Wireless Communications in Fading Multi-user MIMO and CDMA Channels: Optimum Power Allocation and Optimality of Beamforming

Şennur Ulukuş

Department of ECE

University of Maryland

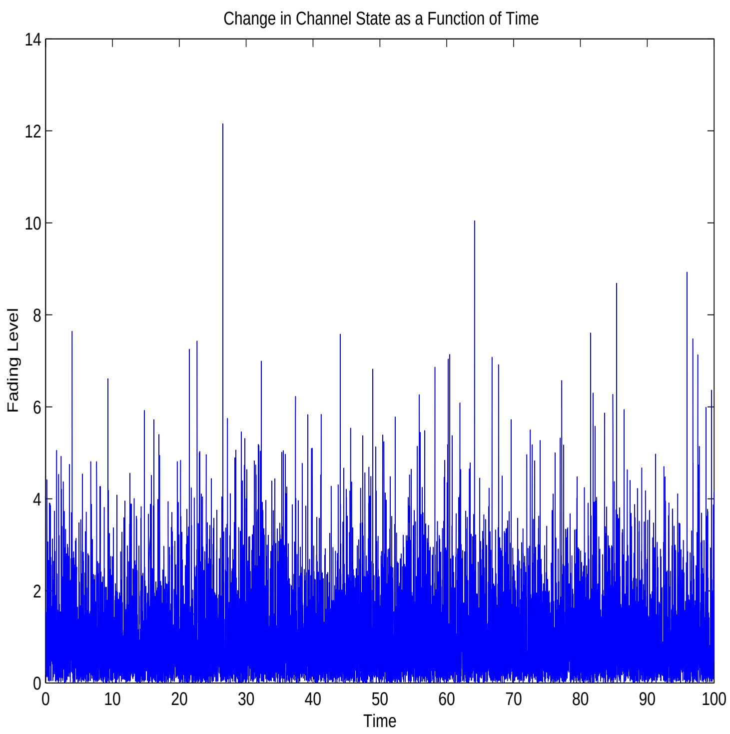
ulukus@umd.edu

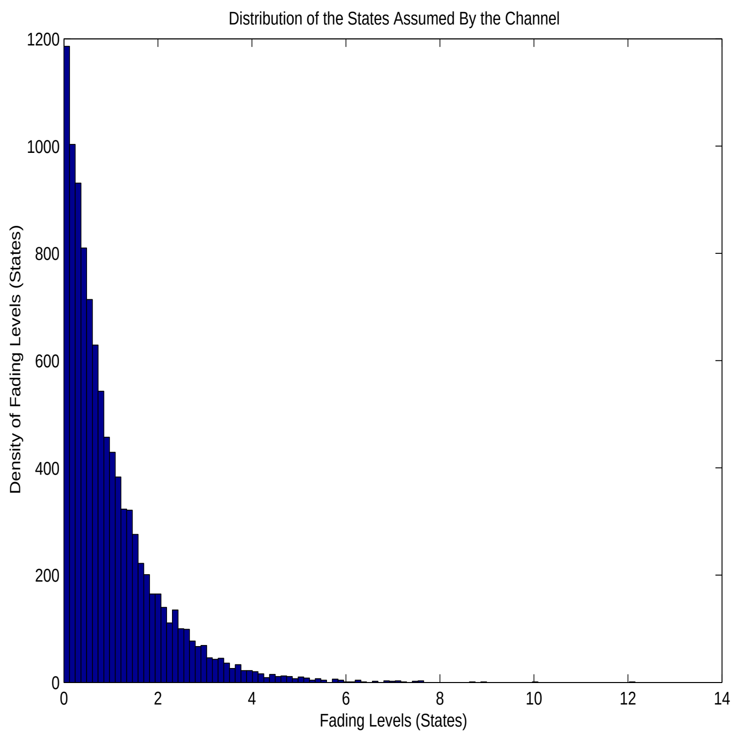
Joint work with Onur Kaya and Alkan Soysal.

Fading

- Fading: random fluctuations in channel gains.

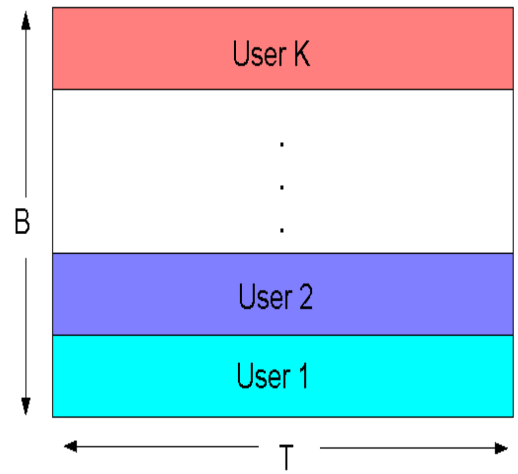
$$r = \sqrt{phx} + n$$



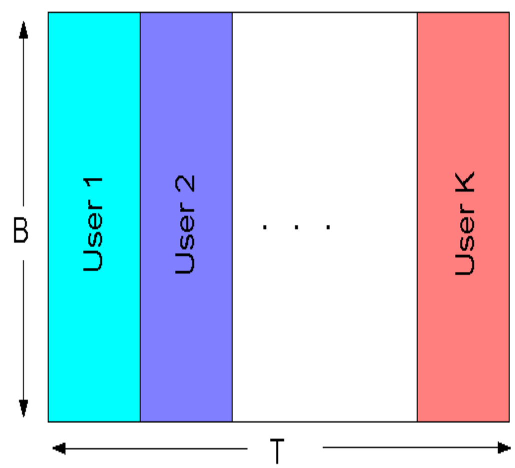


Multiple Access

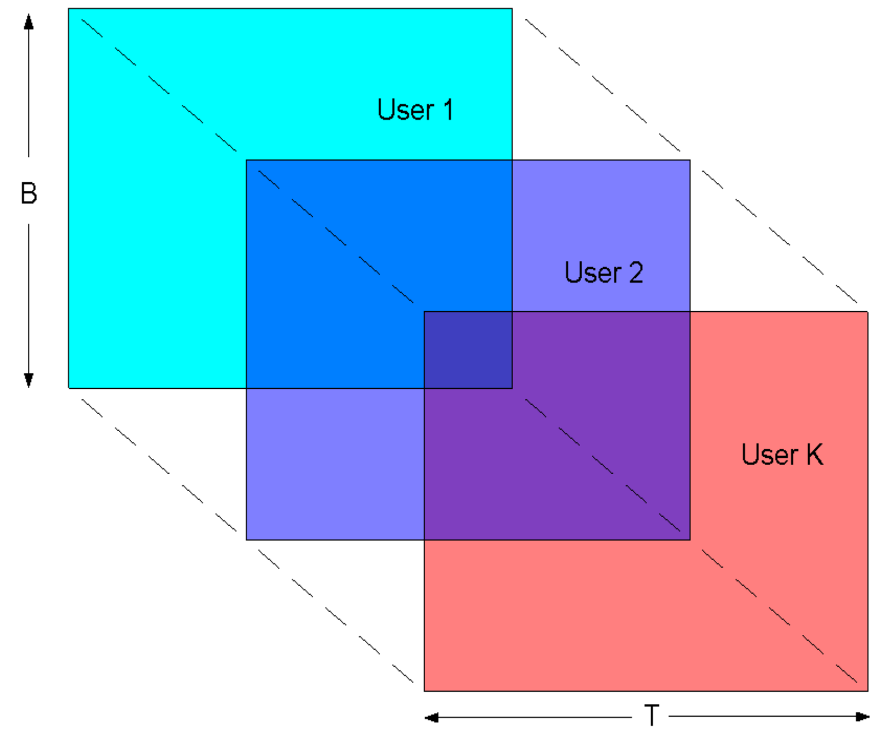
Frequency Division Multiple Access (FDMA)



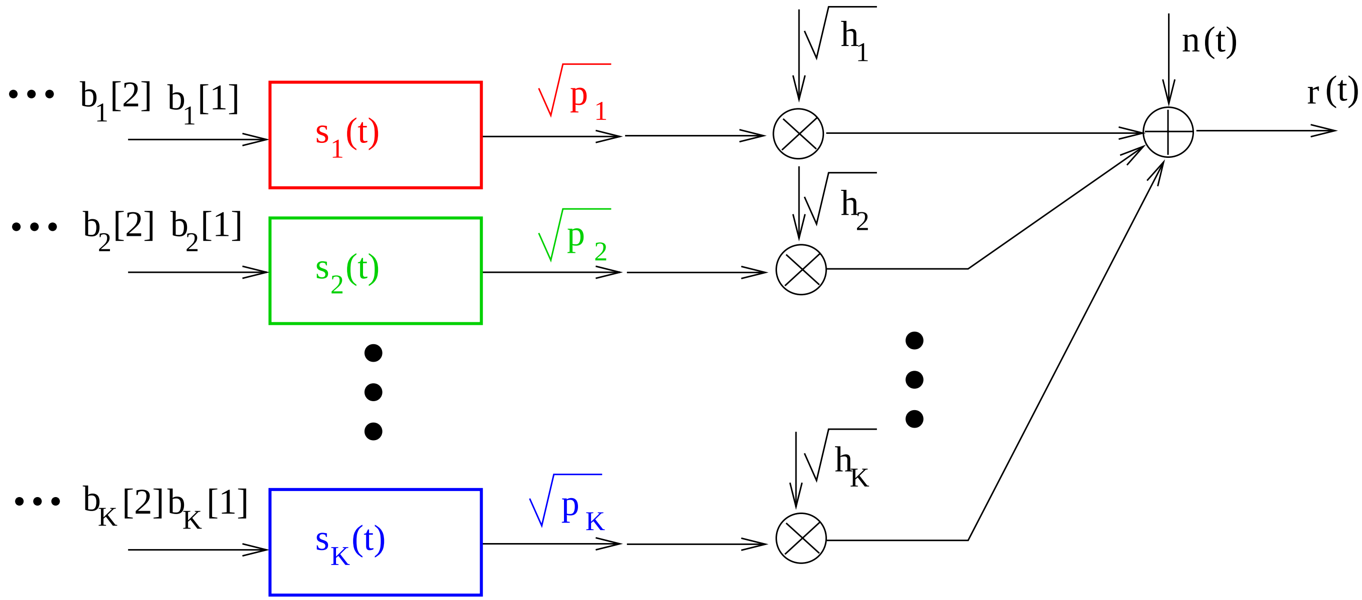
Time Division Multiple Access (TDMA)



