Optimal Selective Encoding for Timely Updates with Empty Symbol

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***Abstract*—An information source generates independent and identically distributed status update messages from an observed random phenomenon which takes n distinct values based on a given pmf. These update packets are encoded at the transmitter to be sent to a receiver which wants to track the observed random variable with as little age as possible. The transmitter implements a selective k encoding policy such that rather than encoding all possible n realizations, the transmitter encodes the most probable k realizations and sends a designated empty symbol when one of the remaining $n-k$ realizations occurs. We consider two scenarios: when the empty symbol does not reset the age and when the empty symbol resets the age. We find the time average age of information and the age-optimal real codeword lengths, including the codeword length for the empty symbol, for both of these scenarios. Through numerical evaluations for arbitrary pmfs, we show that this selective encoding policy yields a lower age at the receiver than encoding every realization and find the corresponding age-optimal k values.**

I. Introduction

We consider a status updating system which consists of a single transmitter and a single receiver (see Fig. 1). The transmitter receives independent and identically distributed time-sensitive status update packets generated by an information source based on an observed random phenomenon that takes n distinct values with a known pmf. This observed random variable could be the position of a UAV in autonomous systems or share prices in the stock market. Arriving status update packets are encoded at the transmitter and sent to the receiver through an error-free noiseless channel. The receiver wants to acquire fresh information regarding the observed random variable which brings up the concept of age of information.

Age of information is an information timeliness metric which tracks the generation time of the most recent status update at the receiver. More precisely, age increases linearly in time such that at time t age $\Delta(t)$ of an update packet which was generated at time $u(t)$ is $\Delta(t) = t - u(t)$. When an update packet is received, age drops to the age of the received update packet. Age of information has been studied in the context of queueing theory, scheduling, optimization, energy harvesting, and remote estimation [1]–[43].

Unlike most of the literature in which the transmission times, also referred to as service times in queueing theory, are based on a given service distribution, in this work, we design transmission times through source coding schemes by choosing the codeword lengths assigned to realizations. That is, the codeword length assigned to each realization represents the service time (transmission time) of that realization. Age of information is analyzed from a source coding perspective in [44]–[47]. Reference [44] considers zero-wait update generation and finds codeword lengths using Shannon codes based on a modified version of the given pmf that achieve the optimal age with a constant gap. References [45] and [46] consider block coding and source coding problems to find age-optimal codes for FIFO queues. Reference [47] introduces a selective encoding policy which only encodes k of the total n realizations and discards the remaining $n-k$ realizations, and shows that this policy yields a lower average age than encoding every realization. Similar k out of n (compared to n out of n) type of schemes are shown to achieve better age performances in the context of multicast networks in which each update packet is transmitted until the earliest k of the n receiver nodes receive that packet [48]–[52].

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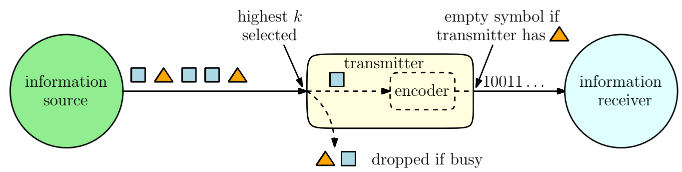

Fig. 1. Update packets that come from the selected portion of the realizations (shown with a square) that find the transmitter idle are sent to the receiver. Non-selected realizations (shown with a triangle) that find the transmitter idle are mapped into an empty symbol.

A disadvantage of the scheme in [47] is the fact that the receiver is not informed when one of the non-selected realizations occurs. In other words, during a period with no arrivals, the receiver cannot differentiate whether there has been no arrivals or if the arrival has taken one of the non-selected values as in either case it does not receive any update packets. Thus, in this work, we take a careful look at the remaining $n-k$ realizations and propose a modified selective encoding policy which still achieves a lower average age than encoding every realization but also informs the receiver when one of the non-selected values is taken by the observed random variable. In our proposed scheme, which we call the *highest k selective encoding with empty symbol*, only the most probable k realizations are encoded and the remaining $n-k$ realizations are mapped into one designated empty symbol such that in the case of these $n-k$ non-selected realizations, this empty symbol is sent to further inform the receiver.

