Dependence Balance Based Outer Bounds for Gaussian Networks with Cooperation and Feedback

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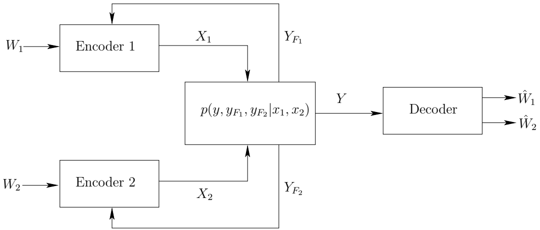
Joint work with Ravi Tandon.

The Multiple Access Channel with Generalized Feedback



- Feedback increases capacity of MAC [Gaarder-Wolf 1975].
- MAC with generalized feedback (MAC-GF) [Carleial 1982].
- Achievable schemes: [Carleial 1982] [Willems-van der Meulen 1983].
- Converse: cut-set outer bound.

Cut-set Outer Bound for MAC-GF

- Cut-set outer bound:

$$\mathcal{CS} = \Big\{(R_1, R_2) : \begin{aligned} R_1 &\leq I(X_1; Y, Y_{F_2} | X_2) \\ R_2 &\leq I(X_2; Y, Y_{F_1} | X_1) \\ R_1 + R_2 &\leq I(X_1, X_2; Y) \end{aligned} \Big\}$$

for some $p(x_1, x_2)$ such that

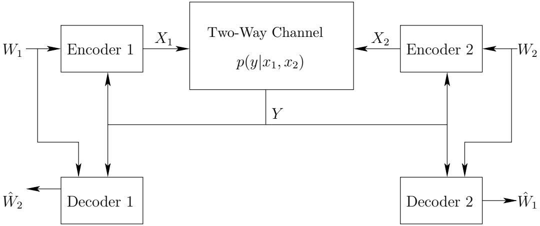
$$p(x_1, x_2, y, y_{F_1}, y_{F_2}) = p(x_1, x_2) p(y, y_{F_1}, y_{F_2} | x_1, x_2).$$

- Cut-set outer bound permits arbitrary input distributions $p(x_1, x_2)$.
- Only optimal for the Gaussian MAC-FB ($Y_{F_1} = Y_{F_2} = Y$) [Ozarow 1984].

Motivation for a New Outer Bound

- Cut-set bound may not be tight in general.
- Possible reason: $\mathcal{CS}$ permits arbitrary correlation between channel inputs.
- Rates of some of these input distributions may not be achievable.
- Need for a converse which restricts the set of input distributions.

Main Idea Behind the New Outer Bound



- MAC with feedback related to single output two-way channel (TWC).
- Dependence balance bounds for the TWC [Hekstra-Willems 1989].
  - Outer bounds for DMC MAC-FB ($Y_{F_1} = Y_{F_2} = Y$) [Tandon-Ulukus 2008].
  - Generalizing this approach: a new outer bound for MAC-GF.

Restricting the Set of Input Distributions

- Proof of the dependence balance constraint:

$$
\begin{aligned}
0 &\leq I(W_1; W_2 | Y_{F_1}^n, Y_{F_2}^n) \\
&= I(W_1; W_2 | Y_{F_1}^n, Y_{F_2}^n) - I(W_1; W_2) \\
&= -I_3(W_1; W_2; Y_{F_1}^n, Y_{F_2}^n) \\
&= \sum_{i=1}^{n} \big[- H(Y_{F_1 i}, Y_{F_2 i} | Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) + H(Y_{F_1 i}, Y_{F_2 i} | W_1, Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) \\
&\quad + H(Y_{F_1 i}, Y_{F_2 i} | W_2, Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) - H(Y_{F_1 i}, Y_{F_2 i} | W_1, W_2, Y_{F_1}^{i-1}, Y_{F_2}^{i-1})\big] \\
&\quad . \\
&\quad . \\
&\quad . \\
&= \sum_{i=1}^{n} I(X_{1i}; X_{2i} | Y_{F_1 i}, Y_{F_2 i}, Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) - \sum_{i=1}^{n} I(X_{1i}; X_{2i} | Y_{F_1}^{i-1}, Y_{F_2}^{i-1}) \\
&= n(I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T_1, T_2) - I(X_1; X_2 | T_1, T_2))
\end{aligned}
$$

Dependence Balance Constraint

- Main steps in the proof:
  - Seemingly trivial inequality $0 \leq I(W_1; W_2 | Y_{F_1}^n, Y_{F_2}^n)$.
  - Symmetry of $I_3(A; B; C) = I(A; B) - I(A; B|C)$.
  - Encoding functions: $X_{1i} = f_{1i}(W_1, Y_{F_1}^{i-1}), X_{2i} = f_{2i}(W_2, Y_{F_2}^{i-1})$.
  - Conditioning reduces entropy.
- Defining:

$$X_1 = X_{1Q}, X_2 = X_{2Q}, Y_{F_1} = Y_{F_1 Q}, Y_{F_2} = Y_{F_2 Q},$$
$$T_1 = (Q, Y_{F_1}^{Q-1}), T_2 = (Q, Y_{F_2}^{Q-1}), Q \sim \mathrm{Unif}\{1, \ldots, n\}.$$

- Resulting *DB* constraint:

$$I(X_1; X_2 | T_1, T_2) \leq I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T_1, T_2)$$

- Interpretation:
  - Dependence consumed: $I(X_1; X_2 | T_1, T_2)$.
  - Dependence produced: $I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T_1, T_2)$.