Code Division Multiple Access (CDMA)



CDMA Channel Model



- Received signal

$$r(t) = \sum_{i=1}^{K} \sqrt{p_i h_i} b_i s_i(t) + n(t), \qquad 0 < t < T$$

- Project on a orthonormal basis of dimensionality N, to obtain the equivalent vector MAC

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i h_i} b_i \mathbf{s}_i + \mathbf{n}$$

Introduction

- Fading: random fluctuations in channel gains.
- If perfect channel state information (CSI) is available at transmitters
  - Dynamic resource allocation to improve quality-of-service or capacity
- Quality-of-service based
  - Provide all users with desired SIR levels
  - Satisfy SIR requirements with minimum transmit power
  - Compensate for channel fading; more power if bad channel, less if good channel
- Capacity based
  - Maximize information theoretic ergodic capacity subject to average power constraints
  - Exploit variations; more power if good channel, less if bad, no power if very bad

Single User Channel (Goldsmith-Varaiya 1994)

- Channel capacity for single user

$$\begin{aligned} C &= \log(1 + SNR) \\ &= \log\left(1 + \frac{p}{\sigma^2}\right) \end{aligned}$$

- In the presence of fading, the capacity for a fixed channel state h,

$$C(h) = \log\left(1 + \frac{p(h)h}{\sigma^2}\right)$$

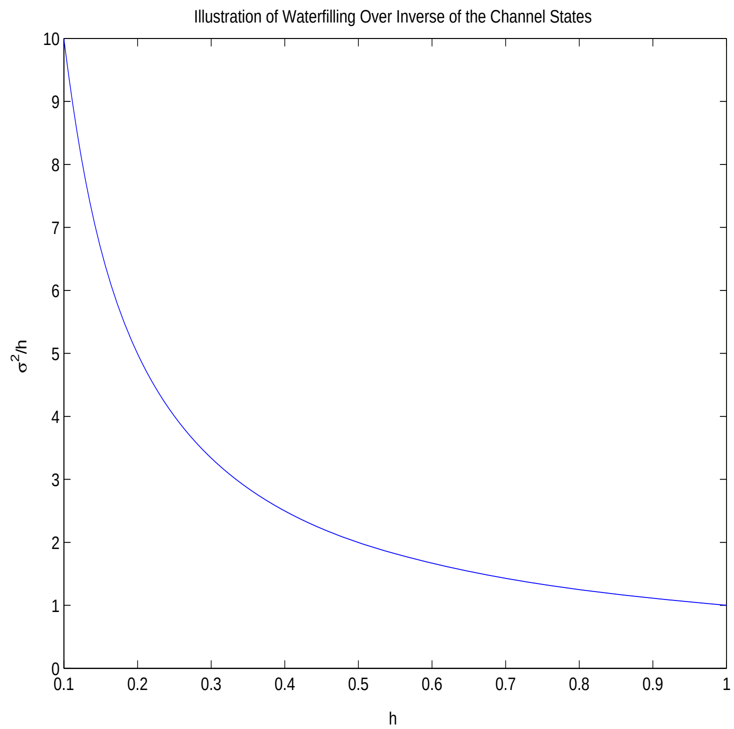
- Maximize the ergodic (expected) capacity, given an average power constraint

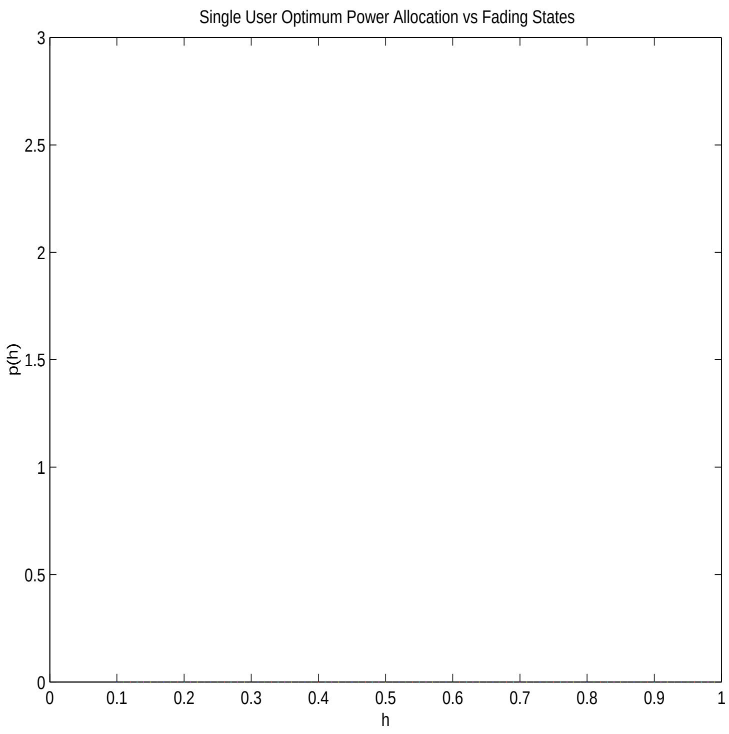
$$\begin{aligned} \max_{\{p(h)\}} \quad & E_h\left[\log\left(1 + \frac{p(h)h}{\sigma^2}\right)\right] \\ \text{s.t.} \quad & E_h[p(h)] \leq \bar{p} \end{aligned}$$

Single User Channel Solution-Waterfilling

- Optimal power allocation: waterfilling of power over time

$$p(h) = \left(\frac{1}{\lambda} - \frac{\sigma^2}{h} \right)^+$$





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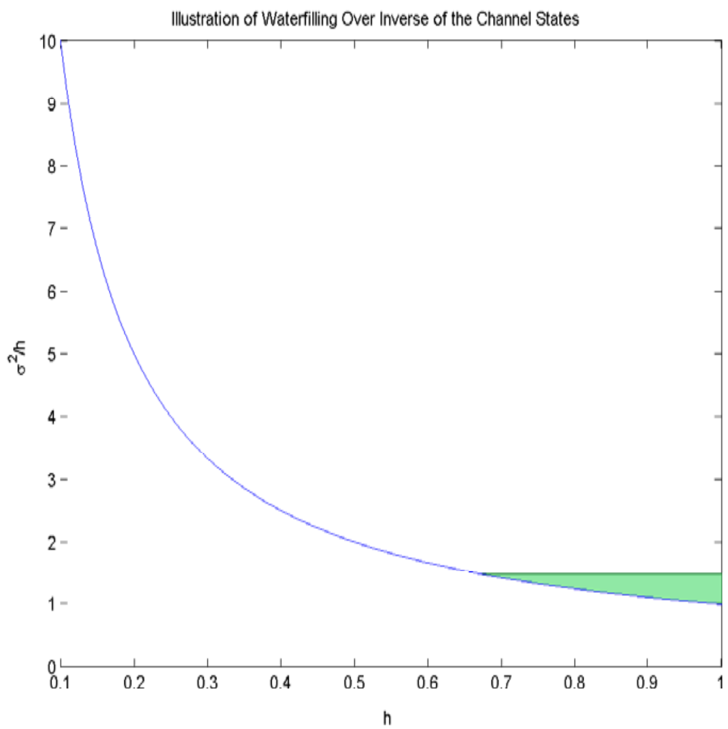

