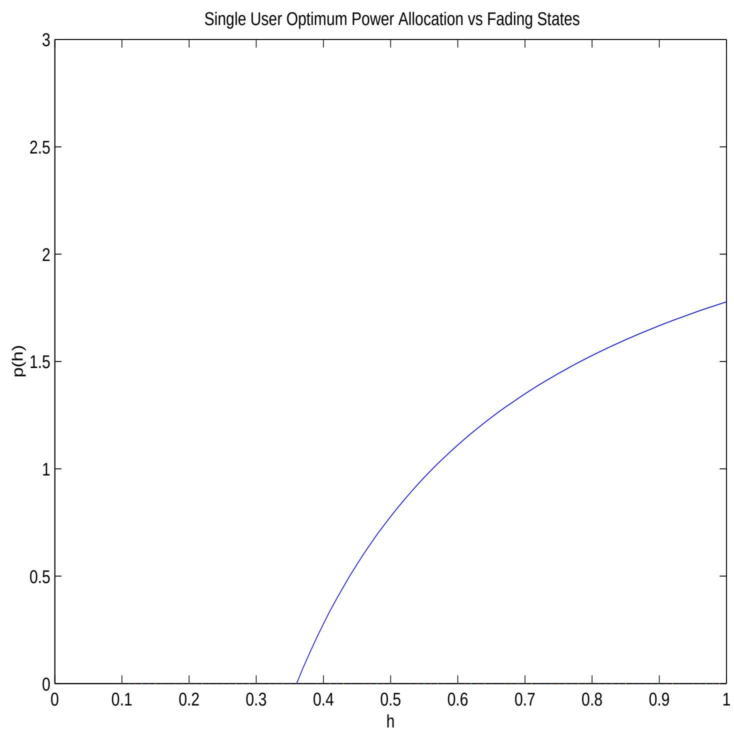

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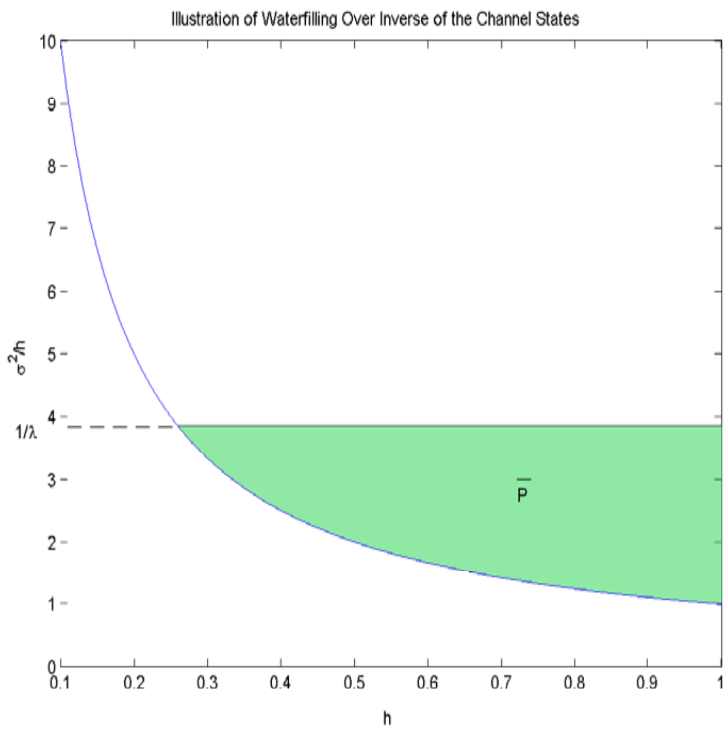
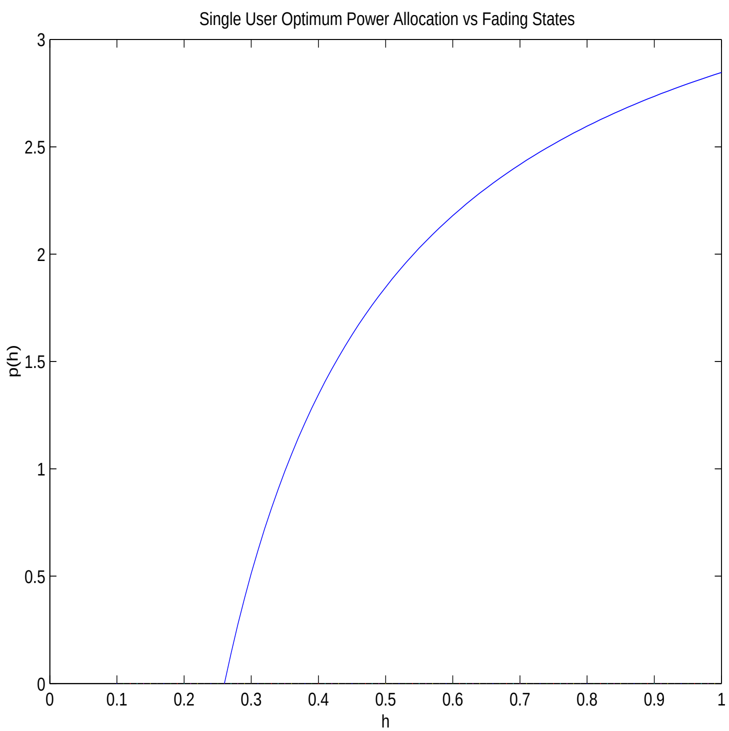




Multiuser Scalar Gaussian Channel (Knopp-Humblet 1995)

- Multiple users, scalar transmissions

$$r = \sum_{i=1}^{K} \sqrt{p_i(\mathbf{h})h_i} x_i + n$$

- Maximize ergodic sum capacity, given average power constraints

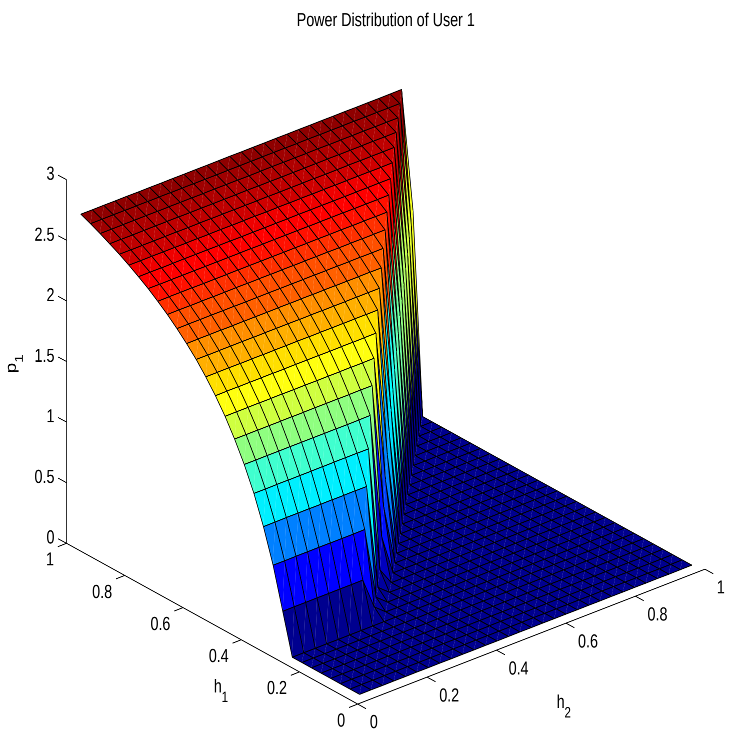
$$\begin{array}{lll} \max\limits_{\{p_i(\mathbf{h})\}} & E_{\mathbf{h}}\left[\log\left(1+\sigma^{-2}\sum_{i=1}^{K} h_i p_i(\mathbf{h})\right)\right] & \\ \text{s.t.} & E_{\mathbf{h}}\left[p_i(\mathbf{h})\right] \le \bar{p}_i, \qquad p_i(\mathbf{h}) \ge 0, & i = 1, \cdots, K \end{array}$$

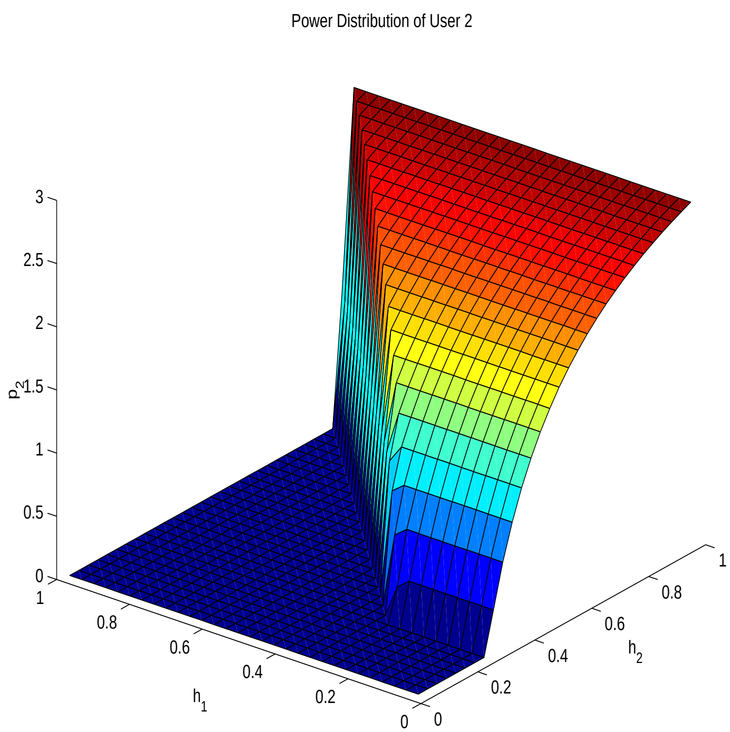
- Optimal power allocation: single user waterfilling on disjoint sets of channel states

$$p_k(\mathbf{h}) = \begin{cases} \left(\frac{1}{\lambda_k} - \frac{\sigma^2}{h_k}\right)^+, & \text{if} \quad h_k/\lambda_k > h_j/\lambda_j, \quad j \neq k \\ 0, & \text{otherwise} \end{cases}$$

- Only the strongest (after some scaling) user transmits at any given time.