In this work, we find the time average age experienced by the receiver and determine the age-optimal real codeword lengths, including the codeword length of the empty symbol by considering two scenarios: when the empty symbol does not reset the age and when the empty symbol resets the age. Through numerical evaluations for given arbitrary pmfs, we show that this selective encoding with empty symbol policy achieves a lower average age than encoding every realization and find the corresponding age-optimal k values.

II. System Model and Problem Formulation

An information source generates independent and identically distributed status update packets from the set $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ with a known pmf $P_X(x_i)$ for $i \in \{1, \dots, n\}$. Without loss of generality, we assume that $P_X(x_m) \geq P_X(x_j)$ for all $m \leq j$, i.e., the probabilities of the realizations are in a non-increasing order. Update packet arrivals at the transmitter follow a Poisson process with parameter λ. The update packets that arrive when the transmitter is busy are blocked and lost, i.e., the transmitter implements a blocking policy. Thus, the transmitter successfully receives only the update packets that arrive when it is idle.

In this paper, unlike [44], we consider a selective encoding mechanism that we call the *highest k selective encoding* such that the transmitter only sends the most probable k realizations, i.e., only the realizations from set $\mathcal{X}_k = \{x_1, \dots, x_k\}$, which have the highest probabilities among possible n updates generated by the source, where $k \in \{1, \dots, n\}$. However, unlike the setting in [47], if an update packet from the remaining non-selected portion of the realizations arrives, the transmitter sends an empty status update denoted as x_e to further inform the receiver at the expense of longer codewords for the selected k realizations.

When an update packet arrives from the set $\mathcal{X}'_k = \mathcal{X}_k \cup \{x_e\}$, the transmitter encodes that update packet with the binary alphabet by using the pmf given as $\{P_X(x_1), P_X(x_2), \dots, P_X(x_k), P_X(x_e)\}$ where $P_X(x_e) = 1 - q_k$ and $q_k \triangleq \sum_{\ell=1}^{k} P_X(x_\ell)$. The transmitter assigns codeword $c(x_i)$ with length $\ell(x_i)$ to realization x_i for $i \in \{1, \dots, k, e\}$. The first and the second moments of the codeword lengths are given by $\mathbb{E}[L] = \sum_{i=1}^{k} P_X(x_i)\ell(x_i) + P_X(x_e)\ell(x_e)$ and $\mathbb{E}[L^2] = \sum_{i=1}^{k} P_X(x_i)\ell(x_i)^2 + P_X(x_e)\ell(x_e)^2$. The channel between the transmitter and the receiver is noiseless. The transmitter sends one bit at a unit time such that it takes $\ell(x_i)$ units of time to send the update packet x_i.

In this paper, we consider the age of information metric to measure the timeliness of the received information. We define the long term average age as,

$$\Delta = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Delta(t) dt, \tag{1}$$

where $\Delta(t)$ represents the instantaneous age at the receiver. Our aim is to find the real-valued codeword lengths that minimize the average age for a given k while satisfying the Kraft inequality [53]. Thus, we formulate the problem as,

$$\begin{aligned} \min_{\{\ell(x_i), \ell(x_e)\}} \quad & \Delta \\ \text{s.t.} \quad & 2^{-\ell(x_e)} + \sum_{i=1}^{k} 2^{-\ell(x_i)} \leq 1 \\ & \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k, e\}. \end{aligned} \tag{2}$$

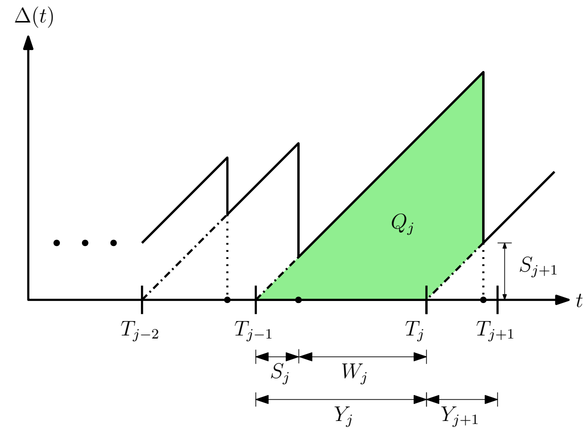

Fig. 2. Sample age evolution $\Delta(t)$ at the receiver. Successful updates are indexed by j. The jth successful update arrives at the transmitter at T_{j-1}. Update cycle at the transmitter is the time in between two successive arrivals and is equal to $Y_j = S_j + W_j = T_j - T_{j-1}$.