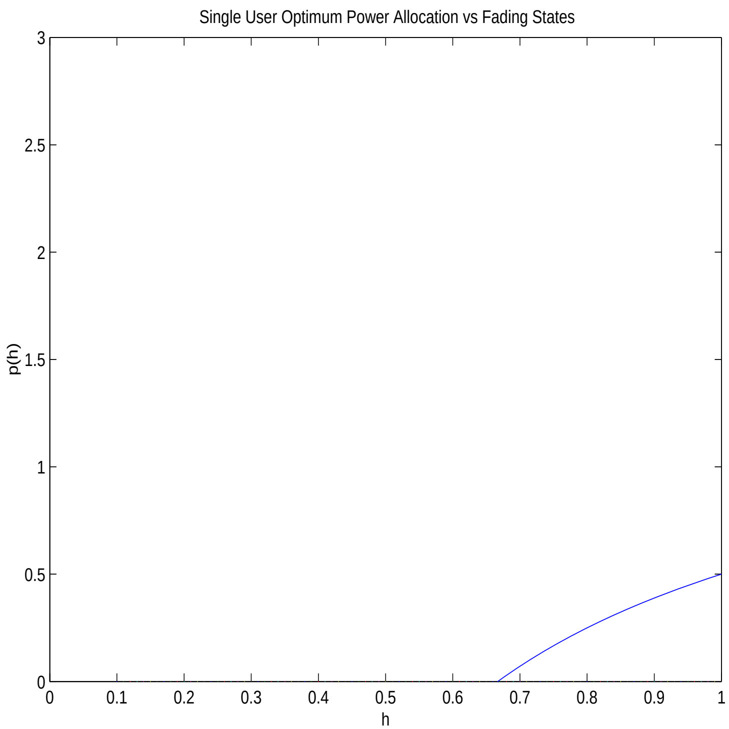

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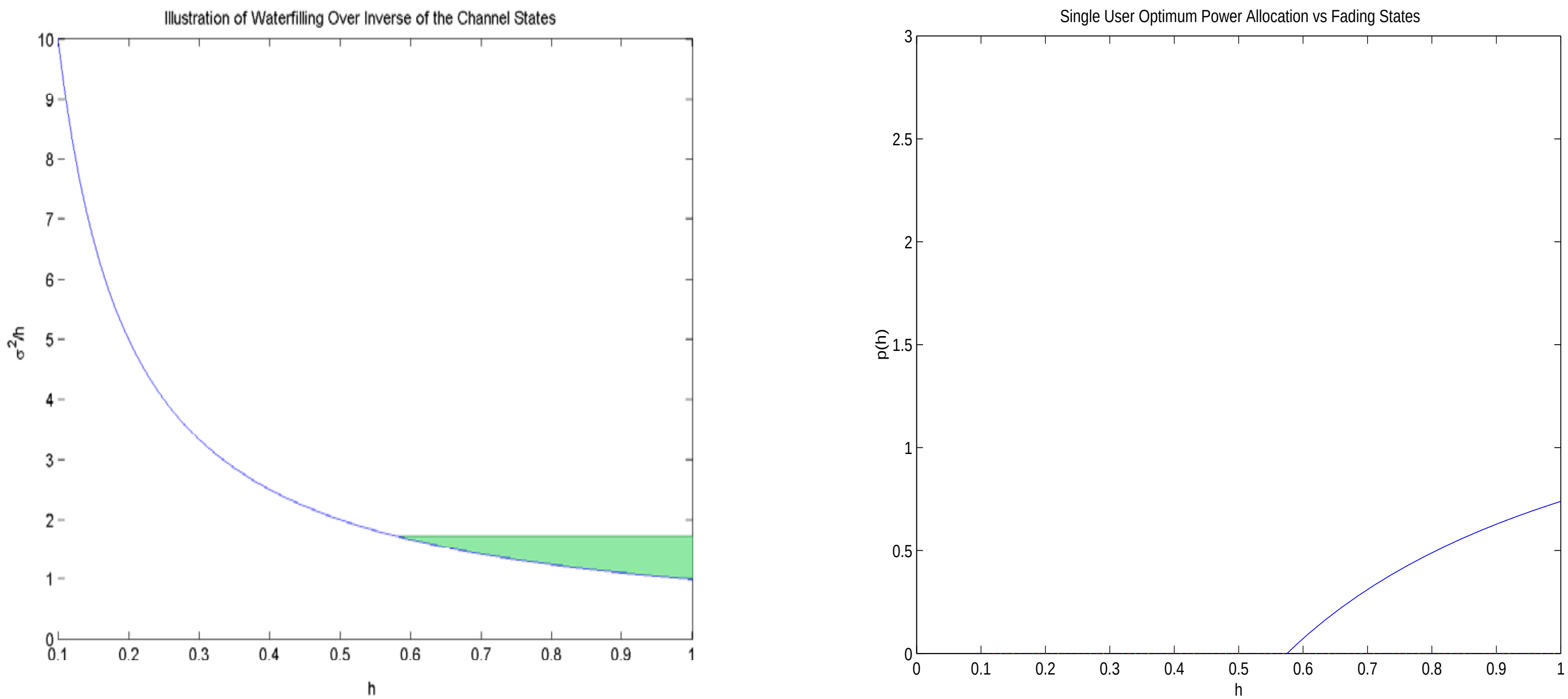

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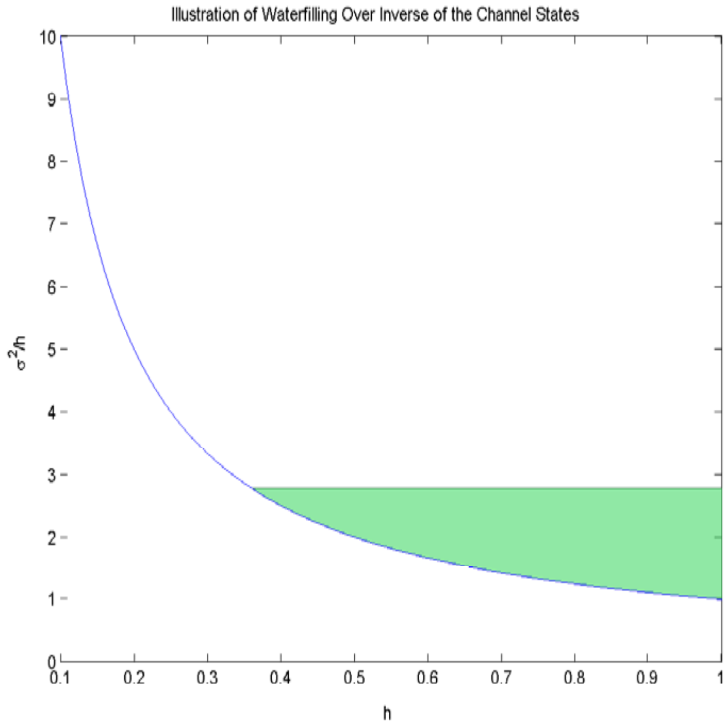

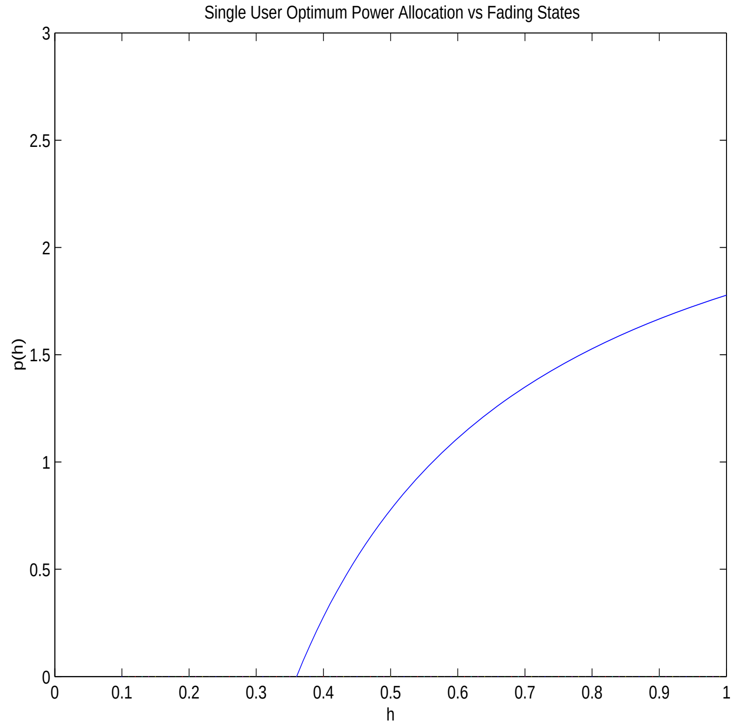

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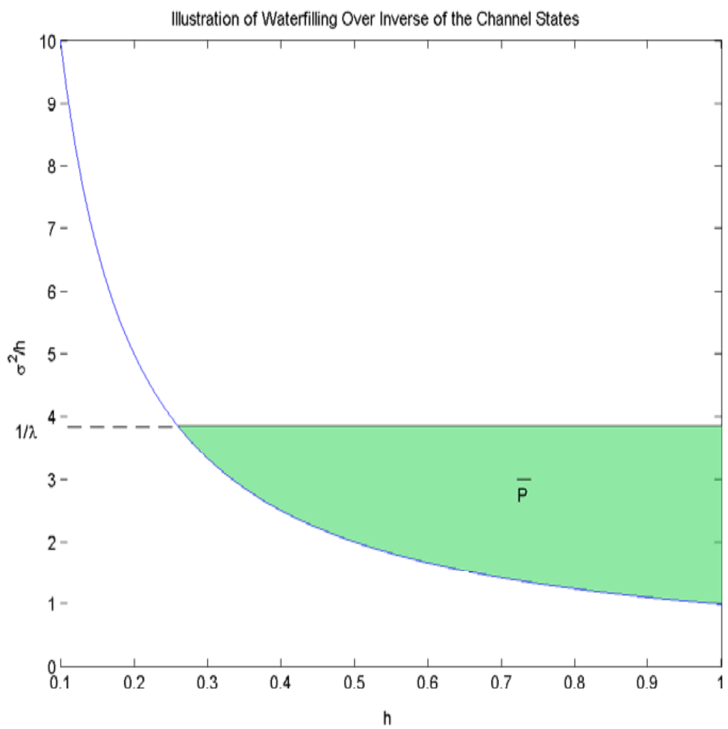
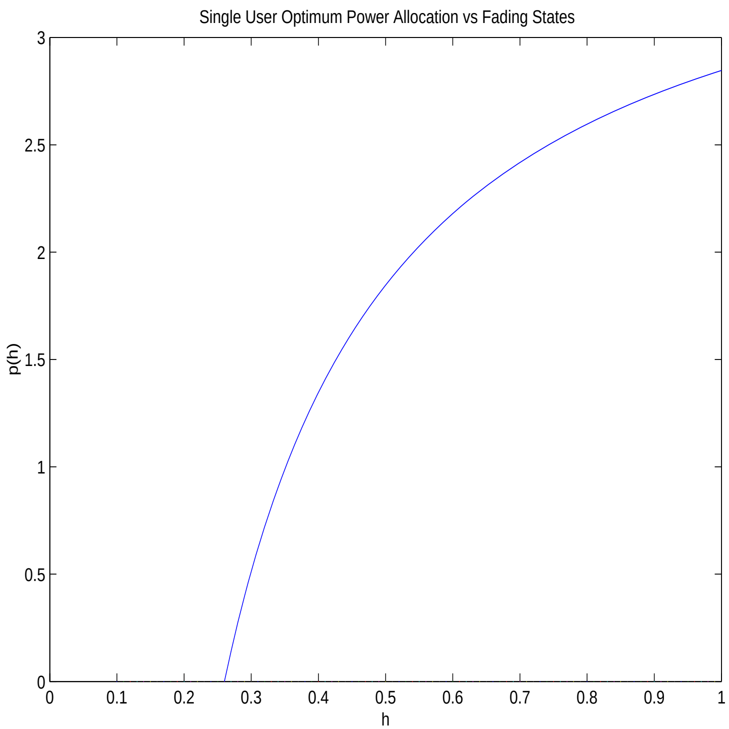




Multiuser Scalar Gaussian Channel (Knopp-Humblet 1995)