Optimum Power Allocation: Scalar Multiuser Channel





Multiuser Vector (Waveform) Gaussian Channel

- Project the received signal onto N basis waveforms.
- CDMA: vector signals modulated by scalar symbols.

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i(\mathbf{h}) h_i} x_i \mathbf{s}_i + \mathbf{n}$$

- Maximize ergodic sum capacity subject to average power constraints

$$\begin{aligned}
\max_{\{\mathbf{p}(\mathbf{h})\}} \quad & E_{\mathbf{h}}\left[\log\left|\mathbf{I}_N + \sigma^{-2}\sum_{i=1}^{K} h_i p_i(\mathbf{h})\mathbf{s}_i\mathbf{s}_i^{\top}\right|\right] \\
\text{s.t.} \quad & E_{\mathbf{h}}[p_i(\mathbf{h})] \le \bar{p}_i, \qquad i = 1, \cdots, K \\
& p_i(\mathbf{h}) \ge 0, \quad \forall\, \mathbf{h}, \quad i = 1, \cdots, K
\end{aligned}$$

Optimal Power Control

- C_{sum} is a concave function of powers. Constraint set is convex.
- Using Lagrange method, optimum powers satisfy (by KKT conditions),

$$\frac{h_k \mathbf{s}_k \mathbf{A}_k^{-1} \mathbf{s}_k}{1 + p_k(\mathbf{h}) h_k \mathbf{s}_k \mathbf{A}_k^{-1} \mathbf{s}_k} \leq \lambda_k, \qquad k = 1, \cdots, K, \qquad \forall\, \mathbf{h} \in R^K$$

with equality iff $p_k > 0$. Here, $\mathbf{A}_k$ is defined as

$$\mathbf{A}_k = \sigma^2 \mathbf{I}_N + \sum_{i \neq k} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top$$

- Optimum power allocation:

$$p_k(\mathbf{h}) = \left(\frac{1}{\lambda_k} - \frac{1}{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k} \right)^+, \qquad k = 1, \cdots, K$$

- Simultaneous waterfilling of powers onto

inverse of the “SIRs with MMSE receivers and unit transmit powers” of users.

Iterative Waterfilling

- Isolate kth user's contribution to sum capacity

$$C_{\text{sum}} = C_k + \overline{C}_k$$

$$C_k = E_{\mathbf{h}}\left[\log\left(1 + h_k p_k(\mathbf{h})\mathbf{s}_k^\top \mathbf{A}_k^{-1}\mathbf{s}_k\right)\right]$$

- Optimize the power of user k only, with the powers of all other users fixed.

$$\begin{aligned} p_k^{n+1} &= \arg\max_{p_k} C_{\text{sum}}\left(p_1^{n+1}, \cdots, p_{k-1}^{n+1}, p_k, p_{k+1}^{n} \cdots, p_K^{n}\right) \\ &= \arg\max_{p_k} C_k\left(p_k\right) \end{aligned}$$

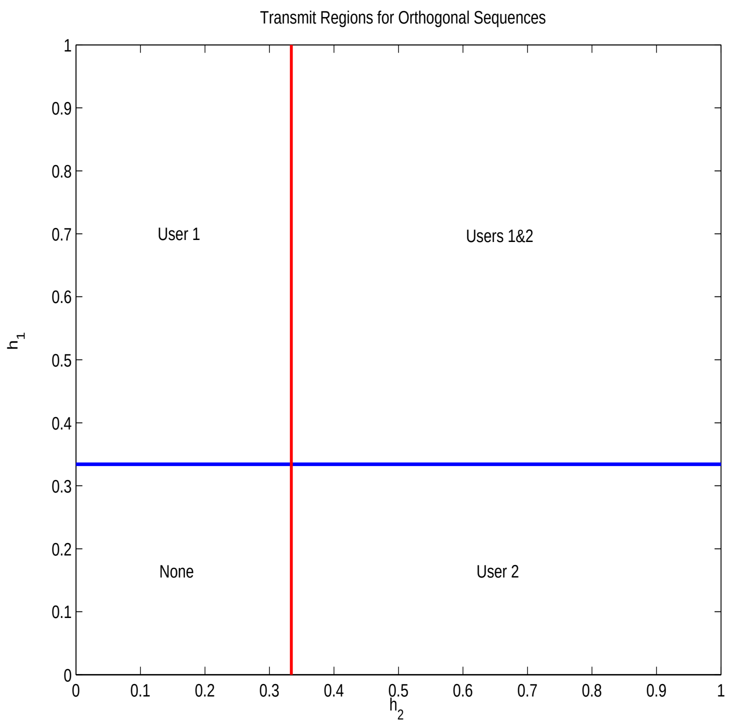
- One-user-at-a-time single user waterfilling:

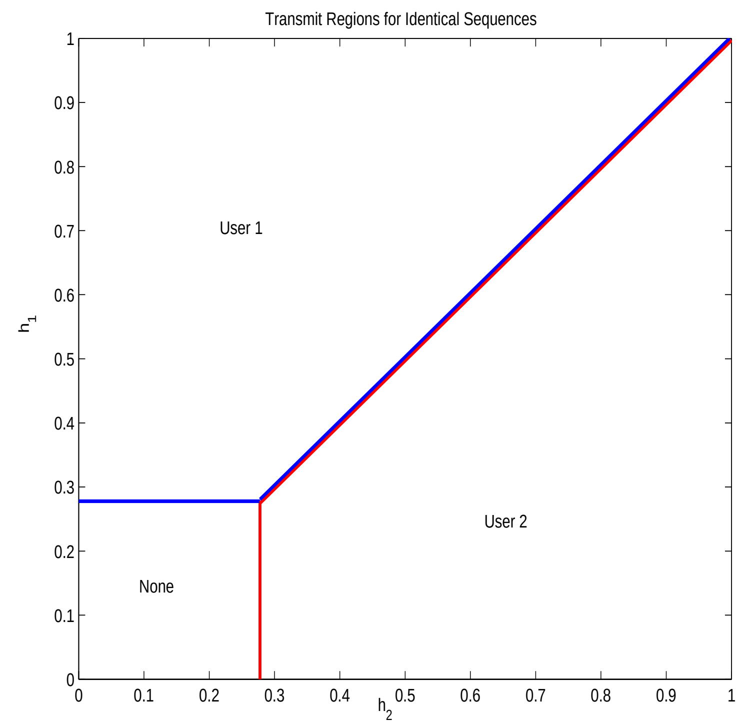
$$p_k(\mathbf{h}) = \left(\frac{1}{\tilde{\lambda}_k} - \frac{1}{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1}\mathbf{s}_k}\right)^+$$

- Converges to global optimum [Bertsekas-Tsitsiklis].

Simultaneous Transmit Regions

- The regions where both users transmit for the two special cases:



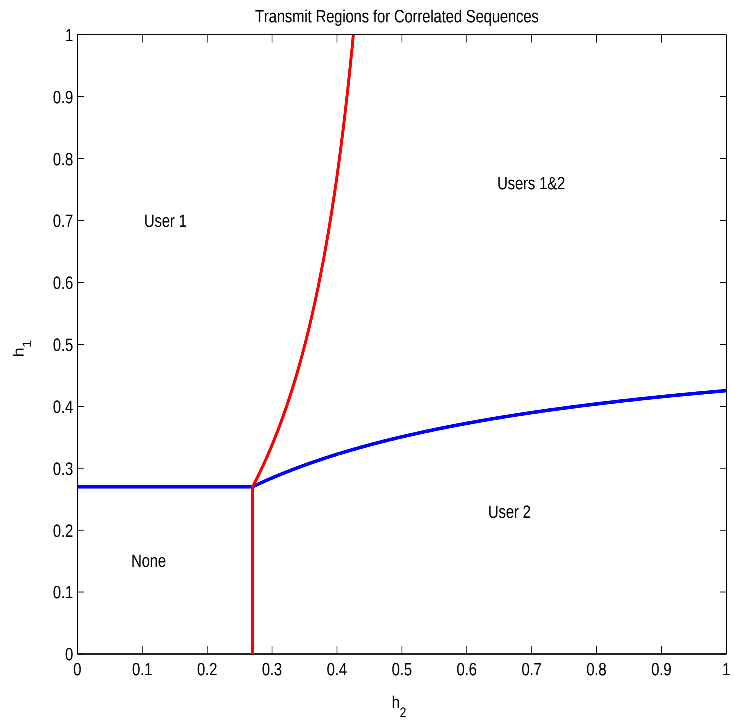


- **Motivation:** for a set of arbitrary signature sequences, is there a set of channel states (with non-zero probability measure) where all users transmit simultaneously?