Parallel Channel Extension of Dependence Balance

- For any $p^+(z|x_1, x_2, y, y_{F_1}, y_{F_2})$
- Starting from

$$0 \leq I(W_1; W_2|Y_{F_1}^n, Y_{F_2}^n, Z^n) - I(W_1; W_2)$$

- Rest of the proof similar to the proof of DB.
- Defining:

$$X_1 = X_{1Q}, X_2 = X_{2Q}, Y_{F_1} = Y_{F_1Q}, Y_{F_2} = Y_{F_2Q}, Z = Z_Q$$
$$T_1 = (Q, Y_{F_1}^{Q-1}, Z^{Q-1}), T_2 = (Q, Y_{F_2}^{Q-1}, Z^{Q-1}), Q \sim \text{Unif}\{1, \ldots, n\}.$$

- Choice of Z effects the definition of auxiliary random variables.
- Resulting DB constraint:

$$I(X_1; X_2|T_1, T_2) \leq I(X_1; X_2|Z, Y_{F_1}, Y_{F_2}, T_1, T_2)$$

- Selecting $Z = \phi$ yields the regular DB constraint.

A Typical Rate Constraint

- Rate upper bounds for MAC-GF:

$$\begin{aligned}
nR_1 &= H(W_1) \\
&= I(W_1; Y^n, Y^n_{F_2}, Z^n) + H(W_1|Y^n, Y^n_{F_2}, Z^n) \\
&\leq I(W_1; Y^n, Y^n_{F_2}, Z^n) + n\epsilon_1^{(n)} \\
&\quad \dots \\
&= \sum_{i=1}^{n} I(X_{1i}; Y_i, Y_{F_2 i}, Z_i|X_{2i}, Y^{i-1}_{F_2}, Z^{i-1}) + n\epsilon_1^{(n)} \\
&= nI(X_1; Y, Y_{F_2}, Z|X_2, T_2) + n\epsilon_1^{(n)} \\
&= nI(X_1; Y, Y_{F_2}|X_2, T_2) + n\,\underbrace{I(X_1; Z|X_2, Y, Y_{F_2}, T_2)}_{L_1} + n\epsilon_1^{(n)}
\end{aligned}$$

- L_1 is the excess rate compared to the rate R_1 for the choice $Z = \phi$.
- Tradeoff in restricting set of $p(t_1, t_2, x_1, x_2)$ and information leaks L_1, L_2.
- $Z = \phi$, yields regular DB constraint and $L_1 = L_2 = 0$.
- $Z = X_1$, smallest permissible set of distributions, but $L_1 \geq 0$.

Application of Dependence Balance to MAC with Output Feedback

- MAC with noiseless output feedback: $Y_{F_1} = Y_{F_2} = Y$.
- Selecting $Z = X_1$ yields the following outer bound:

$$\mathcal{DB} = \bigcup_{X_1 \to T \to X_2} \Big\{(R_1, R_2) : \begin{aligned} R_1 &\leq I(X_1; Y|X_2, T) + H(X_1|Y, X_2, T) \\ R_2 &\leq I(X_2; Y|X_1, T) \\ R_1 &\leq I(X_1; Y|X_2) \\ R_1 + R_2 &\leq I(X_1, X_2; Y) \Big\} \end{aligned}$$

- For the class of MAC such that $X_1 = g(X_2, Y)$, $L_1 = H(X_1|X_2, Y, T) = 0$.
- $\mathcal{DB}$ matches the Cover-Leung achievable rate region [1981].
- $\mathcal{DB}$ equals the feedback capacity region for this class [Willems 1982].
- For the class for which $X_1 \neq g(X_2, Y)$ [Tandon-Ulukus 2008].
  - $\mathcal{DB}$ strictly improves upon the cut-set bound.
  - Explicit evaluation of bounds using composite functions.

A New Outer Bound for MAC-GF using Dependence Balance

- New outer bound:

$$\mathcal{DB}^{MAC} = \big\{(R_1, R_2) : \begin{aligned} R_1 &\leq I(X_1; Y, Y_{F_2}|X_2, T_2) \\ R_2 &\leq I(X_2; Y, Y_{F_1}|X_1, T_1) \\ R_1 + R_2 &\leq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2}|T_1, T_2) \\ R_1 + R_2 &\leq I(X_1, X_2; Y)\big\} \end{aligned}$$

  for some $p(t_1, t_2, x_1, x_2)$ such that

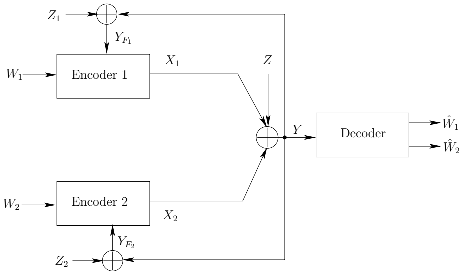
$$p(t_1, t_2, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t_1, t_2, x_1, x_2)p(y, y_{F_1}, y_{F_2}|x_1, x_2)$$

  and such that

$$I(X_1; X_2|T_1, T_2) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T_1, T_2)$$

- Difficult to evaluate, two auxiliary random variables (T_1, T_2).
- Modify $\mathcal{DB}^{MAC}$ according to the channel model in consideration.
  - MAC with noisy feedback (MAC-NF).
  - MAC with user cooperation (MAC-UC).

The Gaussian MAC with Noisy Feedback (MAC-NF)



- Channel model:

$$Y = X_1 + X_2 + Z$$
$$Y_{F_1} = Y + Z_1$$
$$Y_{F_2} = Y + Z_2$$

- Feedback signals (Y_{F_1}, Y_{F_2}) are degraded versions of Y.
- Cut-set bound not sensitive to $\sigma^2_{Z_1}, \sigma^2_{Z_2}$ for MAC-NF.
- For all $\sigma^2_{Z_1}, \sigma^2_{Z_2}$, $\mathcal{CS} = \mathcal{C}_{Ozarow}$, i.e., capacity with noiseless feedback.
- Need a converse reflecting the uselessness of feedback as $\sigma^2_{Z_1}, \sigma^2_{Z_2} \to \infty$.