- Multiple users, scalar transmissions

$$r = \sum_{i=1}^{K} \sqrt{p_i(\mathbf{h})h_i} x_i + n$$

- Maximize ergodic sum capacity, given average power constraints

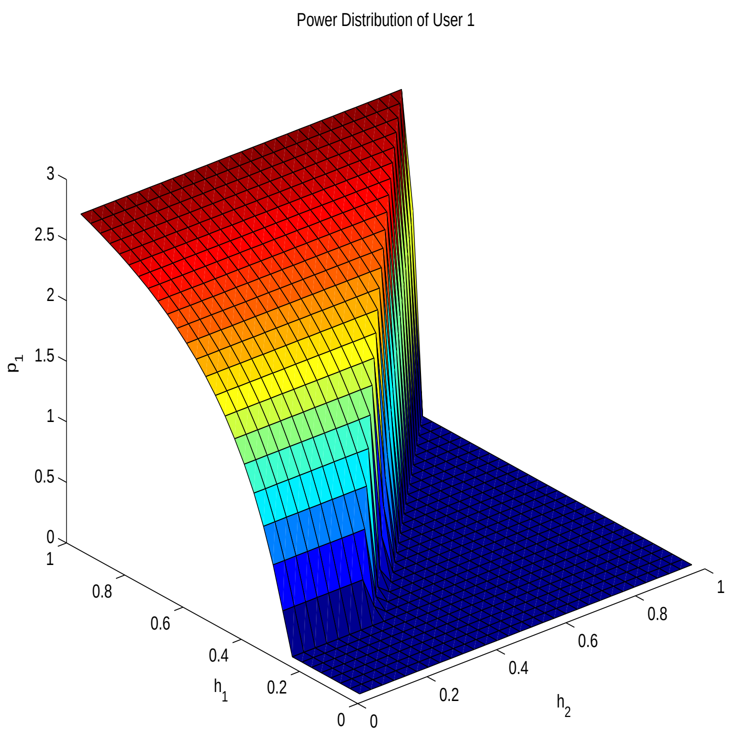
$$\max_{\{p_i(\mathbf{h})\}} \quad E_{\mathbf{h}}\left[\log\left(1+\sigma^{-2}\sum_{i=1}^{K} h_i p_i(\mathbf{h})\right)\right]$$
$$\text{s.t.} \quad E_{\mathbf{h}}\left[p_i(\mathbf{h})\right] \le \bar{p}_i, \qquad p_i(\mathbf{h}) \ge 0, \qquad i = 1, \cdots, K$$

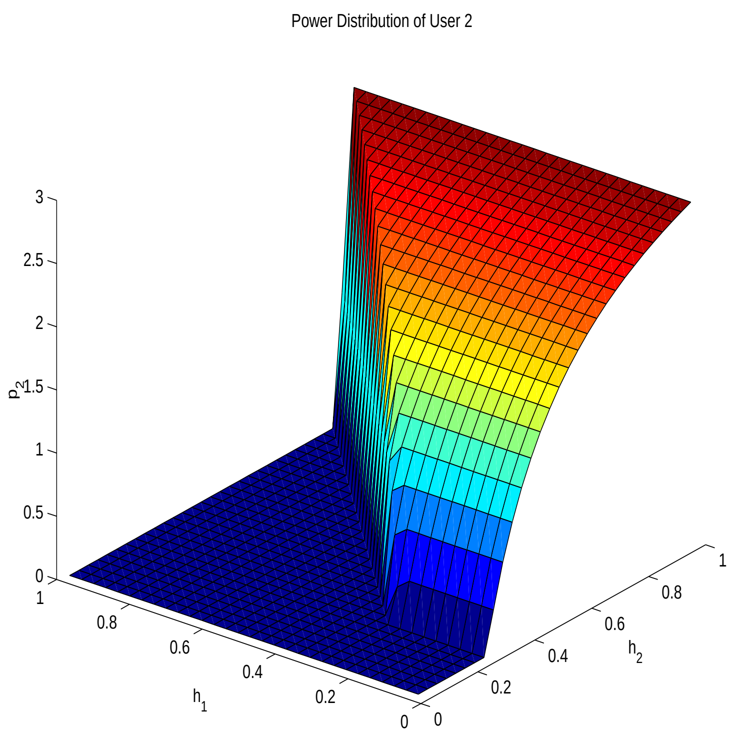
- Optimal power allocation: single user waterfilling on disjoint sets of channel states

$$p_k(\mathbf{h}) = \begin{cases} \left(\frac{1}{\lambda_k} - \frac{\sigma^2}{h_k}\right)^+, & \text{if} \quad h_k/\lambda_k > h_j/\lambda_j, \quad j \neq k \\ 0, & \text{otherwise} \end{cases}$$

- Only the strongest (after some scaling) user transmits at any given time.

Optimum Power Allocation: Scalar Multiuser Channel





Multiuser Vector (Waveform) Gaussian Channel

- Project the received signal onto N basis waveforms.
- CDMA: vector signals modulated by scalar symbols.

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i(\mathbf{h})h_i} x_i \mathbf{s}_i + \mathbf{n}$$

- Maximize ergodic sum capacity subject to average power constraints

$$\begin{aligned} \max_{\{\mathbf{p}(\mathbf{h})\}} \quad & E_{\mathbf{h}}\left[\log\left|\mathbf{I}_N + \sigma^{-2}\sum_{i=1}^{K} h_i p_i(\mathbf{h})\mathbf{s}_i\mathbf{s}_i^{\top}\right|\right] \\ \text{s.t.} \quad & E_{\mathbf{h}}[p_i(\mathbf{h})] \leq \bar{p}_i, \qquad i = 1, \cdots, K \\ & p_i(\mathbf{h}) \geq 0, \quad \forall\, \mathbf{h}, \quad i = 1, \cdots, K \end{aligned}$$

Optimal Power Control

- C_{sum} is a concave function of powers. Constraint set is convex.
- Using Lagrange method, optimum powers satisfy (by KKT conditions),

$$\frac{h_k \mathbf{s}_k \mathbf{A}_k^{-1} \mathbf{s}_k}{1 + p_k(\mathbf{h}) h_k \mathbf{s}_k \mathbf{A}_k^{-1} \mathbf{s}_k} \leq \lambda_k, \qquad k = 1, \cdots, K, \qquad \forall\, \mathbf{h} \in R^K$$

with equality iff $p_k > 0$. Here, $\mathbf{A}_k$ is defined as

$$\mathbf{A}_k = \sigma^2 \mathbf{I}_N + \sum_{i \neq k} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top$$

- Optimum power allocation:

$$p_k(\mathbf{h}) = \left(\frac{1}{\lambda_k} - \frac{1}{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k} \right)^+, \qquad k = 1, \cdots, K$$

- Simultaneous waterfilling of powers onto

  inverse of the "SIRs with MMSE receivers and unit transmit powers" of users.

Iterative Waterfilling

- Isolate kth user's contribution to sum capacity

$$C_{\text{sum}} = C_k + \overline{C}_k$$

$$C_k = E_{\mathbf{h}}\left[\log\left(1 + h_k p_k(\mathbf{h}) \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k\right)\right]$$

- Optimize the power of user k only, with the powers of all other users fixed.

$$\begin{aligned} p_k^{n+1} &= \arg\max_{p_k} C_{\text{sum}}\left(p_1^{n+1}, \cdots, p_{k-1}^{n+1}, p_k, p_{k+1}^n \cdots, p_K^n\right) \\ &= \arg\max_{p_k} C_k\left(p_k\right) \end{aligned}$$

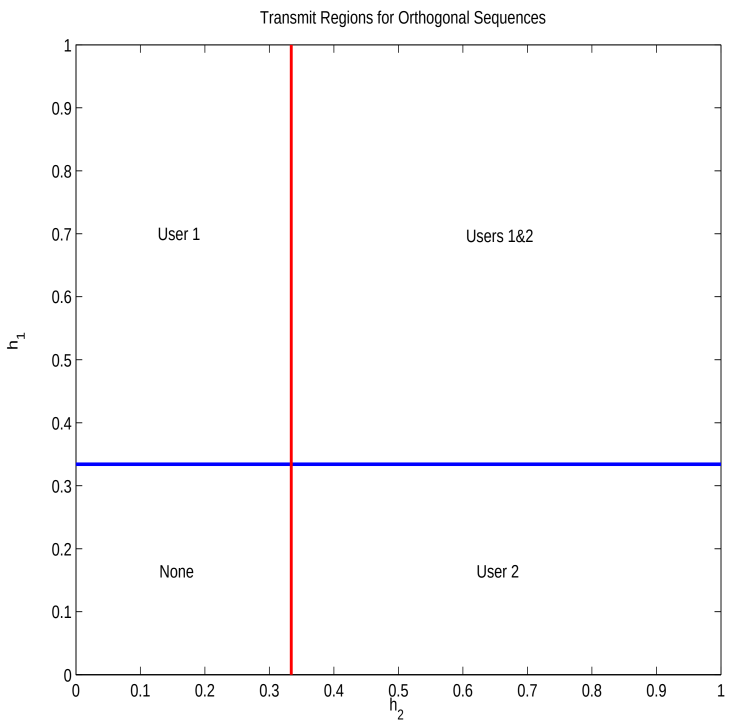
- One-user-at-a-time single user waterfilling:

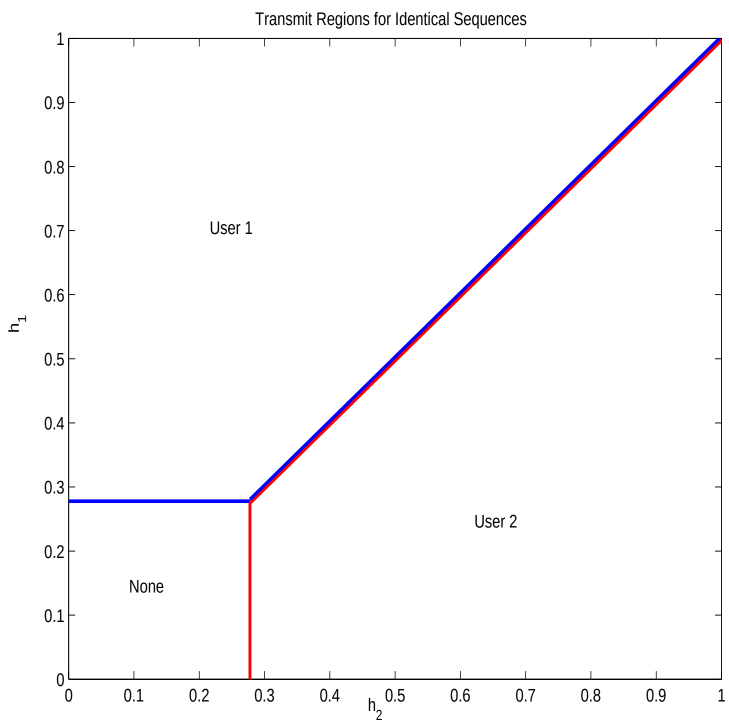
$$p_k(\mathbf{h}) = \left(\frac{1}{\tilde{\lambda}_k} - \frac{1}{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k}\right)^+$$

- Converges to global optimum [Bertsekas-Tsitsiklis].