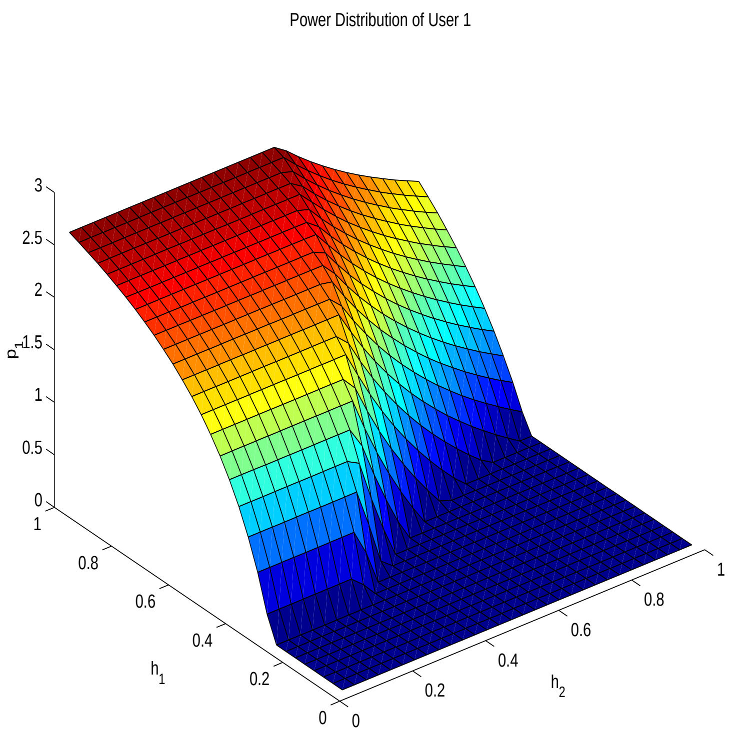
Simultaneous Transmit Condition

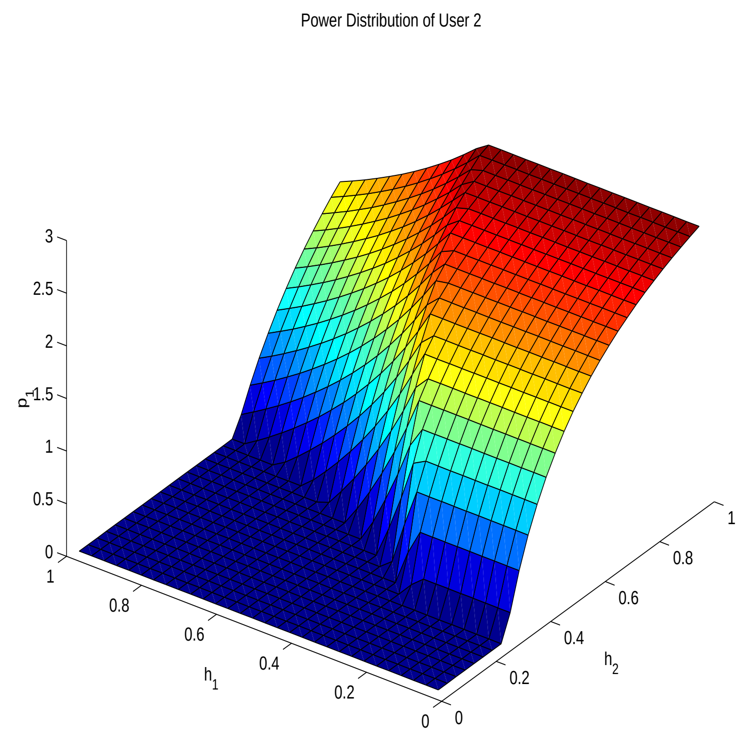
Theorem: There exists a non-zero probability region of fading states $\mathbf{h}$ where all K users in the system transmit simultaneously, if and only if $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent.

Corollary: When $K \leq N$, for a set of K linearly independent signature sequences, there always exists a non-zero probability region of channel states where all K users transmit simultaneously.



Transmit Powers: Correlated Signatures

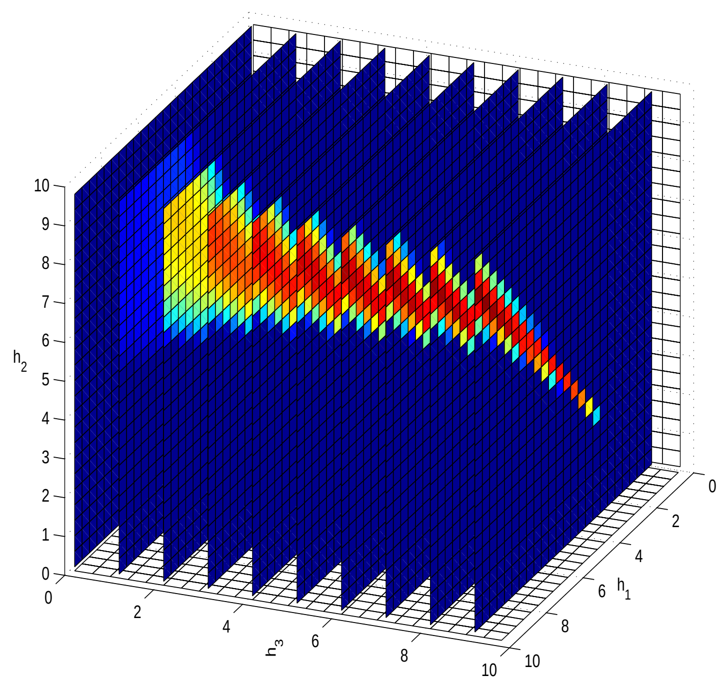




Maximum Number of Simultaneous Transmissions

Corollary: For a set of signature sequences with $\text{rank}(\mathbf{S}) = M \leq \min\{N, K\}$, the number of users that can transmit simultaneously cannot be larger than $M(M+1)/2$.

Example: $N = 2$, $K = 3$.



Signature sequences $\{\mathbf{s}_i\}_{i=1}^K$ are linearly dependent, but $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent.

Jointly Optimal Power and Waveform Allocation in Fading

- Dynamic resource allocation – transmit powers, bandwidth, time slots; or in general waveforms – to combat fading and improve capacity

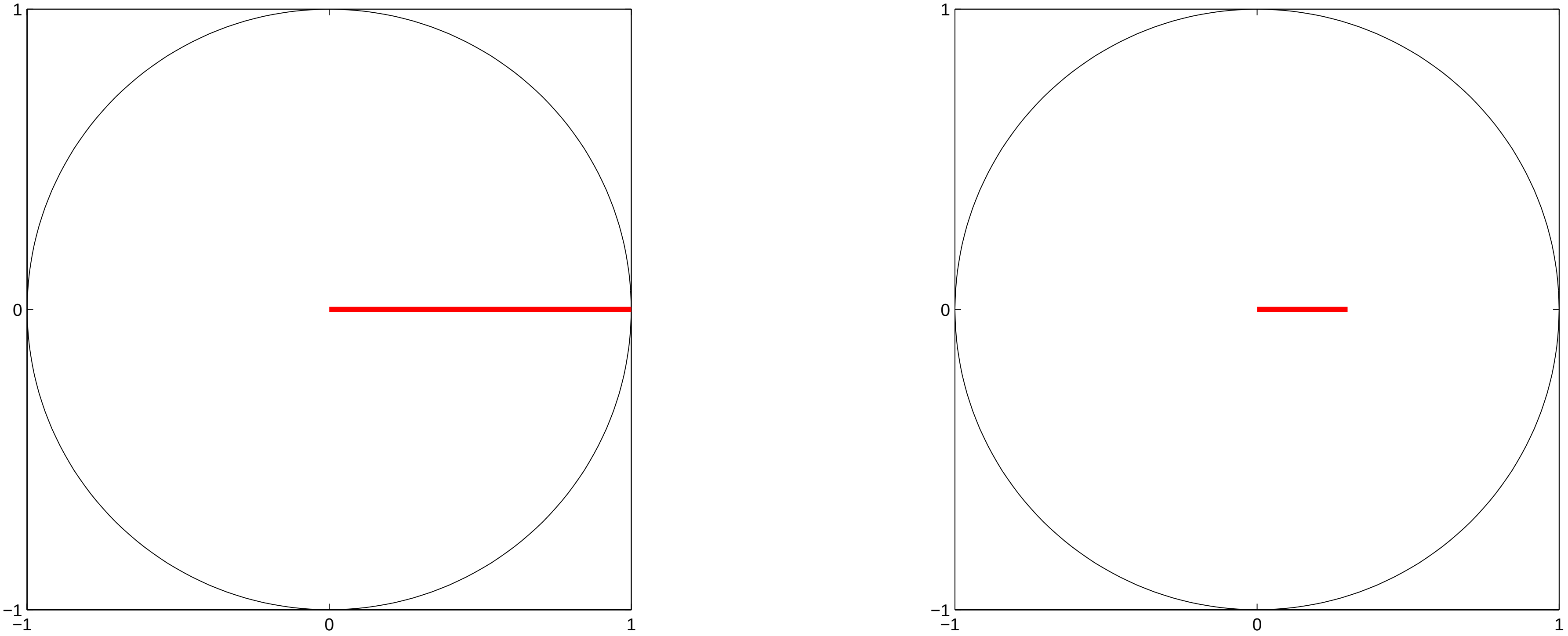
- Vector (waveform) MAC: allocate transmit powers and waveforms to users.

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i h_i} x_i \mathbf{s}_i + \mathbf{n}$$

- Existing literature:
  - Power control only: control powers as a function of CSI in fading [Kaya-Ulukus].
    - maximize sum capacity,
    - achieve any point on the capacity region (maximize weighted sum of rates).
  - Waveform allocation only: find sum-capacity maximizing set of waveforms for a given set of (fixed) powers in no fading [Rupf-Massey, Viswanath-Anantharam].
    - notion of oversized/non-oversized users according to powers,
    - orthogonal waveforms to oversized users, GWBE waveforms to non-oversized users.

Waveform Allocation Only – No Fading, Fixed Powers

Simple example: vectors are signatures with powers.