In the following section, we find an analytical expression for the long term average age, Δ.

III. Average Age Analysis

As described in Section II, status update packets arrive at the transmitter as a Poisson process with rate λ. Update packets that arrive when the transmitter is busy are blocked from entry and dropped. Thus, upon successful delivery of a packet to the receiver, the transmitter idles until the next update packet arrives. This idle waiting period in between two arrivals is denoted by Z which is an exponential random variable with rate λ as update interarrivals at the transmitter are exponential with λ. Not every packet which enters the transmitter is sent to the receiver. The transmitter implements the highest k selective encoding policy in which only the most probable k realizations are encoded. When one of the remaining $n-k$ packets enters the transmitter, however, the transmitter sends a designated empty status update to further inform the receiver about the occurrence of a status update from the remaining $n-k$ realizations. We denote the update packets which arrive when the transmitter is idle and are able to reset the age as the successful update packets. Since the channel is noiseless and there is no preemption, these successful packets are received by the receiver. We denote T_{j-1} as the time instant at which the jth successful update packet is received. We define update cycle denoted by $Y_j = T_j - T_{j-1}$ as the time in between two successive successful update arrivals at the transmitter. Update cycle Y_j consists of a busy cycle and an idle cycle such that

$$Y_j = S_j + W_j, \tag{3}$$

where S_j is the service time of update j and W_j is the overall waiting time in the jth update cycle.

Fig. 2 shows a sample age evolution at the receiver. Here, Q_j denotes the area under the instantaneous age curve in update cycle j and Y_j denotes the length of the jth update cycle as defined earlier. The metric we use, long term average age, is the average area under the age curve which is given by [8]

$$\Delta = \limsup_{n\to\infty} \frac{\frac{1}{n}\sum_{j=1}^{n} Q_j}{\frac{1}{n}\sum_{j=1}^{n} Y_j} = \frac{\mathbb{E}[Q]}{\mathbb{E}[Y]}. \tag{4}$$

By using Fig. 2, we find $Q_j = \frac{1}{2}Y_j^2 + Y_j S_{j+1}$, where Y_j is given in (3). Thus, (4) is equivalent to

$$\Delta = \frac{\mathbb{E}[Y^2]}{2\mathbb{E}[Y]} + \mathbb{E}[S]. \tag{5}$$

In the following section, we find the optimal real-valued codeword lengths for the problem posed in (2).

IV. Optimal Codeword Design

In this section, we calculate the average age by considering two different scenarios for the empty symbol. Operationally, the receiver may not reset its age when x_e is received as it is not a regular update packet and the receiver does not know which realization occurred specifically. On the other hand, the receiver may choose to update its age as this empty symbol carries some information, the fact that the current realization is not one of the k encoded realizations, regarding the observed random variable. Thus, in this section, we consider both of these scenarios[1] and find the age-optimal codeword lengths for the set $\mathcal{X}_k'$ with the pmf $\{P_X(x_1), P_X(x_2), \ldots, P_X(x_k), P_X(x_e)\}$.

A. When the Empty Symbol does not Reset the Age

In this way of operation, the age at the receiver is not updated when the empty status update x_e is received. Thus, sending x_e incurs an additional burden since it does not reset the age but increases the average codeword length of the selected k realizations.

The update cycle is given by (3) with

$$W = (M-1)\ell(x_e) + \sum_{\ell=1}^{M} Z_\ell, \tag{6}$$

where M denotes the total number of update arrivals until the first update from the set $\mathcal{X}_k$ is observed at the transmitter. In other words, there are $M-1$ deliveries of the empty status update x_e in between two successive deliveries from the encoded set $\mathcal{X}_k$. Here, Z is an exponential random variable with rate λ and M is a geometric random variable with parameter q_k such that

$$\mathbb{E}[M] = \frac{1}{q_k}, \quad \mathbb{E}[M^2] = \frac{2-q_k}{q_k^2}. \tag{7}$$

[1]We note that another possible scenario may be to drop the age to an intermediate level between not updating at all and updating fully considering the partial information conveyed by the empty status update. This case is not considered in this paper.

We note that arrival and service processes are independent, i.e., S_j and Z_j are independent, and M is independent of S_j and Z_j.

In the following theorem, we find the long term average age, Δ, when the empty status update does not reset the age.