$\mathcal{DB}$ Outer Bound for Gaussian MAC-NF

- Use the special channel structure.
- Modify $\mathcal{DB}^{MAC}$ to express it in terms of one auxiliary random variable, T.
- Modified outer bound:

$$\mathcal{DB}^{MAC}_{NF} = \{(R_1, R_2) : \begin{aligned} & R_1 \leq I(X_1; Y|X_2, T) \\ & R_2 \leq I(X_2; Y|X_1, T) \\ & R_1 + R_2 \leq I(X_1, X_2; Y|T)\} \end{aligned}$$

for some $p(t, x_1, x_2)$ for which

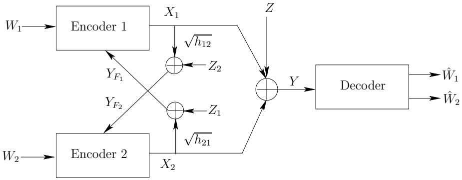
$$p(t, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t, x_1, x_2) p(y|x_1, x_2) p(y_{F_1}|y) p(y_{F_2}|y)$$

and

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$

where the random variable T can be restricted to $|\mathcal{T}| \leq |\mathcal{X}_1||\mathcal{X}_2| + 3$.

The Gaussian MAC with User Cooperation (MAC-UC)



- Channel model:

$$\begin{aligned} Y &= \sqrt{h_{10}}X_1 + \sqrt{h_{20}}X_2 + Z \\ Y_{F_1} &= \sqrt{h_{21}}X_2 + Z_1 \\ Y_{F_2} &= \sqrt{h_{12}}X_1 + Z_2 \end{aligned}$$

- Suitable model for an uplink wireless setting.
- User cooperation diversity [Sendonaris-Erkip-Aazhang 2003].
- Cut-set bound sensitive to $\sigma^2_{Z_1}, \sigma^2_{Z_2}$ but not sensitive enough.
- As $\sigma^2_{Z_1}, \sigma^2_{Z_2} \to \infty$, $\mathcal{CS} \to \mathcal{C}_{Ozarow}$, i.e., capacity with noiseless feedback.

$\mathcal{DB}$ Outer Bound for Gaussian MAC-UC

- Use the special channel structure.
- Modify $\mathcal{DB}^{MAC}$ to express in terms of one auxiliary random variable, T.
- Modified outer bound:

$$\mathcal{DB}_{UC}^{MAC} = \{(R_1, R_2) : \begin{array}{l} R_1 \leq I(X_1; Y, Y_{F_2} | X_2, T) \\ R_2 \leq I(X_2; Y, Y_{F_1} | X_1, T) \\ R_1 + R_2 \leq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2} | T) \\ R_1 + R_2 \leq I(X_1, X_2; Y)\} \end{array}$$

for some $p(t, x_1, x_2)$ for which

$$p(t, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t, x_1, x_2) p(y | x_1, x_2) p(y_{F_1} | x_2) p(y_{F_2} | x_1)$$

and

$$I(X_1; X_2 | T) \leq I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T)$$

where the random variable T can be restricted to $|\mathcal{T}| \leq |\mathcal{X}_1||\mathcal{X}_2| + 3$.

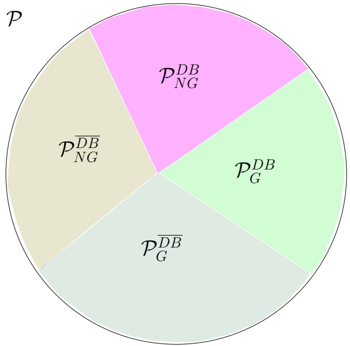
Difficulty in Evaluation of $\mathcal{DB}_{NF}^{MAC}$ and $\mathcal{DB}_{UC}^{MAC}$

- For Gaussian channels, we need to consider joint densities $p(t, x_1, x_2)$.
- The only densities permitted are those which satisfy DB constraint

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$

- Typical difficulty (shortcoming of maximum entropy theorem)
  - Consider a $\mathcal{NG}$ triple (T, X_1, X_2) with density $p(t, x_1, x_2)$ satisfying DB.
  - Covariance matrix of $p(t, x_1, x_2)$ is Q.
  - There does not exist a $\mathcal{JG}$ triple with covariance matrix Q satisfying DB.
- Problem: Maximum entropy theorem fails beyond this point.
- Solution: A new approach to evaluate DB based bounds.

A Systematic Approach for Evaluation of $\mathcal{DB}_{NF}^{MAC}$ and $\mathcal{DB}_{UC}^{MAC}$



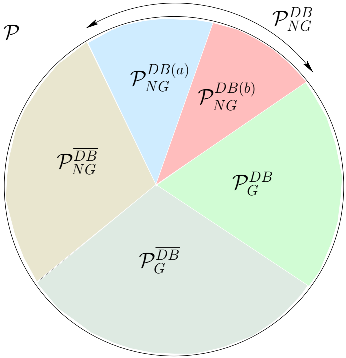
- Partition the set of densities $\mathcal{P}$:

$$\mathcal{P} = \{p(t, x_1, x_2) : E\left[X_1^2\right] \leq P_1, E\left[X_2^2\right] \leq P_2\}$$
$$\mathcal{P}_G = \{p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } \mathcal{JG}\}$$
$$\mathcal{P}_{NG} = \{p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } \mathcal{NG}\}$$

- Further partition $\mathcal{P}_G$ and $\mathcal{P}_{NG}$:

$$\mathcal{P}_G^{DB} = \{p(t, x_1, x_2) \in \mathcal{P}_G : (T, X_1, X_2) \text{ satisfy } (DB)\}$$
$$\mathcal{P}_{NG}^{DB} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG} : (T, X_1, X_2) \text{ satisfy } (DB)\}$$

A Systematic Approach for Evaluation of $\mathcal{DB}_{NF}^{MAC}$ and $\mathcal{DB}_{UC}^{MAC}$



- Further partition the set $\mathcal{P}_{NG}^{DB}$:

$$\mathcal{P}_{NG}^{DB(a)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there exists a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}$$

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there } \textbf{does not exist} \text{ a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}$$

Evaluation of $\mathcal{DB}_{NF}^{MAC}$ and $\mathcal{DB}_{UC}^{MAC}$

- **Claim:** Jointly Gaussian $p(t, x_1, x_2)$ satisfying DB are sufficient to evaluate outer bounds.
- **Proof:**
  - Common step for noisy feedback and user cooperation:
    For every $p(t, x_1, x_2)$ in $\mathcal{P}_{NG}^{DB(a)}$, there exists a jointly Gaussian triple (T_G, X_{1G}, X_{2G}) with covariance matrix Q satisfying DB and yielding larger rates.