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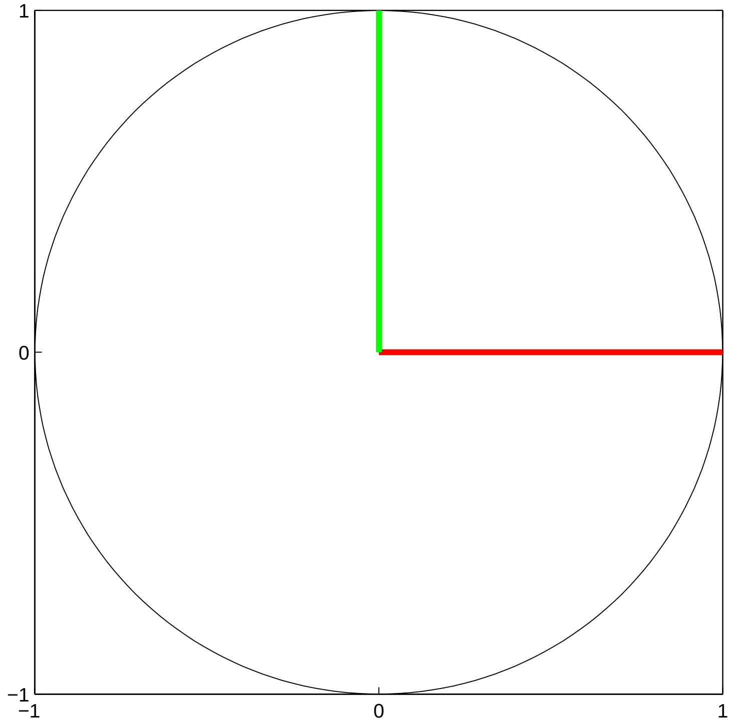

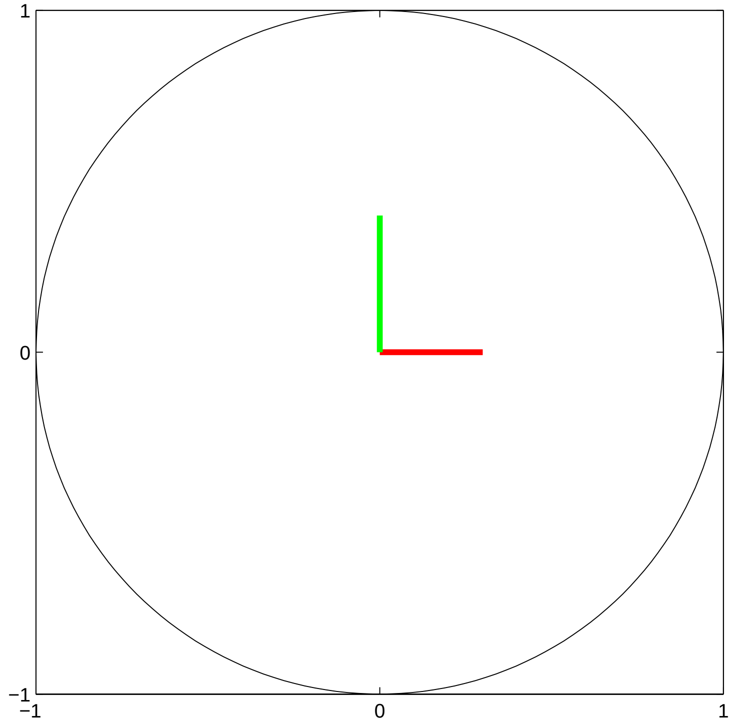

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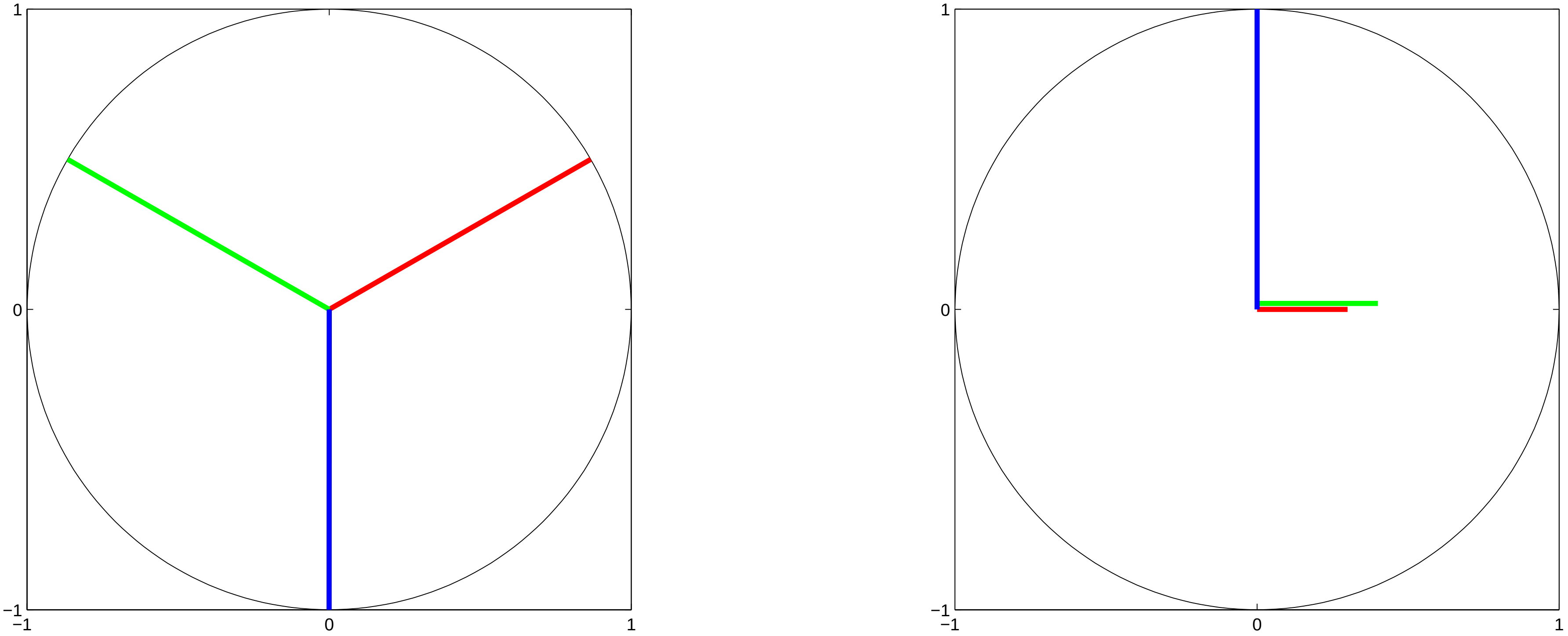


Joint Power and Waveform Allocation

- Consider sum capacity of the network. Perfect CSI at the transmitters.
- Then, both powers and waveforms can be chosen as functions of channel states.

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i(\mathbf{h}) h_i} x_i \mathbf{s}_i(\mathbf{h}) + \mathbf{n}$$

- Ergodic sum capacity maximization problem becomes

$$\begin{aligned}
\max_{\mathbf{p}(\mathbf{h}),\mathbf{S}(\mathbf{h})} \quad & E_{\mathbf{h}}\left[\log\left|\mathbf{I}_N + \sigma^{-2}\sum_{i=1}^{K} h_i p_i(\mathbf{h})\mathbf{s}_i(\mathbf{h})\mathbf{s}_i(\mathbf{h})^{\top}\right|\right] \\
\text{s.t.} \quad & E_{\mathbf{h}}\left[p_i(\mathbf{h})\right] = \bar{p}_i, \quad i = 1, \cdots, K \\
& p_i(\mathbf{h}) \geq 0, \quad \forall\, \mathbf{h}, \quad i = 1, \cdots, K \\
& \mathbf{s}_i(\mathbf{h})^{\top}\mathbf{s}_i(\mathbf{h}) = 1, \quad \forall\, \mathbf{h}, \quad i = 1, \cdots, K
\end{aligned}$$

Waveform Optimized Capacity

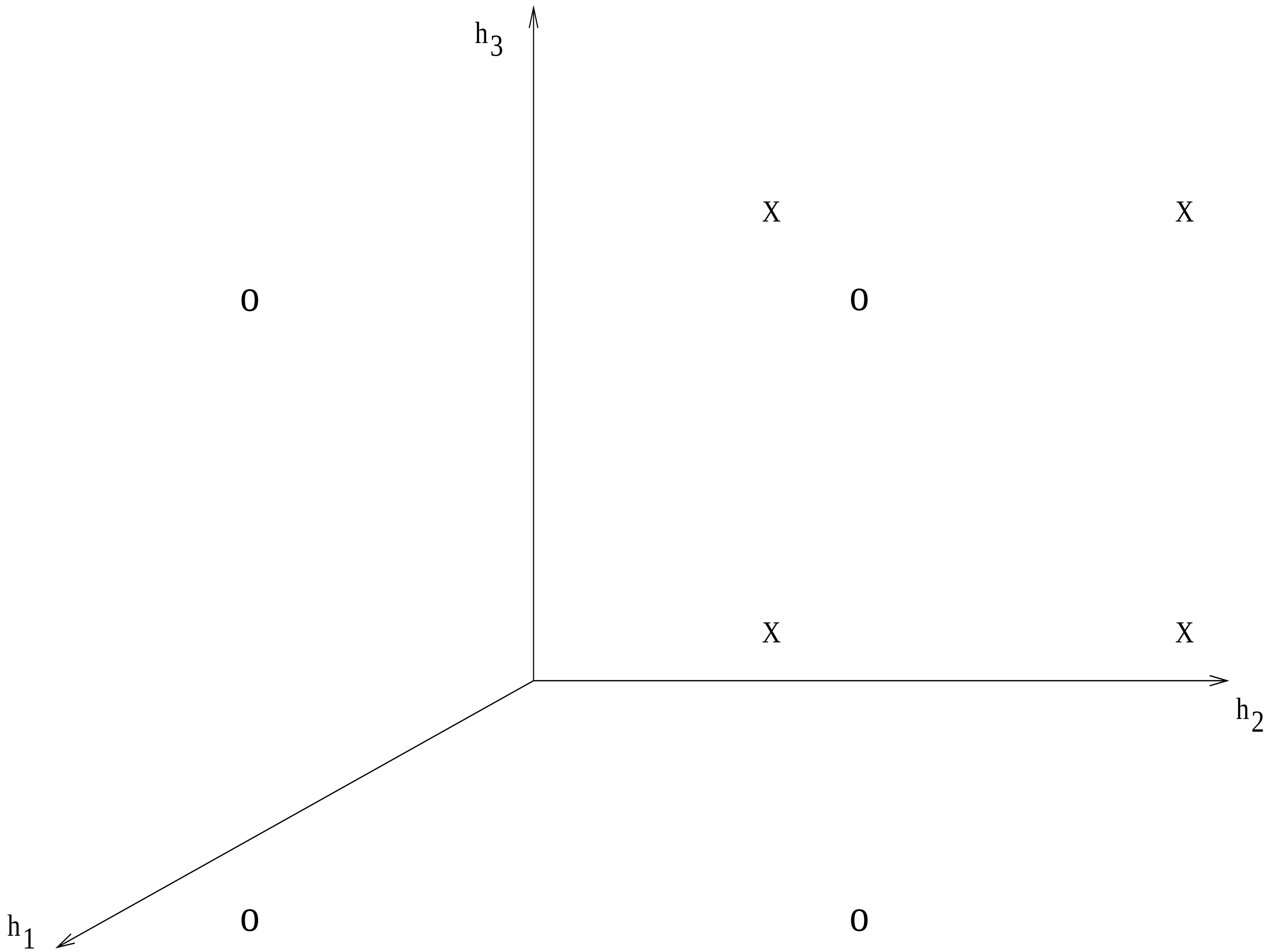
- First, fix an arbitrary valid power allocation over the fading states.
- For each fixed allocation, find the waveforms that maximize the sum capacity at each state **h**.
- Define the waveform-optimized sum capacity at **h**

$$C_{\mathrm{opt}}(\mathbf{h},\mathbf{p}(\mathbf{h})) \triangleq \max_{\mathbf{S}(\mathbf{h})} C_{\mathrm{sum}}(\mathbf{h},\mathbf{p}(\mathbf{h}),\mathbf{S}(\mathbf{h}))$$