Theorem 1 *When the empty status update x_e does not reset the age, the average age under the highest k selective encoding scheme at the receiver is given by*

$$\begin{aligned}\Delta =& \frac{\mathbb{E}[L^2|X_k' \neq x_e] + 2\mathbb{E}[W]\mathbb{E}[L|X_k' \neq x_e] + \mathbb{E}[W^2]}{2\left(\mathbb{E}[L|X_k' \neq x_e] + \mathbb{E}[W]\right)} \\ &+ \mathbb{E}[L|X_k' \neq x_e]. \end{aligned} \tag{8}$$

Proof: We note that the service time of a successful update is equal to its codeword length so that we have

$$\mathbb{E}[S] = \mathbb{E}[L|X_k' \neq x_e] = \sum_{i=1}^{k} P_{X_k}(x_i)\ell(x_i) \tag{9}$$

$$\mathbb{E}[S^2] = \mathbb{E}[L^2|X_k' \neq x_e] = \sum_{i=1}^{k} P_{X_k}(x_i)\ell(x_i)^2 \tag{10}$$

where $P_{X_k}(x_i)$ is defined as $P_{X_k}(x_i) = \frac{P_X(x_i)}{q_k}$ for all $x_i \in \mathcal{X}_k$. By using the independence of M and Z, we find

$$\mathbb{E}[W] = \ell(x_e)\left(\frac{1}{q_k} - 1\right) + \frac{1}{\lambda q_k}, \tag{11}$$

$$\mathbb{E}[W^2] = \frac{(2-q_k)(1-q_k)}{q_k^2}\ell(x_e)^2 + \frac{4(1-q_k)}{\lambda q_k^2}\ell(x_e) + \frac{2}{(\lambda q_k)^2}, \tag{12}$$

where moments of M follow from (7), and Z has exponential distribution with rate λ as discussed earlier. Substituting (9)-(12) in (5) yields the result in (8). ■

We note that Δ in (8) depends on $\ell(x_e)$ only through the overall waiting time W as the age does not change when x_e is received. Next, we write the optimization problem as

$$\begin{aligned}\min_{\{\ell(x_i),\ell(x_e)\}} \quad & \frac{\mathbb{E}[L^2|X_k' \neq x_e] + 2\mathbb{E}[W]\mathbb{E}[L|X_k' \neq x_e] + \mathbb{E}[W^2]}{2\left(\mathbb{E}[L|X_k' \neq x_e] + \mathbb{E}[W]\right)} \\ & + \mathbb{E}[L|X_k' \neq x_e] \\ \text{s.t.} \quad & 2^{-\ell(x_e)} + \sum_{i=1}^{k} 2^{-\ell(x_i)} \leq 1 \\ & \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1,\ldots,k,e\}, \end{aligned} \tag{13}$$

where the objective function is equal to the average age expression in (8). We note that the problem in (13) is not convex due to the middle term in the objective function. However, when $\ell(x_e)$ is fixed, it is a convex problem. Thus, we first solve problem (13) for a fixed $\ell(x_e)$ and then determine the optimal $\ell(x_e)$ numerically in Section V. Thus, (13) becomes

$$\begin{aligned}\min_{\{\ell(x_i)\}} \quad & \frac{\mathbb{E}[L^2|X_k' \neq x_e] + 2\mathbb{E}[W]\mathbb{E}[L|X_k' \neq x_e] + \mathbb{E}[W^2]}{2\left(\mathbb{E}[L|X_k' \neq x_e] + \mathbb{E}[W]\right)} \\ & + \mathbb{E}[L|X_k' \neq x_e]\end{aligned}$$

$$\text{s.t.} \quad \sum_{i=1}^{k} 2^{-\ell(x_i)} \leq 1 - 2^{-c}$$
$$\ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k\}, \tag{14}$$

where $\ell(x_e) = c$. Since the empty status update length $\ell(x_e)$ is fixed and given, we write the Kraft inequality by subtracting the portion allocated for $\ell(x_e)$ for the optimization problem in (14). Similar to [20] and [32], we define $p(\theta)$ as