    All rates of distributions in $\mathcal{P}_{NG}^{DB(a)}$ are covered by distributions in $\mathcal{P}_G^{DB}$.
  - Main step:
    For every $p(t, x_1, x_2)$ in $\mathcal{P}_{NG}^{DB(b)}$ with covariance matrix Q, we show that there exists a jointly Gaussian triple (T_G, X_{1G}, X_{2G}) with covariance matrix S satisfying DB and yielding larger rates.

    All rates of distributions in $\mathcal{P}_{NG}^{DB(b)}$ are covered by distributions in $\mathcal{P}_G^{DB}$.
  - Therefore, it suffices to consider **only** $\mathcal{P}_G^{DB}$ in evaluating our DB bounds.

Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there } \textbf{does} \\ \textbf{not exist} \text{ a } \mathcal{JG} \ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}$$

- Facts at hand:

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$
$$I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)$$

- Maximum entropy theorem tells us:

$$R_1 \leq I(X_1; Y|X_2, T) \leq I^Q(X_{1G}; Y|X_{2G}, T_G)$$
$$R_2 \leq I(X_2; Y|X_1, T) \leq I^Q(X_{2G}; Y|X_{1G}, T_G)$$
$$R_1 + R_2 \leq I(X_1, X_2; Y|T) \leq I^Q(X_{1G}, X_{2G}; Y|T_G)$$

Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t,x_1,x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t,x_1,x_2) \text{ is } Q \text{ and there } \textbf{does not exist} \text{ a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}$$

- Facts at hand:

$$I(X_1;X_2|T) \leq I(X_1;X_2|Y_{F_1},Y_{F_2},T)$$
$$I^{Q}(X_{1G};X_{2G}|T_G) > I^{Q}(X_{1G};X_{2G}|Y_{F_1},Y_{F_2},T_G)$$

- DB with multivariate Costa's EPI [Payaro-Palomar, ISIT 08] tells us:

$$R_1 \leq I(X_1;Y|X_2,T) \leq I^{Q}(X_{1G};Y|X_{2G},T_G)$$
$$R_2 \leq I(X_2;Y|X_1,T) \leq I^{Q}(X_{2G};Y|X_{1G},T_G)$$
$$R_1 + R_2 \leq I(X_1,X_2;Y|T) \leq f(Q) < I^{Q}(X_{1G},X_{2G};Y|T_G)$$

Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{ p(t,x_1,x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t,x_1,x_2) \text{ is } Q \text{ and there } \textbf{does} \\ \textbf{not exist} \text{ a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB) \}$$

- Facts at hand:

$$I(X_1;X_2|T) \leq I(X_1;X_2|Y_{F_1},Y_{F_2},T)$$
$$I^Q(X_{1G};X_{2G}|T_G) > I^Q(X_{1G};X_{2G}|Y_{F_1},Y_{F_2},T_G)$$

- Construct a cov. matrix S such that:

$$R_1 \leq I(X_1;Y|X_2,T) \leq I^Q(X_{1G};Y|X_{2G},T_G)$$
$$R_2 \leq I(X_2;Y|X_1,T) \leq I^Q(X_{2G};Y|X_{1G},T_G)$$
$$R_1+R_2 \leq I(X_1,X_2;Y|T) \leq I^S(X_{1G},X_{2G};Y|T_G) < I^Q(X_{1G},X_{2G};Y|T_G)$$

Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \big\{ p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there } \textbf{does} \\ \textbf{not exist} \text{ a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB) \big\}$$

- Facts at hand:

$$I(X_1; X_2 | T) \leq I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T)$$
$$I^Q(X_{1G}; X_{2G} | T_G) > I^Q(X_{1G}; X_{2G} | Y_{F_1}, Y_{F_2}, T_G)$$

- Construct a cov. matrix S such that:

$$R_1 \leq I(X_1; Y | X_2, T) \leq I^Q(X_{1G}; Y | X_{2G}, T_G) \leq I^S(X_{1G}; Y | X_{2G}, T_G)$$
$$R_2 \leq I(X_2; Y | X_1, T) \leq I^Q(X_{2G}; Y | X_{1G}, T_G) \leq I^S(X_{2G}; Y | X_{1G}, T_G)$$
$$R_1 + R_2 \leq I(X_1, X_2; Y | T) \leq I^S(X_{1G}, X_{2G}; Y | T_G) < I^Q(X_{1G}, X_{2G}; Y | T_G)$$

Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t,x_1,x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t,x_1,x_2) \text{ is } Q \text{ and there } \textbf{does}$$
$$\textbf{not exist} \text{ a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}$$

- Facts at hand:

$$I(X_1;X_2|T) \leq I(X_1;X_2|Y_{F_1},Y_{F_2},T)$$
$$I^{Q}(X_{1G};X_{2G}|T_G) > I^{Q}(X_{1G};X_{2G}|Y_{F_1},Y_{F_2},T_G)$$

- Construct a cov. matrix S such that:

$$R_1 \leq I(X_1;Y|X_2,T) \leq I^{Q}(X_{1G};Y|X_{2G},T_G) \leq I^{S}(X_{1G};Y|X_{2G},T_G)$$
$$R_2 \leq I(X_2;Y|X_1,T) \leq I^{Q}(X_{2G};Y|X_{1G},T_G) \leq I^{S}(X_{2G};Y|X_{1G},T_G)$$
$$R_1 + R_2 \leq I(X_1,X_2;Y|T) \leq I^{S}(X_{1G},X_{2G};Y|T_G)$$

Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there } \textbf{does not exist} \text{ a } \mathcal{JG} \ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}$$

- Facts at hand:

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$
$$I^Q(X_{1G}; X_{2G}|T_G) > I^Q(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)$$