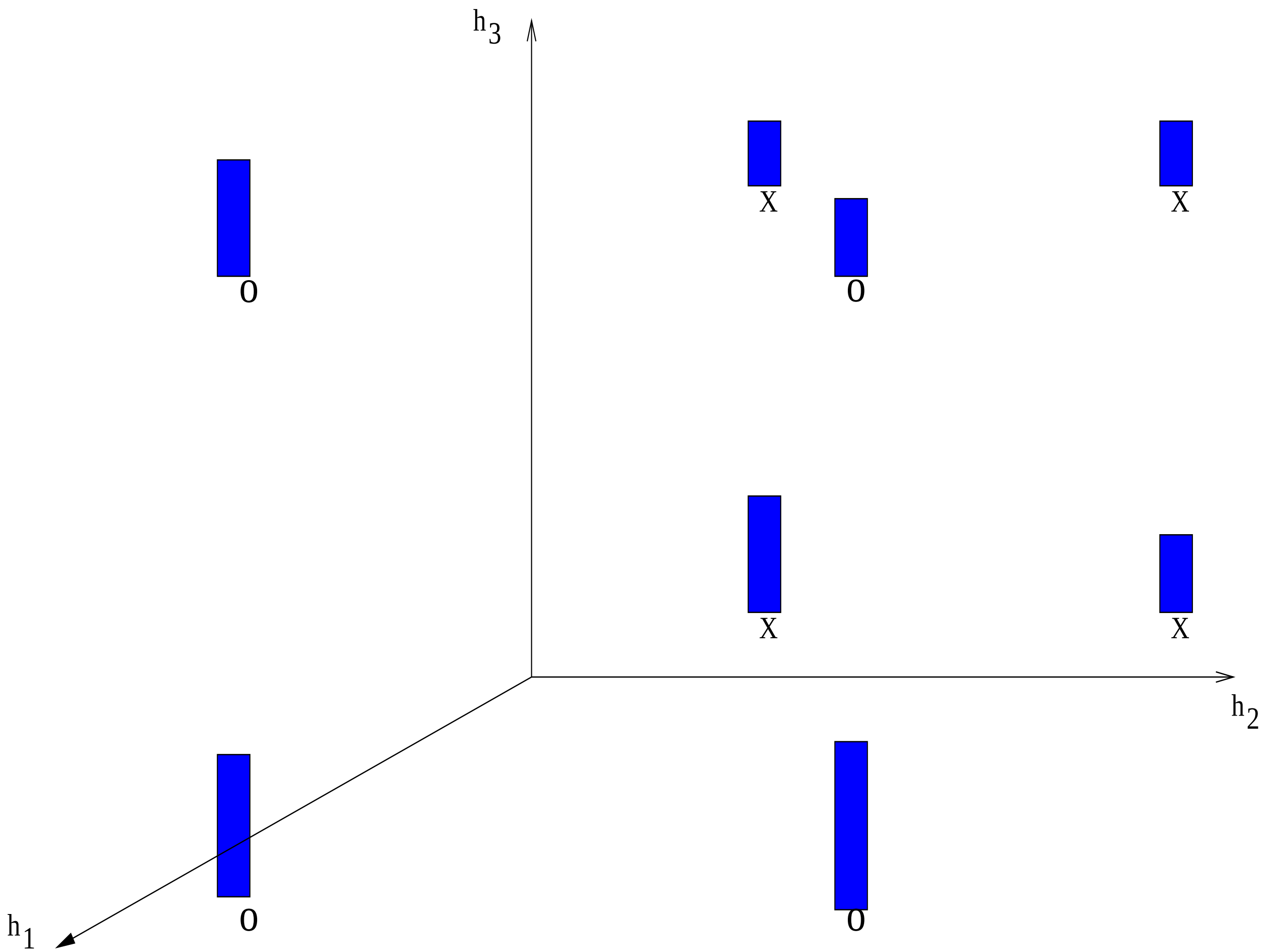
- Then, optimize waveform-optimized sum capacity in terms of the powers,

$$\begin{aligned} \max_{\mathbf{p}(\mathbf{h})} \quad & E_{\mathbf{h}}\left[C_{\mathrm{opt}}(\mathbf{h},\mathbf{p}(\mathbf{h}))\right] \\ \text{s.t.} \quad & E_{\mathbf{h}}\left[p_i(\mathbf{h})\right] = \bar{p}_i, \quad i = 1,\cdots,K \\ & p_i(\mathbf{h}) \geq 0, \quad \forall \mathbf{h}, \quad i = 1,\cdots,K \end{aligned}$$

Choosing the Optimum Waveforms – Illustration



Choosing the Optimum Waveforms – Illustration



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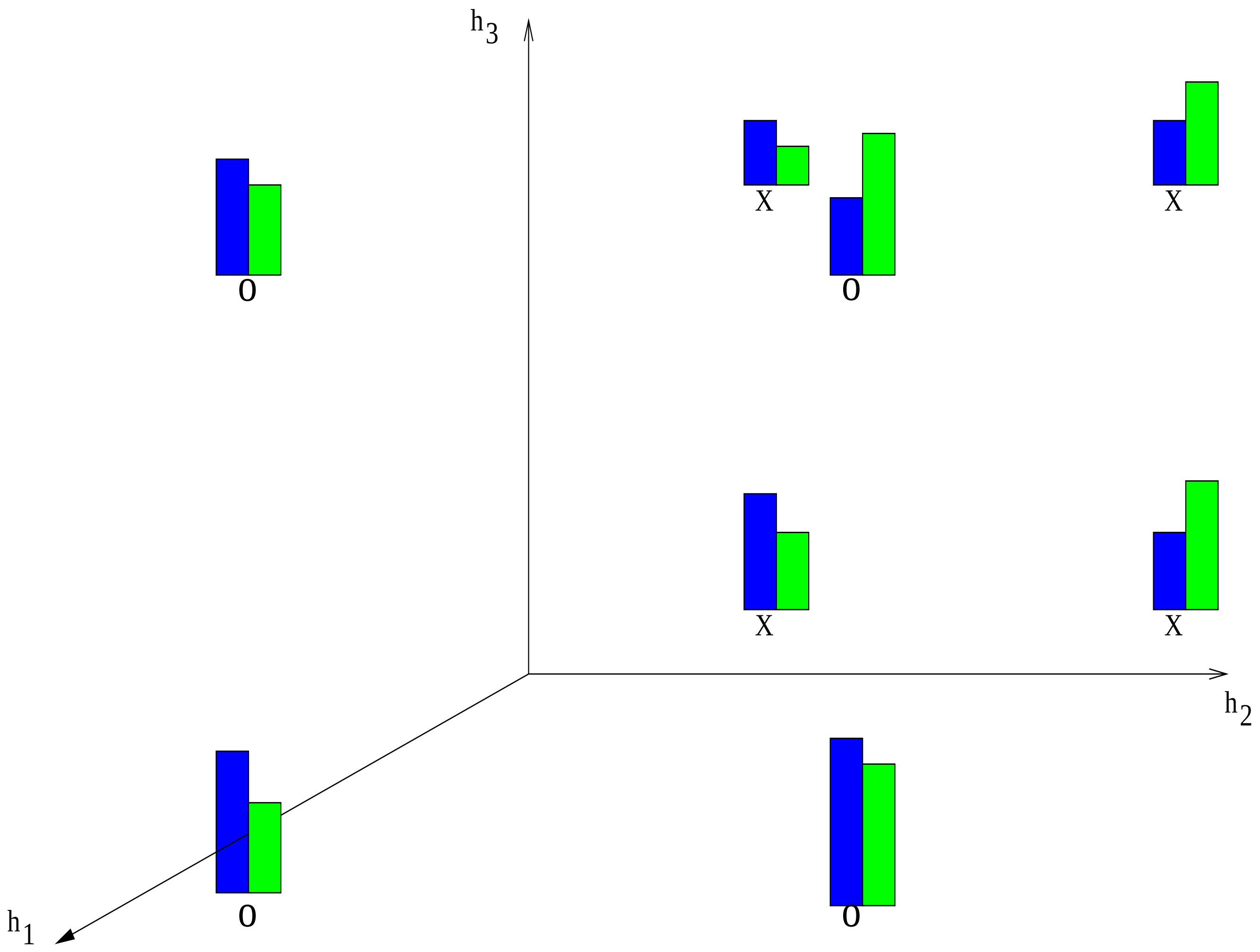