$$\begin{aligned} p(\theta) := \min_{\{\ell(x_i)\}} \; & \frac{1}{2}\mathbb{E}[L^2|X_k' \neq x_e] + \mathbb{E}[L|X_k' \neq x_e]^2 \\ & + (2a-\theta)\mathbb{E}[L|X_k' \neq x_e] + \frac{d}{2} - \theta a \\ \text{s.t.} \quad & \sum_{i=1}^{k} 2^{-\ell(x_i)} \leq 1 - 2^{-c} \\ & \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k\}, \end{aligned} \tag{15}$$

where $a = \mathbb{E}[W]$ and $d = \mathbb{E}[W^2]$. For a fixed and given $\ell(x_e)$, the optimization problem in (15) is convex. We note that $p(\theta)$ is decreasing in θ and the optimal solution is obtained when $p(\theta) = 0$ such that the optimal age for the problem in (14) is equal to θ, i.e., $\Delta^* = \theta$ [54]. Our aim is to find the optimal real valued codeword lengths that minimize the average age expression Δ given in (8) for a fixed $\ell(x_e)$. To this end, we define the Lagrangian [55] function as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\mathbb{E}[L^2|X_k' \neq x_e] + \mathbb{E}[L|X_k' \neq x_e]^2 \\ & + (2a-\theta)\mathbb{E}[L|X_k' \neq x_e] + \frac{d}{2} - \theta a \\ & + \beta\left(\sum_{i=1}^{k} 2^{-\ell(x_i)} + 2^{-c} - 1\right), \end{aligned} \tag{16}$$

where $\beta \geq 0$. The KKT conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \ell(x_i)} = & P_{X_k}(x_i)\ell(x_i) + 2\left(\sum_{j=1}^{k} P_{X_k}(x_j)\ell(x_j)\right) P_{X_k}(x_i) \\ & + (2a-\theta)P_{X_k}(x_i) - \beta(\log 2)2^{-\ell(x_i)} = 0, \quad \forall i, \end{aligned} \tag{17}$$

and the complementary slackness condition is

$$\beta\left(\sum_{i=1}^{k} 2^{-\ell(x_i)} + 2^{-c} - 1\right) = 0. \tag{18}$$

The following lemma shows that the optimal codeword lengths satisfy $\sum_{i=1}^{k} 2^{-\ell(x_i)} = 1 - 2^{-c}$.

Lemma 1 *For the age-optimal real-valued codeword lengths, we must have* $\sum_{i=1}^{k} 2^{-\ell(x_i)} = 1 - 2^{-c}$.

Proof: Assume that the optimal codeword lengths satisfy $\sum_{i=1}^{k} 2^{-\ell(x_i)} < 1 - 2^{-c}$, which implies that $\beta = 0$ due to (18). From (17), we have

$$\begin{aligned} & P_{X_k}(x_i)\ell(x_i) + 2\left(\sum_{j=1}^{k} P_{X_k}(x_j)\ell(x_j)\right) P_{X_k}(x_i) \\ & + (2a-\theta)P_{X_k}(x_i) = 0, \quad \forall i. \end{aligned} \tag{19}$$

By summing (19) over all i, we get $\mathbb{E}[L] = \frac{\theta - 2a}{3}$. Then, we find $\ell(x_i) = \frac{\theta-2a}{3}$ for all $i \in \{1, \dots, k\}$ which makes $p(\theta) = -\frac{\theta^2+2a\theta+4a^2-3d}{6}$. By using $p(\theta) = 0$, we find $\theta = -a + \sqrt{3(d-a^2)}$ which gives $\ell(x_i) = -a + \sqrt{\frac{d-a^2}{3}}$ for $i \in \{1, \dots, k\}$. One can show that θ, hence age, is an increasing function of c. Thus, in the optimal policy, c must be equal to zero. However, choosing $c = 0$ leads to $\sum_{i=1}^{k} 2^{-\ell(x_i)} + 2^{-c} = 1$ and $\ell(x_i) = \infty$ which clearly cannot be the optimal solution thereby leading to a contradiction. ■

Thus, for the age-optimal codeword lengths, we have $\sum_{i=1}^{k} 2^{-\ell(x_i)} = 1 - 2^{-c}$ and $\beta \geq 0$ from (18). By summing (17) over all i and using Lemma 1 we find

$$\mathbb{E}[L|X_k' \neq x_e] = \frac{\theta + \beta \log 2(1-2^{-c}) - 2a}{3}. \tag{20}$$