- Construct a cov. matrix S such that:

$$R_1 \leq I(X_1; Y|X_2, T) \leq I^Q(X_{1G}; Y|X_{2G}, T_G) \leq I^S(X_{1G}; Y|X_{2G}, T_G)$$
$$R_2 \leq I(X_2; Y|X_1, T) \leq I^Q(X_{2G}; Y|X_{1G}, T_G) \leq I^S(X_{2G}; Y|X_{1G}, T_G)$$
$$R_1 + R_2 \leq I(X_1, X_2; Y|T) \leq I^S(X_{1G}, X_{2G}; Y|T_G)$$

- Such that any (T_G, X_{1G}, X_{2G}) with cov. matrix S satisfies,

$$I^S(X_{1G}; X_{2G}|T_G) \leq I^S(X_{1G}; X_{2G}|Y_{F_1}, Y_{F_2}, T_G)$$

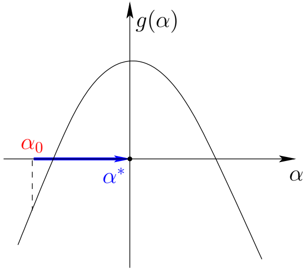
Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Construction of S

- A typical covariance matrix for $P_1 = P_2 = 1$

$$Q = \begin{bmatrix} 1 & \rho_{1T}\rho_{2T} + \alpha_0 & \rho_{1T} \\ \rho_{1T}\rho_{2T} + \alpha_0 & 1 & \rho_{2T} \\ \rho_{1T} & \rho_{2T} & 1 \end{bmatrix}$$

- For $\alpha_0 \leq 0$, and $g(\alpha_0) < 0$, we construct S as,

$$S = \begin{bmatrix} 1 & \rho_{1T}\rho_{2T} & \rho_{1T} \\ \rho_{1T}\rho_{2T} & 1 & \rho_{2T} \\ \rho_{1T} & \rho_{2T} & 1 \end{bmatrix}$$



- The function $g(\alpha)$ is given as

$$g(\alpha) = I^{Q(\alpha)}(X_{1G}; X_{2G} | Y_{F1}, Y_{F2}, T_G) - I^{Q(\alpha)}(X_{1G}; X_{2G} | T_G)$$

- Any (T_G, X_{1G}, X_{2G}) with cov. matrix S satisfies DB.

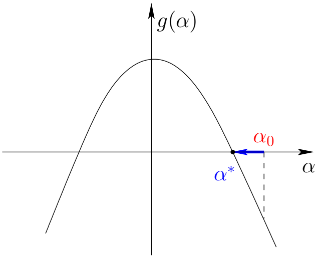
Evaluation of $\mathcal{DB}_{NF}^{MAC}$: Construction of S

- A typical covariance matrix for $P_1 = P_2 = 1$

$$Q = \begin{bmatrix} 1 & \rho_{1T}\rho_{2T} + \alpha_0 & \rho_{1T} \\ \rho_{1T}\rho_{2T} + \alpha_0 & 1 & \rho_{2T} \\ \rho_{1T} & \rho_{2T} & 1 \end{bmatrix}$$

- For $\alpha_0 > 0$, and $g(\alpha_0) < 0$, we construct S as,

$$S = \begin{bmatrix} 1 & \rho_{1T}\rho_{2T} + \alpha^* & \rho_{1T} \\ \rho_{1T}\rho_{2T} + \alpha^* & 1 & \rho_{2T} \\ \rho_{1T} & \rho_{2T} & 1 \end{bmatrix}$$

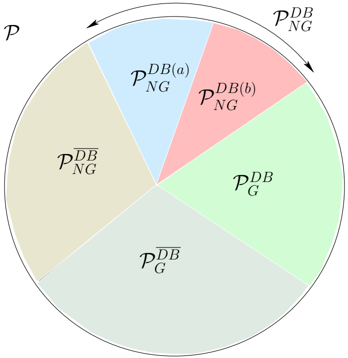


- The function $g(\alpha)$ is given as

$$g(\alpha) = I^{Q(\alpha)}(X_{1G}; X_{2G} | Y_{F1}, Y_{F2}, T_G) - I^{Q(\alpha)}(X_{1G}; X_{2G} | T_G)$$

- Any (T_G, X_{1G}, X_{2G}) with cov. matrix S satisfies DB with equality.

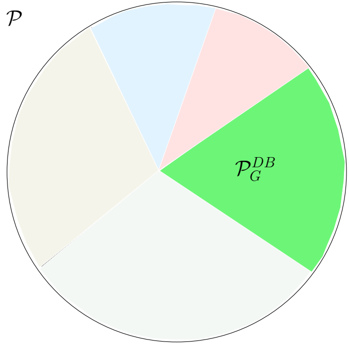
Explicit Evaluation of $\mathcal{DB}_{NF}^{MAC}$



- We only need to consider probability distributions in $\mathcal{P}_G^{DB}$.
- I.e., consider only Gaussian (T, X_1, X_2) satisfying

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T)$$

Explicit Evaluation of $\mathcal{DB}_{NF}^{MAC}$



- We only need to consider probability distributions in $\mathcal{P}_G^{DB}$.
- I.e., consider only Gaussian (T, X_1, X_2) satisfying

$$I(X_1; X_2 | T) \leq I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T)$$

$\mathcal{DB}_{NF}^{MAC}$ Outer Bound

- Final expression for the outer bound:

$$\mathcal{DB}_{NF}^{MAC} = \bigcup_{Q \in \mathcal{Q}^{DB}} \left\{ (R_1, R_2) : \begin{array}{l} R_1 \leq \frac{1}{2} \log\left(1 + \frac{f_1(Q)}{\sigma_Z^2}\right) \\ R_2 \leq \frac{1}{2} \log\left(1 + \frac{f_2(Q)}{\sigma_Z^2}\right) \\ R_1 + R_2 \leq \frac{1}{2} \log\left(1 + \frac{f_3(Q)}{\sigma_Z^2}\right) \end{array} \right\}$$

where

$$f_1(Q) = \mathrm{Var}(X_{1G}|X_{2G}, T_G), \quad f_2(Q) = \mathrm{Var}(X_{2G}|X_{1G}, T_G)$$
$$f_3(Q) = \mathrm{Var}(X_{1G}|T_G) + \mathrm{Var}(X_{2G}|T_G) + 2\mathrm{Cov}(X_{1G}, X_{2G}|T_G)$$