Simultaneous Transmit Regions

- The regions where both users transmit for the two special cases:



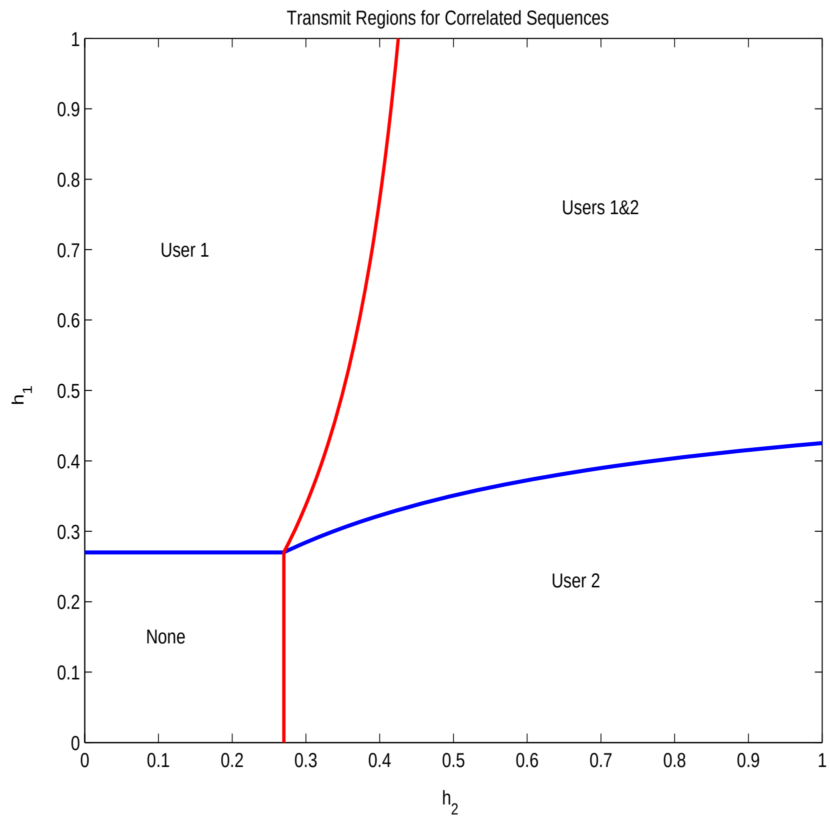


- **Motivation:** for a set of arbitrary signature sequences, is there a set of channel states (with non-zero probability measure) where all users transmit simultaneously?

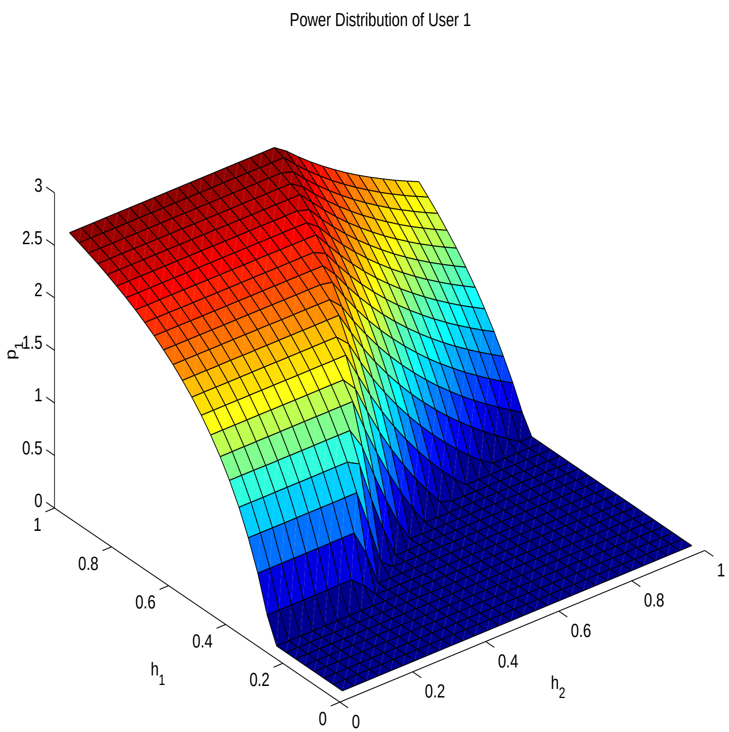
Simultaneous Transmit Condition

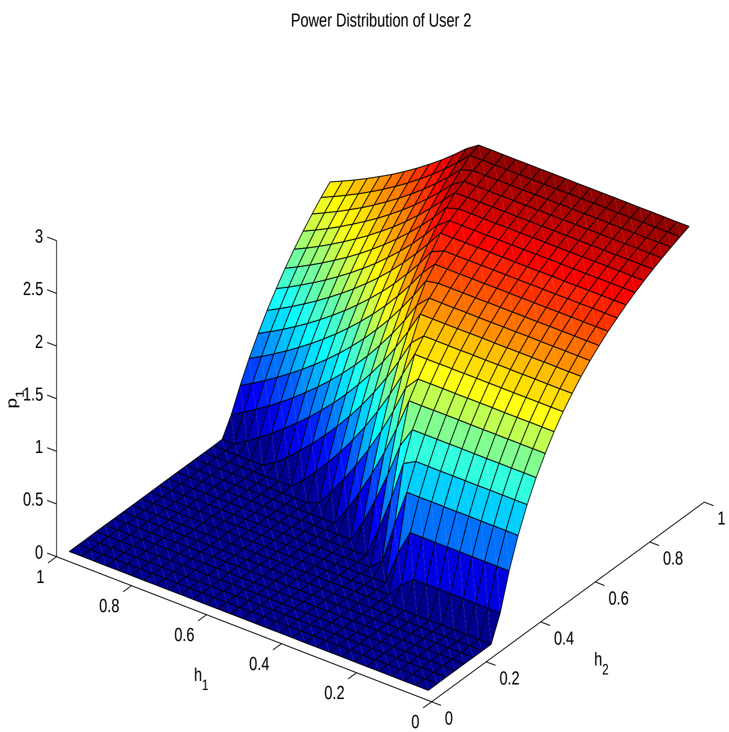
Theorem: There exists a non-zero probability region of fading states $\mathbf{h}$ where all K users in the system transmit simultaneously, if and only if $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent.

Corollary: When $K \leq N$, for a set of K linearly independent signature sequences, there always exists a non-zero probability region of channel states where all K users transmit simultaneously.