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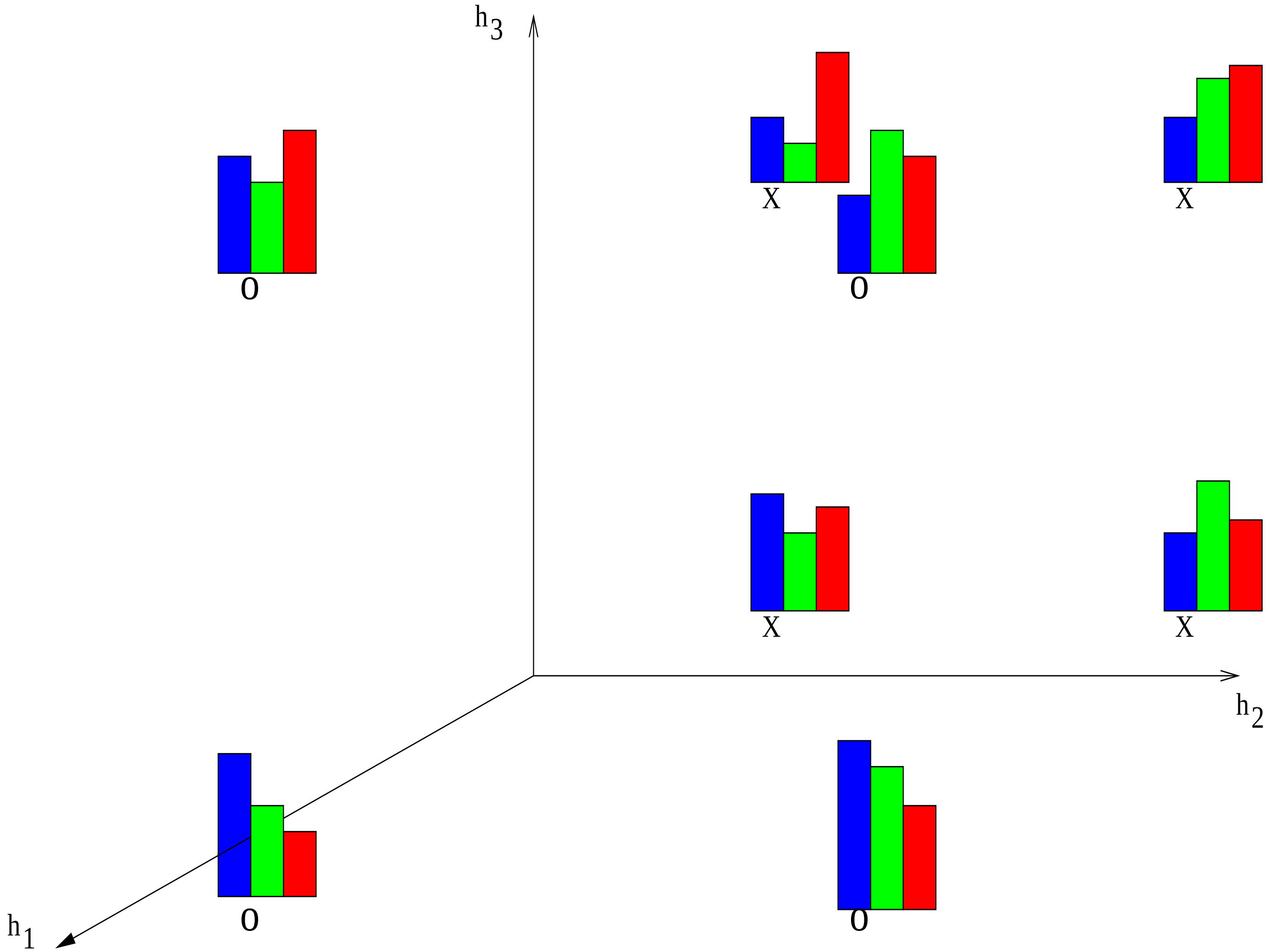

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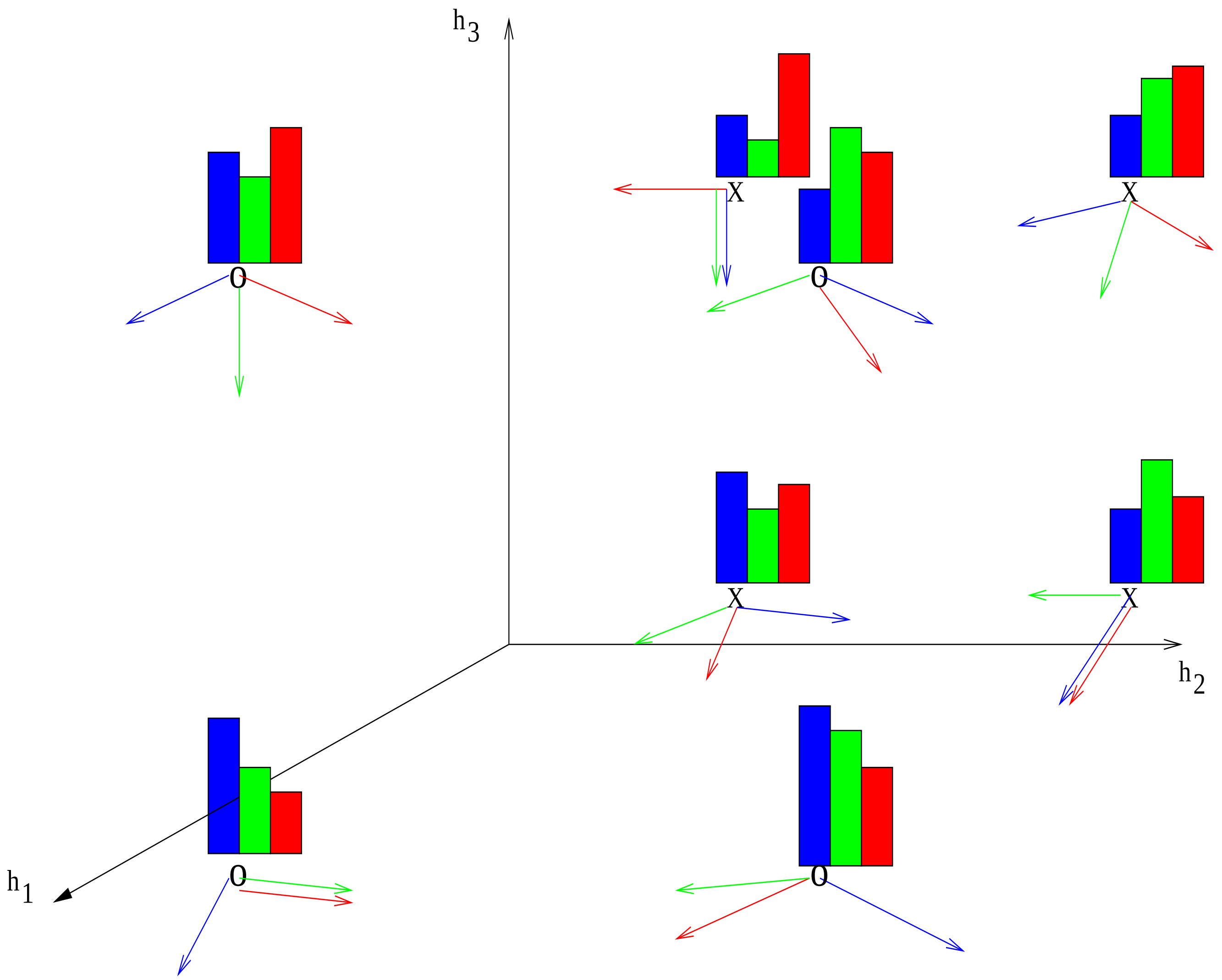

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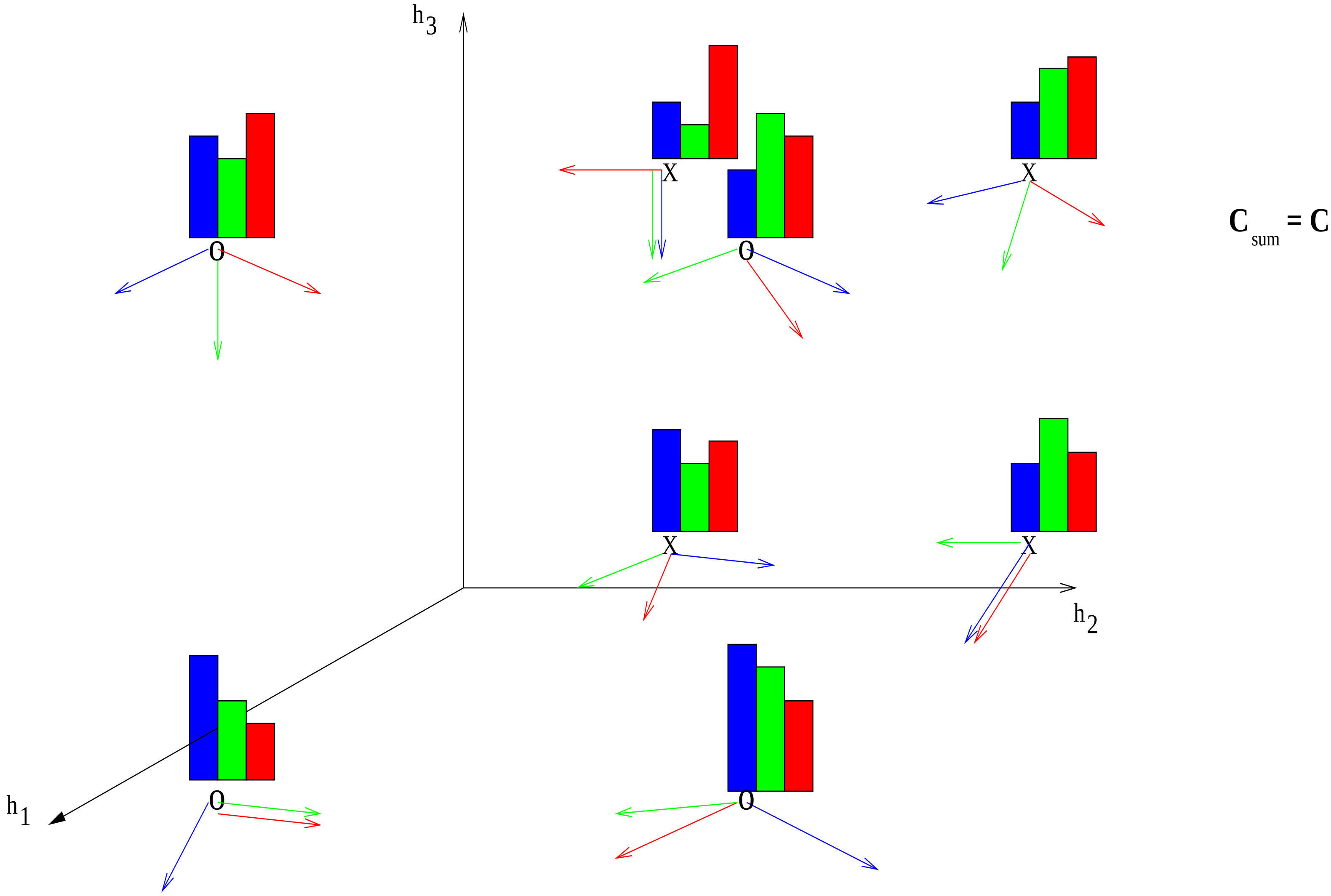