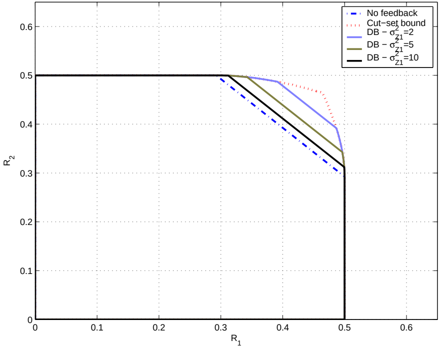
and $\mathcal{Q}^{DB}$ is the set of such 3×3 matrices satisfying

$$f_3(Q) \leq f_1(Q) + f_2(Q) + \frac{f_1(Q) f_2(Q)}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)}\right)}$$

Limiting Behavior of $\mathcal{DB}_{NF}^{MAC}$ and $\mathcal{CS}$ Bounds

- Cut-set bound is not sensitive to $\sigma_{Z_1}^2, \sigma_{Z_2}^2$.
  - For all $\sigma_{Z_1}^2, \sigma_{Z_2}^2$, $\mathcal{CS} = \mathcal{C}_{Ozarow}$, i.e., capacity with noiseless feedback.
- Our bound depends on the noise variances.
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to 0$:
    We recover capacity with noiseless feedback [Ozarow 1984].
    $$f_3(Q) \leq f_1(Q) + f_2(Q) + \frac{f_1(Q) f_2(Q)}{\sigma_Z^2}$$
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to \infty$:
    We recover capacity without feedback.
    $$f_3(Q) \leq f_1(Q) + f_2(Q)$$

Illustration of Bounds for Gaussian MAC with Noisy Feedback



$$P_1 = P_2 = \sigma_Z^2 = 1 \text{ and } \sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 2, 5, 10.$$

- Cut-set bound is insensitive to the feedback noise variances $\sigma_{Z_1}^2, \sigma_{Z_2}^2$.
- $\mathcal{DB}_{NF}^{MAC}$ depends on the feedback noise variances $\sigma_{Z_1}^2, \sigma_{Z_2}^2$.
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to 0$, $\mathcal{DB}_{NF}^{MAC} \to \mathcal{CS} = \mathcal{C}_{Ozarow}$.
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to \infty$, $\mathcal{DB}_{NF}^{MAC} \to \mathcal{C}_{No-Feedback}$.

Evaluation of $\mathcal{DB}_{UC}^{MAC}$: Main Step

- For user cooperation, DB is equivalent to

$$I(Y_{F_1}; X_1 | T) = 0$$
$$I(Y_{F_2}; X_2 | Y_{F_1}, T) = 0$$

- A jointly Gaussian (T, X_1, X_2) satisfies DB iff $X_1 \to T \to X_2$.
  - Equivalently, covariance matrix Q of (T, X_1, X_2) is such that $\rho_{12} = \rho_{1T}\rho_{2T}$.

- Consider any $p(t, x_1, x_2) \in \mathcal{DB}_{NG}^{DB(b)}$ with a cov. matrix Q satisfying

$$I(Y_{F_1}; X_1 | T) = 0$$

  - This implies $\rho_{12} \neq \rho_{1T}\rho_{2T}$.

Evaluation of $\mathcal{DB}_{UC}^{MAC}$: Construction of S

- Construct a triple $(T^{'}, X_1, X_2)$ with a covariance matrix S by selecting

$$T^{'} = E[X_1|T]$$

- The selection related to [Bross-Wigger-Lapidoth ISIT 2008].
- For such selection,

$$E[X_1 X_2] = \frac{E[X_1 T^{'}]E[X_2 T^{'}]}{Var(T^{'})}$$

- We also have,

$$E[X_1 X_2] = \rho_{12}\sqrt{P_1 P_2}$$
$$E[X_1 T^{'}] = \rho_{1T'}\sqrt{P_1 P_{T'}}$$
$$E[X_2 T^{'}] = \rho_{2T'}\sqrt{P_2 P_{T'}}$$

- Implying that covariance matrix S of $(T^{'}, X_1, X_2)$ satisfies

$$\rho_{12} = \rho_{1T'}\rho_{2T'}$$

- Any jointly Gaussian $(T^{'}_G, X_{1G}, X_{2G})$ with covariance matrix S satisfies DB.

Evaluation of $\mathcal{DB}_{UC}^{MAC}$: Main Step

- Consider any jointly Gaussian $(T_G^{'}, X_{1G}, X_{2G})$ with covariance matrix S.

$$
\begin{aligned}
I(X_{1G}; Y, Y_{F_2}|X_{2G}, T_G^{'}) &= h(Y, Y_{F_2}|X_{2G}, T_G^{'}) - h(Y, Y_{F_2}|X_{1G}, X_{2G}, T_G^{'}) \\
&= h(\sqrt{h_{10}}X_{1G} + Z, \sqrt{h_{12}}X_{1G} + Z_2|X_{2G}, T_G^{'}) \\
&\quad - h(Y, Y_{F_2}|X_{1G}, X_{2G}, T_G^{'}) \\
&\geq h(\sqrt{h_{10}}X_1 + Z, \sqrt{h_{12}}X_1 + Z_2|X_2, T^{'})) \\
&\quad - h(Y, Y_{F_2}|X_{1G}, X_{2G}, T_G^{'}) \\
&\geq h(\sqrt{h_{10}}X_1 + Z, \sqrt{h_{12}}X_1 + Z_2|X_2, T^{'}, T)) \\
&\quad - h(Y, Y_{F_2}|X_1, X_2, T) \\
&= I(X_1; Y, Y_{F_2}|X_2, T)
\end{aligned}
$$