Transmit Powers: Correlated Signatures

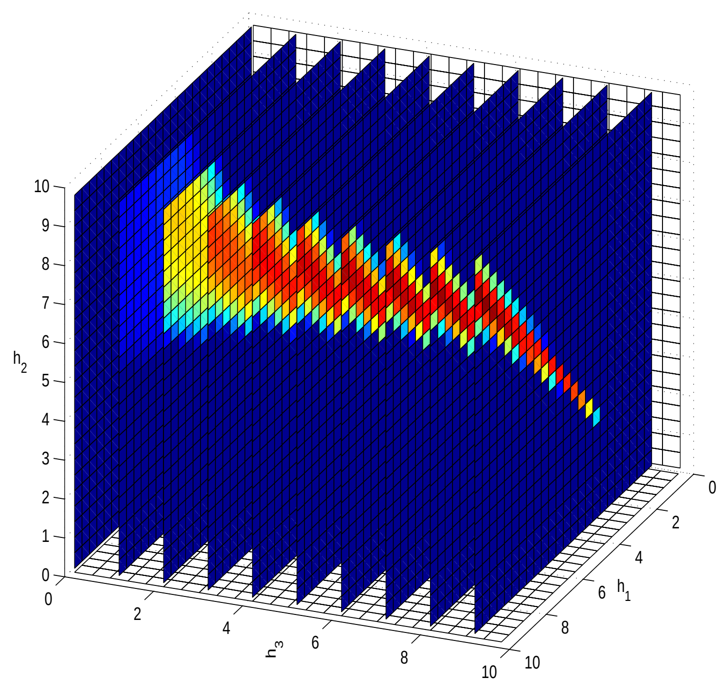




Maximum Number of Simultaneous Transmissions

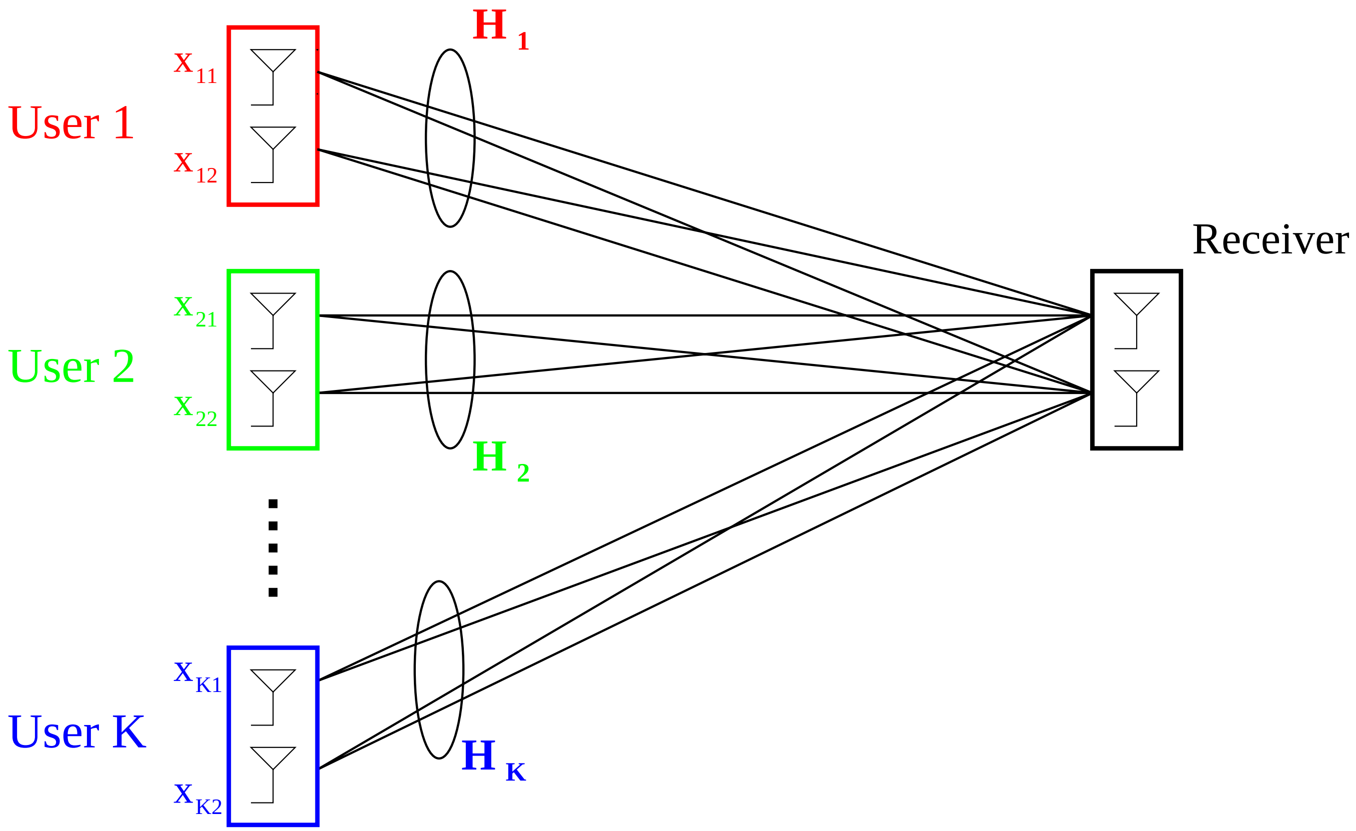
Corollary: For a set of signature sequences with $\text{rank}(\mathbf{S}) = M \leq \min\{N, K\}$, the number of users that can transmit simultaneously cannot be larger than $M(M+1)/2$.

Example: $N = 2$, $K = 3$.



Signature sequences $\{\mathbf{s}_i\}_{i=1}^K$ are linearly dependent, but $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent.

Multi-user MIMO Multiple Access Channel



- The received vector at the receiver, $\mathbf{r}$:

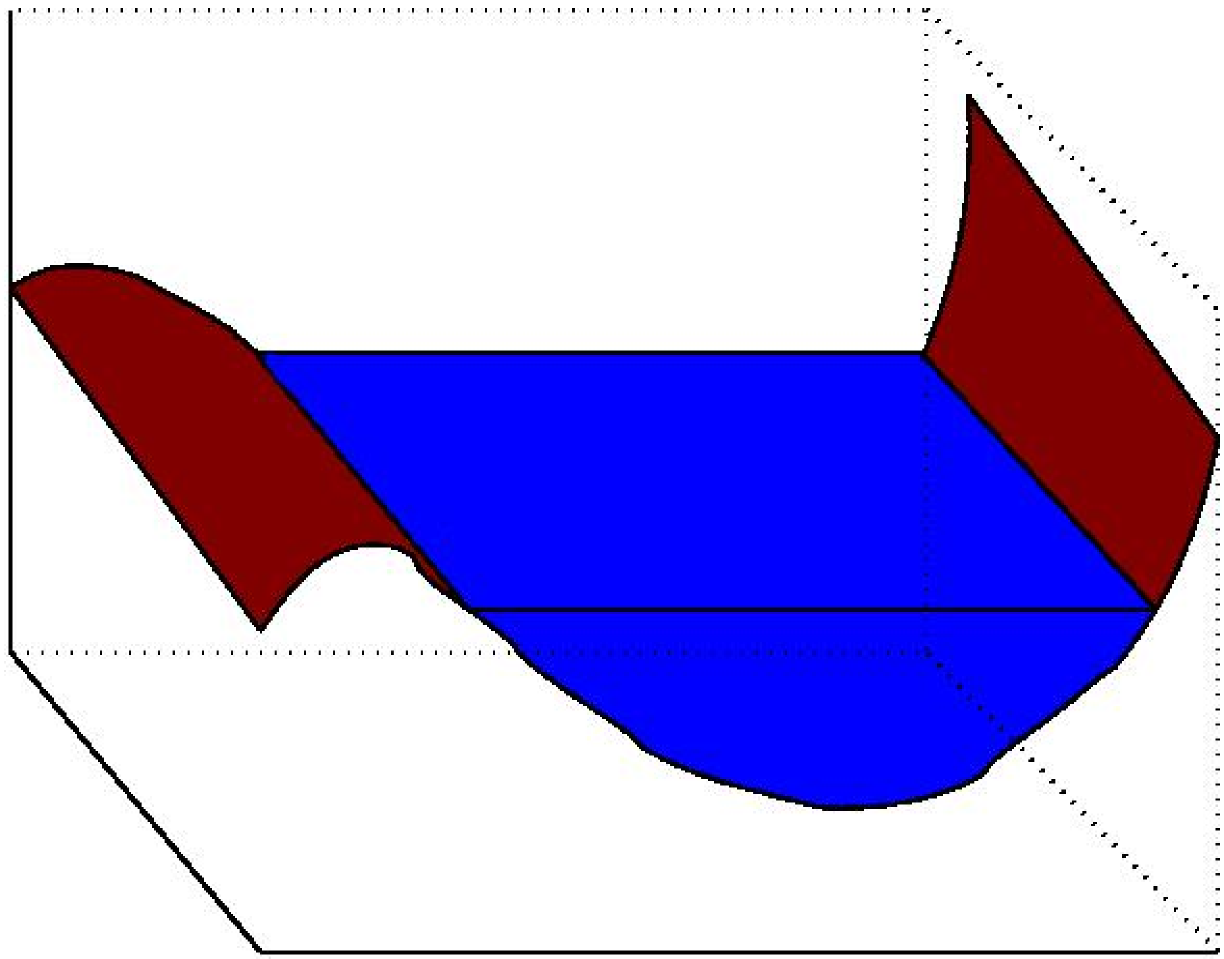
$$\mathbf{r} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n}$$

Problem Formulation

- $\mathbf{H}_k$ is $n_R \times n_T$ matrix, entries of which are complex Gaussian random variables.
- $\mathbf{x}_k$ is the transmitted vector by user k.
- $\mathbf{Q}_k = E[\mathbf{x}_k \mathbf{x}_k^\dagger]$ is the transmit covariance matrix for user k.
- Perfect CSI:
  - Choose $\mathbf{Q}_k$ at all channel states, subject to average power constraint.
  - Multi-dimensional iterative waterfilling (Yu-Rhee-Cioffi, 2001)
- Partial CSI:
  - Choose a constant (independent of the channel state) $\mathbf{Q}_k$.
  - The optimization problem:

$$C_{sum} = \max_{\substack{\mathrm{tr}(\mathbf{Q}_k) \leq P \\ k=1,\ldots,K}} E\left[\log\left|\mathbf{I}_{n_R} + \sum_{k=1}^{K} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^\dagger\right|\right]$$

Multidimensional Waterfilling



Transmit Directions

- The matrix $\mathbf{Q}_k$ has the eigenvalue decomposition

$$\mathbf{Q}_k = \mathbf{U}_{Q_k} \mathbf{\Lambda}_{Q_k} \mathbf{U}_{Q_k}^{\dagger} = \sum_{i=1}^{n_T} \lambda_{ki}^{Q} \mathbf{u}_{ki} \mathbf{u}_{ki}^{\dagger}$$

- Let $\mathbf{U}_{Q_k} = [\mathbf{u}_{k1}, \mathbf{u}_{k2}, \ldots, \mathbf{u}_{kn_T}]$, we have

$$\mathbf{Q}_k = \sum_{i=1}^{n_T} \lambda_{ki}^{Q} \mathbf{u}_{ki} \mathbf{u}_{ki}^{\dagger}$$

  – $\mathbf{u}_{ki}$ is the i^{th} transmit direction

  – λ_{ki}^{Q} is the power allocated along the i^{th} direction of user k

- In general, multiple transmit directions are needed

  – either vector coding

  – or parallel processing of scalar codes

  both of which are very complex.

Beamforming

- Beamforming is a scalar coding strategy, in which

$$\mathbf{Q}_k = \lambda_{k1}^{Q} \mathbf{u}_{k1} \mathbf{u}_{k1}^{\dagger}$$

  – either it is forced that $\lambda_{ki} = 0$, for $i \geq 2$ (not optimal),

  – or the optimum solution happens to be that $\lambda_{ki} = 0$, for $i \geq 2$

- If the transmitters do not have any CSI, then beamforming is not optimal.

- If the transmitters have some form of CSI, beamforming might be optimal.

- Determine the conditions under which beamforming is optimal for all users in MIMO-MAC.

Partial CSI with covariance feedback

- Correlation between signals transmitted by or received at different antenna elements.
- The channel is modeled by *Chuah et al.* [Trans. on Information Theory, 2002] as

$$\mathbf{H}_k = \mathbf{\Phi}_k^{1/2} \mathbf{Z}_k \mathbf{\Sigma}_k^{1/2}$$

  where
  - receive correlation $\mathbf{\Phi}_k$ does not depend on the transmit antenna,
  - transmit correlation $\mathbf{\Sigma}_k$ does not depend on the receive antenna.
- In MAC, we have a single receiver, and therefore $\mathbf{\Phi}_k = \mathbf{\Phi}$ for all users
- WLOG, $\mathbf{\Phi}_k = \mathbf{\Phi} = \mathbf{I}$ for all users.

$$\mathbf{H}_k = \mathbf{Z}_k \mathbf{\Sigma}_k^{1/2}$$

Partial CSI with mean feedback

- Line of sight between the users and the receiver.
- The channel of user k is given as

$$\mathbf{H}_k = \mathbf{H}_{\mu_k} + \mathbf{Z}_k$$

- $\mathbf{H}_{\mu_k}$ is modeled to be of unit-rank [*Lozano, et al.* VTC Spring, 2001] with

$$\mathbf{H}_{\mu_k} = \mathbf{a}_{R_k}\mathbf{a}_{T_k}^{\dagger}$$

- The transmitted signals arrive at the receiver in phase, i.e., $\mathbf{a}_{R_k} = \mathbf{a}_R$,

$$\mathbf{H}_{\mu_k} = \mathbf{a}_R\mathbf{a}_{T_k}^{\dagger}$$

Covariance Feedback - Previous Work (Single-user)

- Single-user MISO and MIMO systems are analyzed by
  - *Visotsky* and *Madhow* [Trans. on Information Theory, 2001]
  - *Jafar* and *Goldsmith* [Trans. on Wireless Comm., 2004]
- Transmit directions are the eigenvectors of the channel covariance matrix

$$\mathbf{U}_Q = \mathbf{U}_\Sigma, \quad \text{where } \boldsymbol{\Sigma} = \mathbf{U}_\Sigma \boldsymbol{\Lambda}_\Sigma \mathbf{U}_\Sigma^\dagger$$

- Condition for the optimality of beamforming is found to be

$$P\lambda_2^\Sigma \leq \frac{1 - E\left[\frac{1}{1+P\lambda_1^\Sigma \mathbf{z}^\dagger \mathbf{z}}\right]}{n_R - 1 + E\left[\frac{1}{1+P\lambda_1^\Sigma \mathbf{z}^\dagger \mathbf{z}}\right]}$$

where λ_1^Σ and λ_2^Σ are the largest and the second largest eigenvalues of the channel covariance matrix, $\mathbf{z}$ is $n_R \times 1$ and $\mathcal{N}(\mathbf{0}, \mathbf{I})$.