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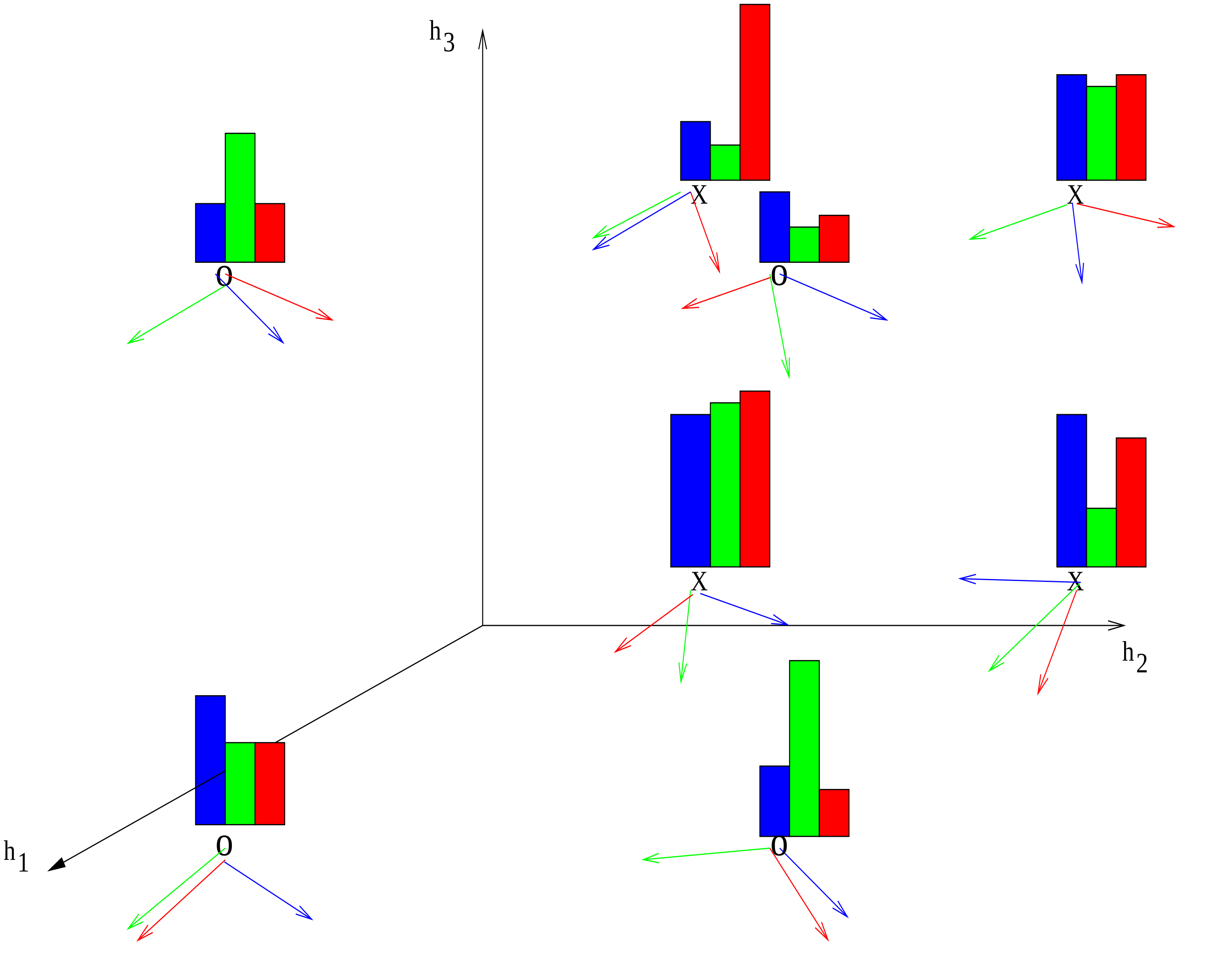


$C_{sum} = C'$

Joint Power and Waveform Allocation – $K \leq N$

- Optimal waveforms constitute an orthogonal set for any power allocation.
- Problem reduces to K independent single user [Goldsmith-Varaiya] problems, i.e.,

$$\begin{aligned} \max_{\mathbf{p}(\mathbf{h})} \quad & E_{\mathbf{h}}\left[\sum_{i=1}^{K} \log\left(1+\frac{p_i(\mathbf{h})h_i}{\sigma^2}\right)\right] \\ \text{s.t.} \quad & E_{\mathbf{h}}\left[p_i(\mathbf{h})\right] = \bar{p}_i, \quad i = 1, \cdots, K \end{aligned}$$

- Concave maximization over an affine set of constraints, using KKT conditions,

$$p_i^*(\mathbf{h}) = \left(\frac{1}{\lambda_i} - \frac{\sigma^2}{h_i}\right)^+, \qquad i = 1, \cdots, K$$

- Channel non-adaptive waveform selection is as good as any channel adaptive selection.

Joint Power and Waveform Allocation – $K > N$

- For a given power control policy $\mathcal{P}(\mathbf{h})$, let $L(\mathbf{h})$ and $\bar{L}(\mathbf{h})$ be sets of oversized and non-oversized users respectively, for a given $\mathbf{h}$.
- Define $\mathbf{D} \triangleq \text{diag}(p_1 h_1, \cdots, p_K h_K)$. Optimum waveforms satisfy,

$$\mathbf{SDS}^\top \mathbf{s}_i(\mathbf{h}) = \mu_i(\mathbf{h}) \mathbf{s}_i(\mathbf{h})$$

$$\mu_i(\mathbf{h}) = \begin{cases} \frac{\sum_{j \in \bar{L}(\mathbf{h})} p_j h_j}{N - |L(\mathbf{h})|}, & i \in \bar{L}(\mathbf{h}) \\ p_i h_i, & i \in L(\mathbf{h}) \end{cases}$$

- The waveform-optimized ergodic sum-capacity is then

$$E_{\mathbf{h}} \left[\sum_{i \in L(\mathbf{h})} \log\left(1 + \frac{p_i(\mathbf{h}) h_i}{\sigma^2}\right) + (N - |L(\mathbf{h})|) \log\left(1 + \frac{\sum_{i \in \bar{L}(\mathbf{h})} p_i(\mathbf{h}) h_i}{\sigma^2 (N - |L(\mathbf{h})|}\right) \right]$$

Maximum Number of Simultaneously Transmitting Users

Theorem 1 *Let* $\bar{K}(\mathbf{h})$ *be a subset of* $\{1, \cdots, K\}$, *such that* $\forall i \in \bar{K}(\mathbf{h})$, $p_i^*(\mathbf{h}) > 0$, *where* $\mathbf{p}^*(\mathbf{h})$ *is the maximizer of* $E_{\mathbf{h}}\left[C_{opt}(\mathbf{h}, \mathbf{p}(\mathbf{h}))\right]$. *Then, with probability 1,* $|\bar{K}(\mathbf{h})| \leq N$.

Proof:

- $C_{\text{opt}}(\mathbf{h}, \mathbf{p}(\mathbf{h}))$ is concave [Viswanath-Anantharam]
- Power constraint set is convex (affine).
- $\mathbf{p}^*(\mathbf{h})$ achieves the global optimum of the sum-capacity $\Leftrightarrow$ it satisfies the KKT conditions.

$$\frac{h_i}{\mu_i(\mathbf{h}) + \sigma^2} \leq \lambda_i, \qquad \forall \mathbf{h} \qquad \text{w.e. if } p_i(\mathbf{h}) > 0$$

- Let $|\bar{K}(\mathbf{h})| > N$. Then, at least $|\bar{K}(\mathbf{h})| - N + 1$ users have the same eigenvalue $\mu_i(\mathbf{h})$.
- Then, $h_i/\lambda_i = h_j/\lambda_j$ for $i \neq j$, $i, j \in \bar{K}(\mathbf{h})$ for at least $|\bar{K}(\mathbf{h})| - N + 1$ users.
- This event has zero probability, therefore, with probability one, $|\bar{K}(\mathbf{h})| \leq N$.