- Similarly,

$$
\begin{aligned}
I(X_{2G}; Y, Y_{F_1}|X_{1G}, T_G^{'}) &\geq I(X_2; Y, Y_{F_1}|X_1, T) \\
I(X_{1G}, X_{2G}; Y, Y_{F_1}, Y_{F_2}|T_G^{'}) &\geq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2}|T) \\
I(X_{1G}, X_{2G}; Y) &\geq I(X_1, X_2; Y)
\end{aligned}
$$

Evaluation of $\mathcal{DB}_{UC}^{MAC}$: Main Step

- Recall the definition of the set $\mathcal{P}_{NG}^{DB(b)}$:

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t,x_1,x_2) \in \mathcal{P}_{NG}^{DB} : \text{ cov. matrix of } p(t,x_1,x_2) \text{ is } Q \text{ and there } \textbf{does not exist} \text{ a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with cov. matrix } Q \text{ satisfying } (DB)\}$$

- Facts at hand:

$$I(X_1;X_2|T) \leq I(X_1;X_2|Y_{F_1},Y_{F_2},T)$$

$$I^Q(X_{1G};X_{2G}|T_G) > I^Q(X_{1G};X_{2G}|Y_{F_1},Y_{F_2},T_G)$$

- Construct a cov. matrix S such that:

$$R_1 \leq I^Q(X_{1G};Y,Y_{F_2}|X_{2G},T_G) \leq I^S(X_{1G};Y,Y_{F_2}|X_{2G},T_G)$$

$$R_2 \leq I^Q(X_{2G};Y,Y_{F_1}|X_{1G},T_G) \leq I^S(X_{2G};Y,Y_{F_1}|X_{1G},T_G)$$

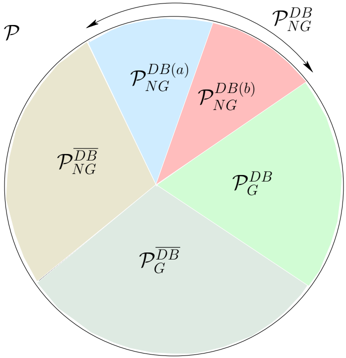
$$R_1+R_2 \leq I^Q(X_{1G},X_{2G};Y,Y_{F_1},Y_{F_2}|T_G) \leq I^S(X_{1G},X_{2G};Y,Y_{F_1},Y_{F_2}|T_G)$$

$$R_1+R_2 \leq I^Q(X_{1G},X_{2G};Y) \leq I^S(X_{1G},X_{2G};Y)$$

- Such that any (T_G, X_{1G}, X_{2G}) with cov. matrix S satisfies,

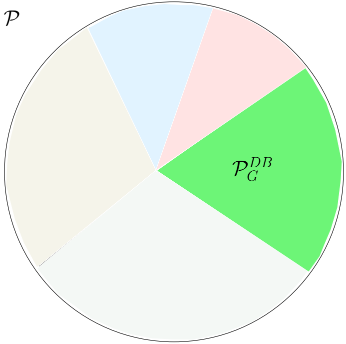
$$I^S(X_{1G};X_{2G}|T_G) = I^S(X_{1G};X_{2G}|Y_{F_1},Y_{F_2},T_G)$$

Explicit Evaluation of $\mathcal{DB}_{UC}^{MAC}$



- We only need to consider probability distributions in $\mathcal{P}_{G}^{DB}$.
- I.e., consider only Gaussian (T, X_1, X_2) satisfying $X_1 \rightarrow T \rightarrow X_2$.

Explicit Evaluation of $\mathcal{DB}_{UC}^{MAC}$



- We only need to consider probability distributions in $\mathcal{P}_G^{DB}$.
- I.e., consider only Gaussian (T, X_1, X_2) satisfying $X_1 \to T \to X_2$.

$\mathcal{DB}_{UC}^{MAC}$ Outer Bound

- Final expression for the outer bound:

$$\mathcal{DB}_{UC}^{MAC} = \bigcup_{(\rho_{1T},\rho_{2T})\in[0,1]\times[0,1]} \Big\{ (R_1, R_2) : \begin{aligned} R_1 &\le \frac{1}{2}\log\left(1 + f_1(\rho_{1T})\right) \\ R_2 &\le \frac{1}{2}\log\left(1 + f_2(\rho_{2T})\right) \\ R_1 + R_2 &\le \frac{1}{2}\log\left(1 + f_3(\rho_{1T}, \rho_{2T})\right) \\ R_1 + R_2 &\le \frac{1}{2}\log\left(1 + f_4(\rho_{1T}, \rho_{2T})\right) \end{aligned} \Big\}$$

where

$$f_1(\rho_{1T}) = (1-\rho_{1T}^2)P_1\left(\frac{h_{10}}{\sigma_Z^2} + \frac{h_{12}}{\sigma_{Z_2}^2}\right), \quad f_2(\rho_{2T}) = (1-\rho_{2T}^2)P_2\left(\frac{h_{20}}{\sigma_Z^2} + \frac{h_{21}}{\sigma_{Z_1}^2}\right)$$

$$f_3(\rho_{1T}, \rho_{2T}) = f_1(\rho_{1T}) + f_2(\rho_{2T}) + (1-\rho_{1T}^2)(1-\rho_{2T}^2)P_1P_2\beta$$

$$f_4(\rho_{1T}, \rho_{2T}) = \frac{\left(h_{10}P_1 + h_{20}P_2 + 2\rho_{1T}\rho_{2T}\sqrt{h_{10}h_{20}P_1P_2}\right)}{\sigma_Z^2}$$

and

$$\beta = \frac{\left(h_{12}h_{21}\sigma_Z^2 + h_{20}h_{12}\sigma_{Z_1}^2 + h_{10}h_{21}\sigma_{Z_2}^2\right)}{\sigma_Z^2\sigma_{Z_1}^2\sigma_{Z_2}^2}$$