Covariance Feedback - Our Results (MIMO-MAC)

- **Optimum transmit directions:**

  Let $\mathbf{\Sigma}_k = \mathbf{U}_{\Sigma_k} \mathbf{\Lambda}_{\Sigma_k} \mathbf{U}_{\Sigma_k}^\dagger$ *be the spectral decomposition of the channel covariance matrix of user* k*. Then, the optimum input covariance matrix* $\mathbf{Q}_k$ *of user* k *has the form* $\mathbf{Q}_k = \mathbf{U}_{\Sigma_k} \mathbf{\Lambda}_{Q_k} \mathbf{U}_{\Sigma_k}^\dagger$*, for all users.*

  Users transmit along the eigenvectors of their own channel covariance matrices. The single-user solution is also optimal in a multi-user setting.

- **Conditions for optimality of beamforming:**

  In a MIMO-MAC system where the transmitters have partial CSI in the form of covariance feedback, the transmit covariance matrices of all users have unit-rank (i.e., beamforming is optimal for all users) if and only if

$$P\lambda_{k2}^{\Sigma} \leq \frac{1 - E\left[\frac{1}{1+P\lambda_{k1}^{\Sigma}\mathbf{z}_k \mathbf{A}_k^{-1}\mathbf{z}_k}\right]}{n_R - K + \sum_{l=1}^{K} E\left[\frac{1}{1+P\lambda_{l1}^{\Sigma}\mathbf{z}_l \mathbf{A}_l^{-1}\mathbf{z}_l}\right]}, \quad k = 1, \ldots, K$$

  where $\mathbf{A} = \mathbf{I}_{n_R} + P\sum_{l=1}^{K} \lambda_{l1}^{\Sigma}\mathbf{z}_l\mathbf{z}_l^\dagger$, $\mathbf{A}_k = \mathbf{A} - P\lambda_{k1}^{\Sigma}\mathbf{z}_k\mathbf{z}_k^\dagger$, λ_{ki} *is the* i^{th} *largest eigenvalue of the channel covariance matrix of user* k*, and* $\mathbf{z}_l$ *are* $n_R \times 1$ *and* $\mathcal{N}(\mathbf{0}, \mathbf{I})$*.*

MAC vs Single-user

- Single-user case

$$P\lambda_2^{\Sigma} \leq \frac{1 - E\left[\frac{1}{1+P\lambda_1^{\Sigma}\mathbf{z}^{\dagger}\mathbf{z}}\right]}{n_R - 1 + E\left[\frac{1}{1+P\lambda_1^{\Sigma}\mathbf{z}^{\dagger}\mathbf{z}}\right]}$$

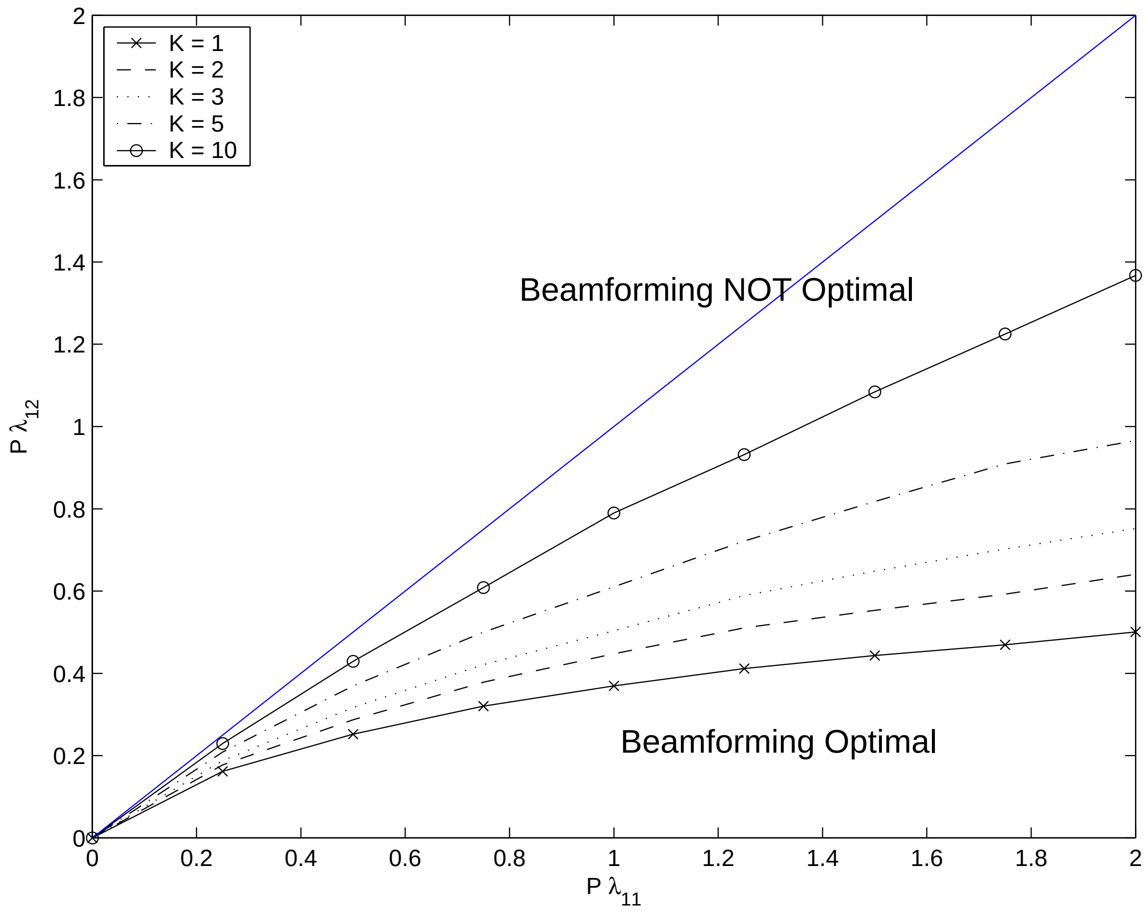
- Our MAC result

$$P\lambda_{k2}^{\Sigma} \leq \frac{1 - E\left[\frac{1}{1+P\lambda_{k1}^{\Sigma}\mathbf{z}_k^{\dagger}\mathbf{A}_k^{-1}\mathbf{z}_k}\right]}{n_R - K + \sum_{l=1}^{K} E\left[\frac{1}{1+P\lambda_{l1}^{\Sigma}\mathbf{z}_l^{\dagger}\mathbf{A}_l^{-1}\mathbf{z}_l}\right]}, \quad k = 1, \ldots, K$$

where $\mathbf{A}_k = \mathbf{I}_{n_R} + P\sum_{l \neq k}^{K} \lambda_{l1}^{\Sigma}\mathbf{z}_l\mathbf{z}_l^{\dagger}$.

Numerical Results

maximum possible λ_{12}^{Σ} for a range of λ_{11}^{Σ}



Mean Feedback - Previous Work (Single-user)

- Single-user MISO and MIMO systems are analyzed by
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- Transmit directions are the eigenvectors of the channel mean matrix

$$\mathbf{U}_Q = \mathbf{V}_\mu \quad \text{where } \mathbf{H}_\mu = \mathbf{U}_\mu \mathbf{\Lambda}_\mu^{1/2} \mathbf{V}_\mu^\dagger$$

- Condition for the optimality of beamforming is found to be

$$P \leq \frac{1 - E\left[\frac{1}{1+P\hat{\mathbf{z}}^\dagger\hat{\mathbf{z}}}\right]}{n_R - 1 + E\left[\frac{1}{1+P\hat{\mathbf{z}}^\dagger\hat{\mathbf{z}}}\right]}$$

where $\hat{\mathbf{z}}$ is an $n_R \times 1$ and $\mathcal{N}(\lambda_\mu^{1/2}\mathbf{e}_1, \mathbf{I})$

Mean Feedback - Our Results (MIMO-MAC)

- **Optimum transmit directions:**

  Let $\mathbf{H}_{\mu_k} = \mathbf{U}_\mu \mathbf{\Lambda}_{\mu_k}^{1/2} \mathbf{V}_{\mu_k}^\dagger$ *be the singular value decomposition of the channel mean matrix of user* k*. Then, the optimum input covariance matrix* $\mathbf{Q}_k$ *of user* k *may be expressed as* $\mathbf{Q}_k = \mathbf{V}_{\mu_k} \mathbf{\Lambda}_k \mathbf{V}_{\mu_k}^\dagger$*, for all users.*

- **Optimality of beamforming:**

  In a MIMO-MAC system where the transmitters have partial CSI in the form of mean feedback, the transmit covariance matrices of all users have unit-rank (i.e., beamforming is optimal for all users) if and only if

$$P \leq \frac{1 - E\left[\frac{1}{1+P\hat{\mathbf{z}}_k^\dagger \mathbf{B}_k^{-1}\hat{\mathbf{z}}_k}\right]}{n_R - K + \sum_{l=1}^K E\left[\frac{1}{1+P\hat{\mathbf{z}}_l^\dagger \mathbf{B}_l^{-1}\hat{\mathbf{z}}_l}\right]}, \quad k = 1, \ldots, K$$

where $\mathbf{B} = \mathbf{I}_{n_R} + P\sum_{l=1}^K \hat{\mathbf{z}}_l \hat{\mathbf{z}}_l^\dagger$, $\mathbf{B}_k = \mathbf{B} - P\hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^\dagger$, *and* $\hat{\mathbf{z}}_k = (\lambda_{k1}^\mu)^{1/2}\mathbf{e}_1 + \mathbf{z}_k$.