From (17), we obtain

$$-\ell(x_i) + \frac{\beta \log 2}{P_{X_k}(x_i)} 2^{-\ell(x_i)} = 2\mathbb{E}[L|X_k' \neq x_e] + 2a - \theta. \tag{21}$$

Thus, we find the unique solution for $\ell(x_i)$ as

$$\ell(x_i) = -\frac{\log\left(\frac{P_{X_k}(x_i)}{\beta(\log 2)^2} W\left(\frac{\beta(\log 2)^2}{P_{X_k}(x_i)} 2^{\frac{-\theta+2\beta\log 2(1-2^{-c})+2a}{3}}\right)\right)}{\log 2}, \tag{22}$$

for $i \in \{1, \dots, k\}$, where W denotes the principle branch of the Lambert W function [56]. In order to find the optimal codeword lengths, we solve (22) for (θ, β) such that $p(\theta) = 0$ and $\sum_{i=1}^{k} 2^{-\ell(x_i)} = 1 - 2^{-c}$. We start from an arbitrary (θ, β) pair and if $p(\theta) > 0$ (or $p(\theta) < 0$), we increase (or respectively decrease) θ in the next iteration as $p(\theta)$ is a decreasing function of θ. Then, we update β by using (20). We repeat this procedure until $p(\theta) = 0$ and $\sum_{i=1}^{k} 2^{-\ell(x_i)} = 1 - 2^{-c}$.

We note that the average age achieved at the receiver depends on $\ell(x_e)$. In Section V, we provide numerical results where we vary $\ell(x_e)$ over all possible values and choose the one that yields the least average age for given arbitrary pmfs.

B. When the Empty Symbol Resets the Age

In this subsection, we consider the case where the empty symbol resets the age as it carries *partial* status information as in [25]. In other words, each update which arrives when the transmitter idles is accepted as a successful update. In the following theorem, we find the average age, Δ.

Theorem 2 *When the empty status update x_e is able to reset the age, the average age under the highest k selective encoding scheme at the receiver is given by*

$$\Delta = \frac{\mathbb{E}[L^2] + 2\frac{1}{\lambda}\mathbb{E}[L] + \frac{2}{\lambda^2}}{2\left(\mathbb{E}[L] + \frac{1}{\lambda}\right)} + \mathbb{E}[L]. \tag{23}$$

Proof: Different from the previous sections, the moments for the waiting time are equal to $\mathbb{E}[W] = \frac{1}{\lambda}$ and $\mathbb{E}[W^2] = \frac{2}{\lambda^2}$ as

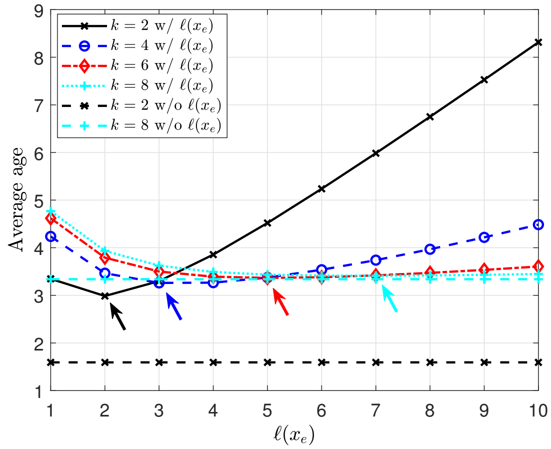

Fig. 3. Average age with the age-optimal codeword lengths with respect to $\ell(x_e)$ with the pmf in (26) for $n = 10$ when the empty symbol does not reset the age. Arrows indicate the age-optimal $\ell(x_e)$ values. We also provide the optimal age without sending the empty symbol for $k = 2$ and $k = 8$.

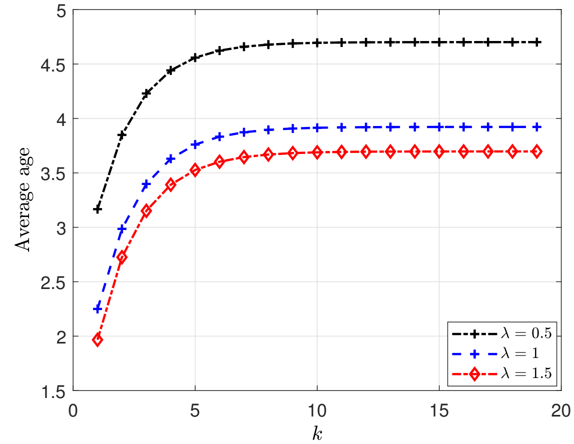

Fig. 4. Average age with the age-optimal codeword lengths for varying k with the pmf in (26) for $n = 20$ when the empty symbol resets the age.