Limiting Behavior of $\mathcal{DB}_{UC}^{MAC}$ and $\mathcal{CS}$ Bounds

- As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to 0$:
  - Both $\mathcal{DB}_{UC}^{MAC}$ and $\mathcal{CS}$ degenerate to the total cooperation line.

$$R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{h_{10}P_1 + h_{20}P_2 + 2\sqrt{h_{10}h_{20}P_1P_2}}{\sigma_Z^2}\right)$$

- As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to \infty$:
  - $\mathcal{DB}_{UC}^{MAC} \to$ capacity region without cooperation.

$$f_1(\rho_{1T}) = \frac{(1-\rho_{1T}^2)h_{10}P_1}{\sigma_Z^2} \quad (1)$$

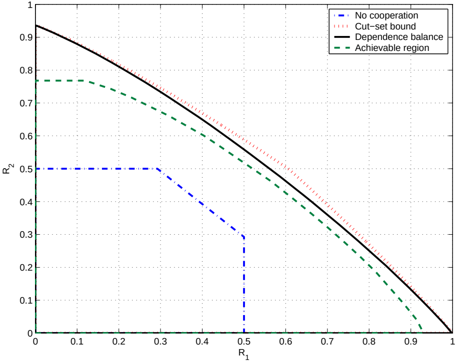
$$f_2(\rho_{2T}) = \frac{(1-\rho_{2T}^2)h_{20}P_2}{\sigma_Z^2} \quad (2)$$

$$f_3(\rho_{1T}, \rho_{2T}) = f_1(\rho_{1T}) + f_2(\rho_{2T}) \quad (3)$$

$$< \frac{(h_{10}P_1 + h_{20}P_2 + 2\rho_{1T}\rho_{2T}\sqrt{h_{10}h_{20}P_1P_2})}{\sigma_Z^2} \quad (4)$$

  - $\mathcal{CS} \to$ capacity region with output feedback [Ozarow 1984].

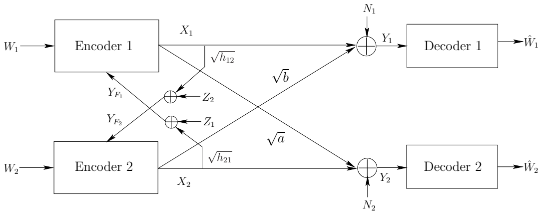
Illustration of Bounds for Gaussian MAC with User Cooperation



$$P_1 = P_2 = \sigma_Z^2 = \sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 1 \text{ and } h_{10} = h_{20} = 1, h_{12} = 3, h_{21} = 2.$$

- Cut-set bound is sensitive to cooperation noise variances $\sigma_{Z_1}^2, \sigma_{Z_2}^2$.
- $\mathcal{DB}_{UC}^{MAC}$ is more sensitive to feedback noise variances $\sigma_{Z_1}^2, \sigma_{Z_2}^2$.
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to 0$, $\mathcal{DB}_{UC}^{MAC} \to \mathcal{CS}$ (degenerates to the total cooperation line).
  - As $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \to \infty$, $\mathcal{DB}_{UC}^{MAC} \to \mathcal{C}_{No-Cooperation}$.

The Gaussian IC with User Cooperation (IC-UC)

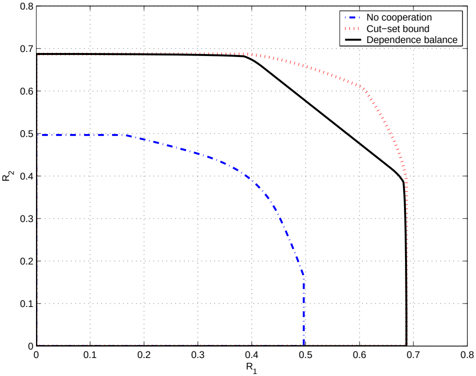


- Channel model:

$$Y_1 = X_1 + \sqrt{b}X_2 + N_1$$
$$Y_1 = \sqrt{a}X_1 + X_2 + N_2$$
$$Y_{F_1} = \sqrt{h_{21}}X_2 + Z_1$$
$$Y_{F_2} = \sqrt{h_{12}}X_1 + Z_2$$

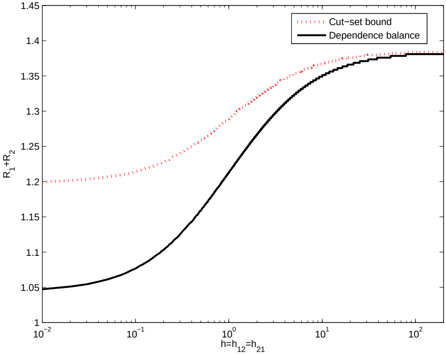
- Similar outer bound for IC-UC.
- It suffices to consider $\mathcal{P}_G^{DB}$ when evaluating $\mathcal{DB}_{UC}^{IC}$.
- Sum-rate $\mathcal{DB}$ based bound for IC-NF [Gastpar-Kramer 2006].

Illustration of Bounds for Gaussian IC with User Cooperation



$P_1 = P_2 = \sigma^2_{N_1} = \sigma^2_{N_2} = \sigma^2_{Z_1} = \sigma^2_{Z_2} = 1$ and $a = b = 0.5$, $h_{12} = h_{21} = 0.1$.

Illustration of Bounds for Gaussian IC with User Cooperation



$P_1 = P_2 = \sigma^2_{N_1} = \sigma^2_{N_2} = \sigma^2_{Z_1} = \sigma^2_{Z_2} = 1$ and $a = b = 0.5, h = h_{12} = h_{21}$.

Conclusions

- Obtained a new outer bound for MAC-GF.
- Application of the new outer bound for two channel models:
  - Gaussian MAC with noisy feedback.
  - Gaussian MAC with user cooperation.
- Similar results for IC-GF.
- A new approach for evaluating bounds involving auxiliary random variables.
- For all non-zero values of $\sigma^2_{Z_1}, \sigma^2_{Z_2}$, our $\mathcal{DB}$ bounds strictly improve over the cut-set bound.