MAC vs Single-user

- Single-user case

$$P \leq \frac{1 - E\left[\frac{1}{1+P\hat{\mathbf{z}}^\dagger\hat{\mathbf{z}}}\right]}{n_R - 1 + E\left[\frac{1}{1+P\hat{\mathbf{z}}^\dagger\hat{\mathbf{z}}}\right]}$$

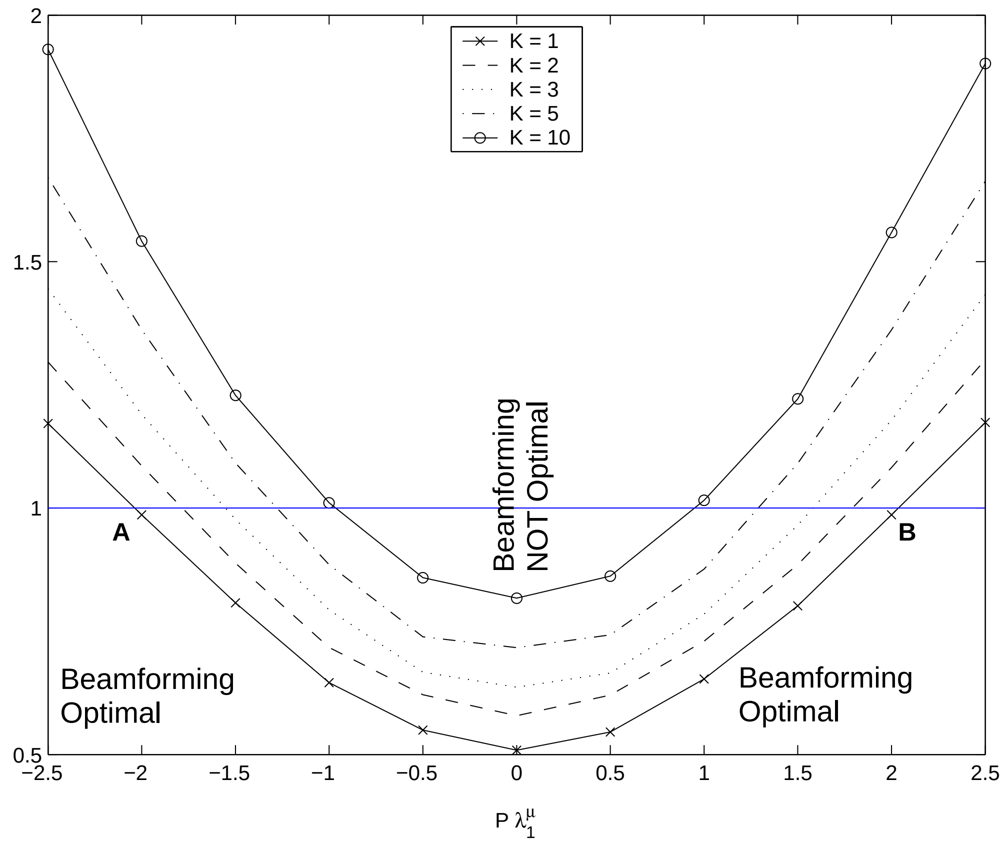
- Our MAC result

$$P \leq \frac{1 - E\left[\frac{1}{1+P\hat{\mathbf{z}}_k^\dagger\mathbf{B}_k^{-1}\hat{\mathbf{z}}_k}\right]}{n_R - K + \sum_{l=1}^{K} E\left[\frac{1}{1+P\hat{\mathbf{z}}_l^\dagger\mathbf{B}_l^{-1}\hat{\mathbf{z}}_l}\right]}, \quad k = 1, \ldots, K$$

where $\mathbf{B}_k = \mathbf{I}_{n_R} + P\sum_{l\neq k}^{K} \hat{\mathbf{z}}_l\hat{\mathbf{z}}_l^\dagger$.

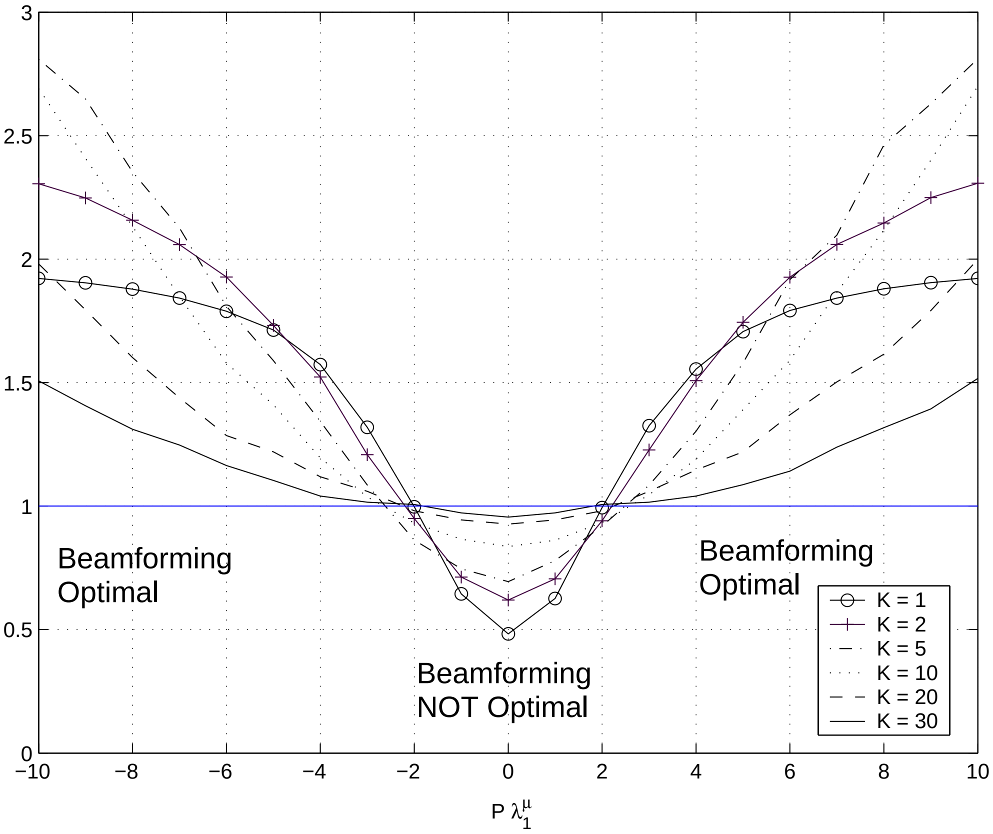
Numerical Results 1

an example where the region under which beamforming is optimal increases



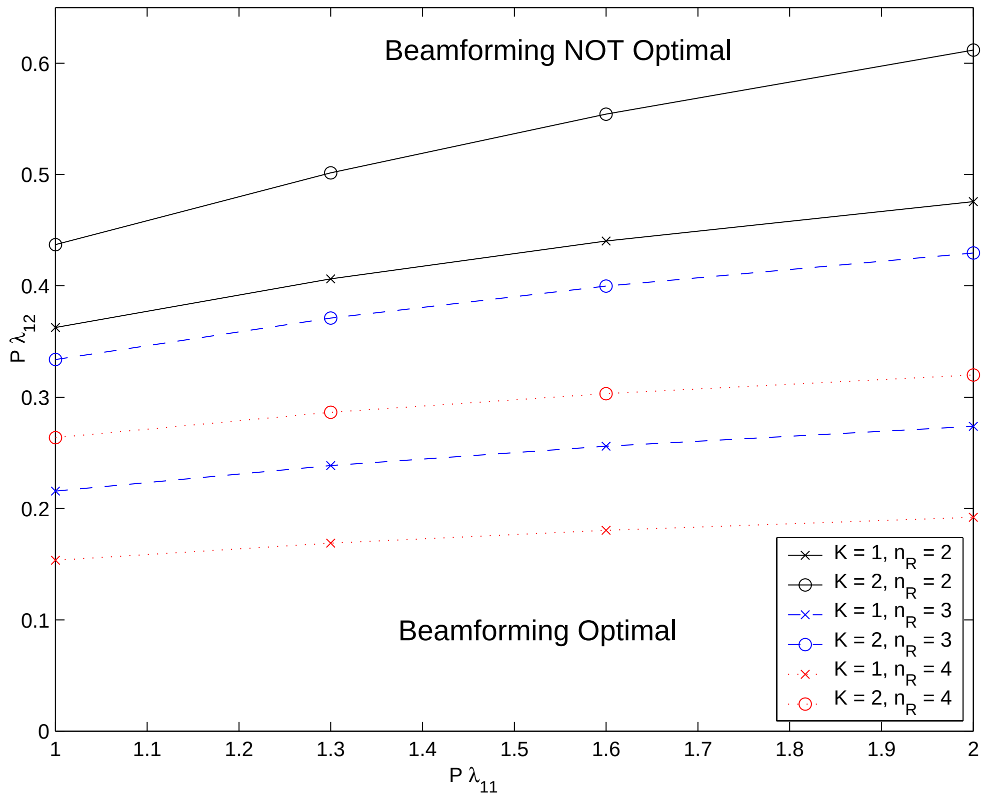
Numerical Results 2

an example where the region under which beamforming is optimal does not increase



Numerical Results 3

behavior of the region with varying number of receive antennas



Other Research Directions

- Sensor networks:
  - Coding and transmission of correlated data
  - Asymptotic analysis: order-optimal schemes in large sensor networks
- User cooperation diversity:
  - Power control for user cooperation
- Multi-user wireless security:
  - Physical layer wireless security
  - Jamming, correlated (informed) jamming, eavesdropping