Jointly Optimum Waveforms and Powers – $K > N$

- At most N users transmit: assign orthogonal waveforms to those users.
- Optimum power allocation is similar to single user waterfilling

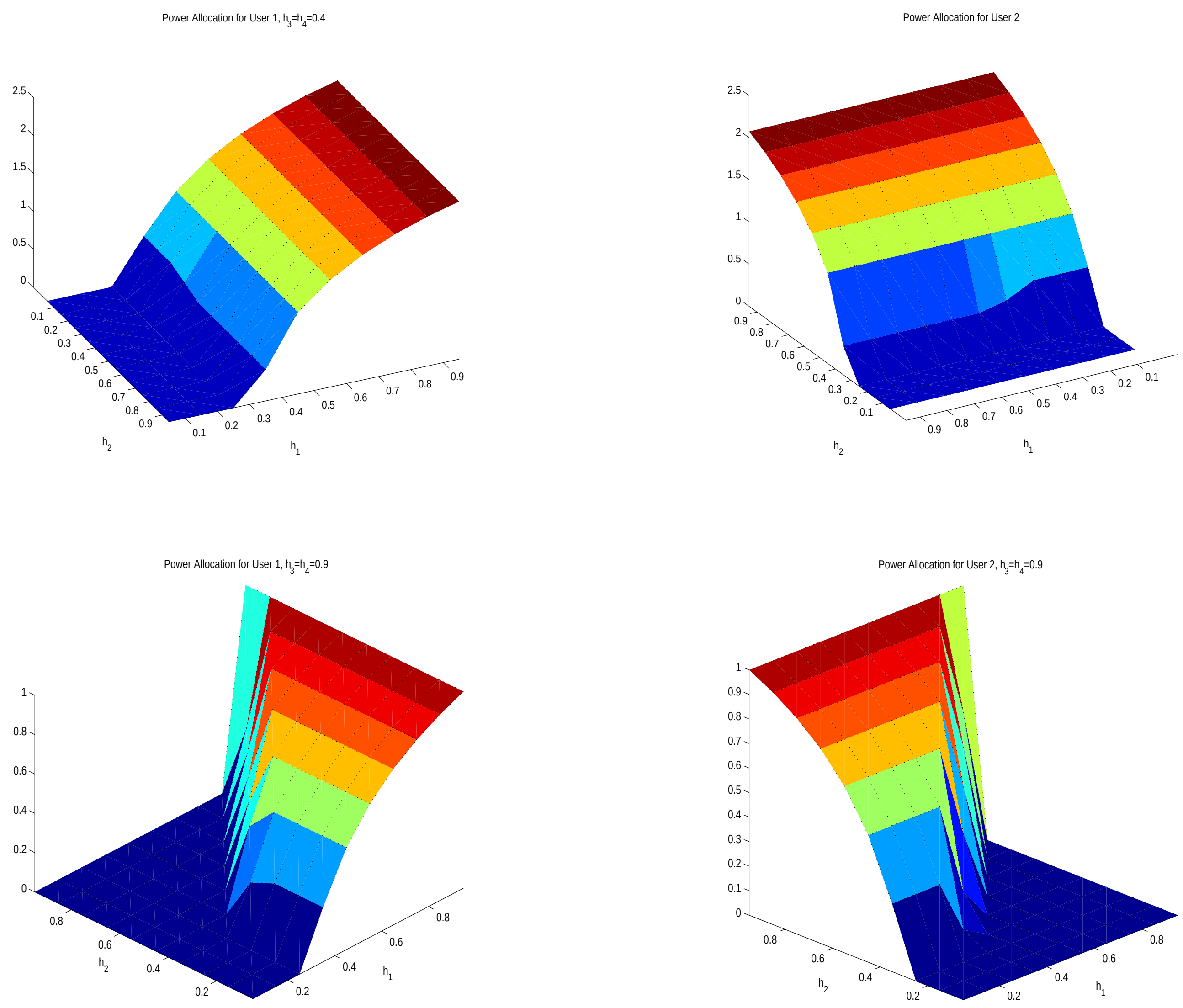
$$p_i^*(\mathbf{h}) = \begin{cases} \left(\frac{1}{\lambda_i} - \frac{\sigma^2}{h_i}\right), & i \in \bar{K}(\mathbf{h}) \\ 0, & \text{otherwise} \end{cases}$$

- Here, a channel adaptive allocation of orthogonal waveforms is necessary.
- Define $\gamma_i = h_i/\lambda_i$, and let $\{\gamma_{[i]}\}_{i=1}^K$ be the order statistics for γ_is, and let for given $\mathbf{h}$

$$\gamma_{[1]} \geq \cdots \geq \gamma_{[n]} > \sigma^2 \geq \gamma_{[n+1]} \geq \cdots \geq \gamma_{[K+1]} = 0$$

- If $n \leq N$, the users with highest n γ_i's transmit with powers $p_i^*(\mathbf{h})$.
- If $n > N$, by Theorem 1, the users with highest N γ_i's transmit with positive powers.

Optimum Power Allocation: $K = 4, N = 3$



Iterative Power and Waveform Optimization

- Already characterized a "closed form" solution for optimal powers and waveforms.
- The optimum resource allocation still depends on $\lambda_i, \ i = 1, \cdots, K$.
- Instead of simultaneously solving for all powers, we propose the following algorithm:

repeat

    for $i = 1$ *to* K *and for all* $\mathbf{h}$

      -find oversized users

      -compute waveforms for all users

      -update ith user's power using waterfilling keeping other powers fixed

    end

until $\mathbf{p}(\mathbf{h})$ *converges.*

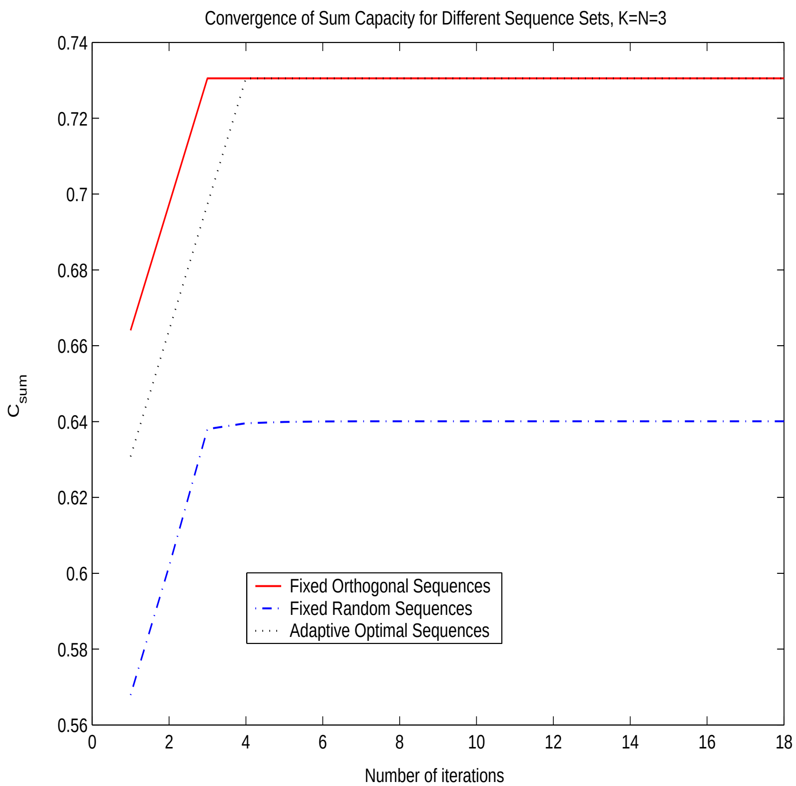
Convergence of the Iterative Algorithm

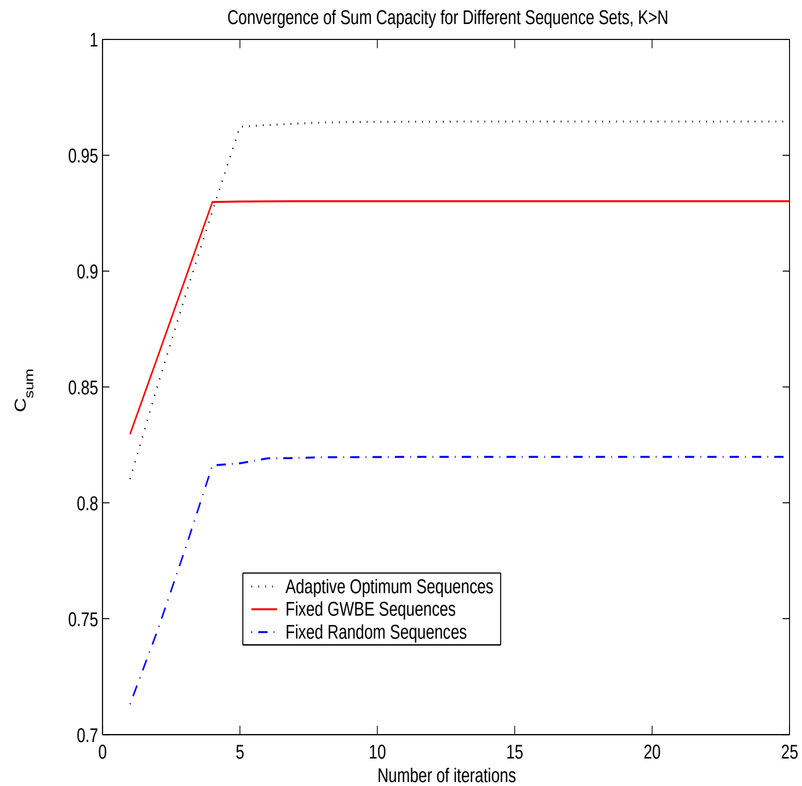
- This algorithm corresponds to iteration of the best waveform-only update for all users and best power-only update for one user, so sum capacity values obtained are non-decreasing.
- The sum capacity is also bounded from above, so this algorithm converges to a limit.
- Same algorithm can be seen as an iterative update directly from powers-to-powers

$$p_k^{n+1}(\mathbf{h}) = \left(\frac{1}{\lambda_k} - \frac{\sigma^2 + \mu_k^n(\mathbf{h}) - h_k p_k^n(\mathbf{h})}{h_k}\right)^+$$

- The fixed point $\mathbf{p}^{n+1}(\mathbf{h}) = \mathbf{p}^n(\mathbf{h})$ satisfies the KKT conditions for the optimization problem.
- Algorithm converges to the jointly optimum power and waveform allocation.
- **Remark:** Optimum power allocation is unique, optimum waveform allocation is not.

Convergence and Comparison to Non-Adaptive Policies





Summary

- Characterized optimum power allocation in fading waveform channels
  - Developed an iterative waterfilling algorithm; proved its convergence to global optimum
  - All users transmit simul. with non-zero prob. iff $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent
    * $K \leq N$, signatures independent: all users transmit simultaneously with > 0 probability.
    * Maximum number of users that can transmit simul. is $M(M+1)/2$; $M = \text{rank}(\mathbf{S})$.
- Characterized jointly optimum power and waveform adaptation policy
  - Optimal policy dictates orthogonal transmissions, achieved by
    * time division across fading states [Knopp-Humblet-like]
    * orthogonal waveforms for multiple users transmitting at a given state
  - Developed an iterative algorithm; proved its convergence to global optimum
- The results may be interpreted as
  - Opportunistic scheduling in waveform channels
  - Cross-layer design: interacting/cooperating physical and MAC layers