each successful symbol is able to reset the age. Thus, substituting $\mathbb{E}[W]$ and $\mathbb{E}[W^2]$ in (5) and noting that $\mathbb{E}[S] = \mathbb{E}[L]$ yields the result. ■

Next, we formulate the optimization problem as

$$\min_{\{\ell(x_i),\ell(x_e)\}} \quad \frac{\mathbb{E}[L^2] + 2\tilde{a}\mathbb{E}[L] + 2\tilde{a}^2}{2(\mathbb{E}[L] + \tilde{a})} + \mathbb{E}[L]$$
$$\text{s.t.} \quad 2^{-\ell(x_e)} + \sum_{i=1}^{k} 2^{-\ell(x_i)} \leq 1$$
$$\ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k, e\}, \tag{24}$$

where $\tilde{a} = \frac{1}{\lambda}$. We follow a similar solution technique to that given in Section IV-A to get

$$\ell(x_i) = -\frac{\log\left(\frac{P_X(x_i)}{\beta(\log 2)^2} W\left(\frac{\beta(\log 2)^2}{P_X(x_i)} 2^{\frac{-\theta+2\beta\log 2+2\tilde{a}}{3}}\right)\right)}{\log 2}, \tag{25}$$

for $i \in \{1, \dots, k, e\}$.

The value of k affects $\ell(x_e)$ such that when k is close to n, the probability of the empty symbol becomes small which leads to a longer $\ell(x_e)$, whereas when k is small, the probability of the empty symbol becomes large which results in a shorter $\ell(x_e)$. In Section V, we numerically determine the optimal k selection which achieves the lowest average age for a given arbitrary distribution.

V. Numerical Results

In this section, we provide simulations to numerically characterize the optimal k values that minimize the average age for a given arbitrary pmf. We use the following pmf

$$P_X(x_i) = \begin{cases} 2^{-i}, & i = 1, \dots, n-1 \\ 2^{-n+1}, & i = n. \end{cases} \tag{26}$$

In the first simulation, we consider the pmf in (26) for $n = 10$ and take $\lambda = 5$. We find the optimal codeword length of the empty symbol, $\ell(x_e)$, when the empty symbol does not reset the age (see Fig. 3). We observe that when k is small, the probability of sending the empty symbol becomes large so that a shorter codeword is preferable for x_e. For example, we observe in Fig. 3 that choosing $\ell(x_e) = 2$ when $k = 2$ and $\ell(x_e) = 3$ when $k = 4$ is optimal. Similarly, when k is larger, a longer codeword is desirable for x_e. We observe in Fig. 3 that choosing $\ell(x_e) = 5$ when $k = 6$ and $\ell(x_e) = 7$ when $k = 8$ is optimal. Further, we note in Fig. 3 that the average age increases when we send the empty symbol in the case of the remaining $n-k$ realizations as the empty symbol increases the total waiting time for the next successful arrival as well as the codeword lengths for the encoded k realizations. For smaller k values, i.e., when $k = 2$, this effect is significant as the empty symbol has a large probability whereas when k is larger, i.e., when $k = 8$, sending an empty status update increases the age slightly (especially when $\ell(x_e)$ is high) as the empty symbol has a small probability.

In the second simulation shown in Fig. 4, we consider the case when the empty symbol x_e resets the age. We observe that the minimum age is achieved when $k = 1$, i.e., only the most probable realization is encoded. This is because the overall waiting time is independent of k and larger k values result in larger codewords which in turn increases transmission times. Thus, in this case, only the most probable realization is received separately since all others are embedded into the empty symbol. We note that this selection results in significant information loss at the receiver which is not captured by the age metric alone. This problem can be addressed by introducing a distortion constraint which measures the information loss together with the age metric which measures freshness [57].

VI. Conclusions

We studied the problem of timely source coding and proposed the highest k selective encoding with empty symbol scheme. We determined the average age under this scheme and found the age-optimal real-valued codeword lengths. Through simulations, we determined the age-optimal k values. Our results indicate that this selective encoding scheme results in a lower average age than encoding every realization.

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