On Gaussian MIMO Broadcast Channels with Common and Private Messages

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***Abstract*— We study the Gaussian multiple-input multiple-output (MIMO) broadcast channel with common and private messages. We first obtain an outer bound for the capacity region of the two-user discrete memoryless broadcast channel with common and private messages. We next show that if jointly Gaussian random variables are sufficient to evaluate this outer bound for the Gaussian MIMO broadcast channel, the dirty-paper coding (DPC) region is the capacity region of the Gaussian MIMO broadcast channel with common and private messages. However, we can evaluate only a loosened version of this outer bound, which yields the result that extending the DPC region in the common message rate direction by a fixed amount is an outer bound for the capacity region of the Gaussian MIMO broadcast channel with common and private messages. However, this fixed amount, i.e., the gap, is not finite for all channels. We derive the necessary and sufficient conditions for this gap to be finite.**

I. Introduction

We study the two-user Gaussian multiple-input multiple-output (MIMO) broadcast channel with common and private messages, in which the transmitter sends a private message to each user in addition to a common message that is directed to both users. The capacity region of this channel is not known completely. However, when one of these three messages is absent, the corresponding capacity region is known. In particular, the capacity region is known i) when there is no common message, i.e., each user gets only a private message [1], and ii) for the degraded message set case, i.e., there is a common message directed to both users, and only one of the users gets a private message [2], [3].

The Gaussian MIMO broadcast channel with common and private messages is first studied in [4], where an achievable rate region that relies on dirty-paper coding (DPC) is proposed. This achievable rate region is called the DPC region. Moreover, [4] obtains the capacity region when the Gaussian MIMO broadcast channel is equivalent to a set of parallel independent Gaussian channels. The Gaussian MIMO broadcast channel with common and private messages is further studied in [2], [3], where the optimality of the DPC region is shown partially. References [2], [3] first propose an outer bound for the capacity region of the Gaussian MIMO broadcast channel with common and private messages, which matches the DPC region partially, i.e., is partially tight. Moreover, [2], [3] show that for a given common message rate, the private message sum rate capacity is attained by DPC. Finally, [2], [3] show the optimality of the DPC region when the common message rate is beyond a certain threshold. Despite these partial results, it is unknown whether the DPC region is equal to the capacity region or not.

We first consider the two-user discrete memoryless broadcast channel with common and private messages, and obtain an outer bound for the capacity region of this channel. We next consider the evaluation of this outer bound for the Gaussian MIMO broadcast channel. We first observe that if jointly Gaussian auxiliary random variables and the channel input are sufficient to evaluate this outer bound, the DPC region is the capacity region for the Gaussian MIMO broadcast channel with common and private messages. Unfortunately, we could not show the sufficiency of the jointly Gaussian auxiliary random variables and the channel input to evaluate this outer bound. Instead, we evaluate a loosened version of this outer bound which states that extending the DPC region in the common message rate direction by a fixed amount yields an outer bound for the capacity region of the Gaussian MIMO broadcast channel with common and private messages. However, this fixed amount of gap between the outer bound provided here and the DPC region is not necessarily finite for all channels, i.e., this gap may scale with the available power at the transmitter for some channels. We derive necessary and sufficient conditions for the gap between our outer bound and the DPC region to be finite irrespective of the available power at the transmitter. We show that these necessary and sufficient conditions are weaker than the degradedness condition. Moreover, we characterize the degrees of freedom region of Gaussian MIMO broadcast channels satisfying these necessary and sufficient conditions for the finite gap. Finally, we establish the connections between the outer bound we obtained for the capacity region of the Gaussian MIMO broadcast channel with common and private messages and the outer bound provided in [2], [3].

II. An Outer Bound for Broadcast Channels with Common and Private Messages

We study the discrete two-user memoryless broadcast channel for the scenario where each user gets a private message that is intended to only this user in addition to a common message which is directed to both users. The transition probability of

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the channel is $p(y_1, y_2|x)$ where $x \in \mathcal{X}$ is the channel input, $y_j \in \mathcal{Y}_j$ is the channel output of the jth user, $j = 1, 2$.

An $(n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})$ code for this channel consists of three message sets $\mathcal{W}_j = \{1, \ldots, 2^{nR_j}\},\ j = 0, 1, 2$, one encoder $f_n : \mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2 \to \mathcal{X}^n$, one decoder at each receiver $g_n^j : \mathcal{Y}_j^n \to \mathcal{W}_0 \times \mathcal{W}_j,\ j = 1, 2$. The probability of error is defined as $P_e^n = \max\{P_{e1}^n, P_{e2}^n\}$, where $P_{ej}^n = \Pr[g_n^j(f_n(W_0, W_j)) \neq (W_0, W_j)],\ j = 1, 2$, and W_j is a uniformly distributed random variable in $\mathcal{W}_j,\ j = 0, 1, 2$.

A rate triple (R_0, R_1, R_2) is said to be achievable if there exists an $(n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})$ code which has $\lim_{n\to\infty} P_e^n = 0$. The capacity region $\mathcal{C}$ is defined as the convex closure of all achievable rate triples (R_0, R_1, R_2). The capacity region is convex due to time-sharing, hence, the boundary of the capacity region can be written as the solution of

$$\max_{(R_0,R_1,R_2)\in\mathcal{C}} R_0 + \mu_1 R_1 + \mu_2 R_2 \tag{1}$$

where $\mu_j \in [0, \infty),\ j = 1, 2$. Here, our main focus is on the part of the boundary of the capacity region where all rates are strictly positive, and we want to obtain an outer bound for this part of the capacity region. To this end, we introduce the following two lemmas that give the conditions on μ_1, μ_2 for which the maximum in (1) is attained by $R_j \neq 0$, for all $j = 0, 1, 2$.

Lemma 1 *If* $\max\{\mu_1, \mu_2\} > 1$, *we have*

$$\max_{(R_0,R_1,R_2)\in\mathcal{C}} R_0 + \mu_1 R_1 + \mu_2 R_2 = \max_{(0,R_1,R_2)\in\mathcal{C}} \mu_1 R_1 + \mu_2 R_2 \tag{2}$$

Lemma 2 *If* $\mu_1 + \mu_2 < 1$, *we have*

$$\begin{aligned}&\max_{(R_0,R_1,R_2)\in\mathcal{C}} R_0 + \mu_1 R_1 + \mu_2 R_2 \\ &= \max\left\{ \max_{(R_0,0,R_2)\in\mathcal{C}} R_0 + \mu_2 R_2,\ \max_{(R_0,R_1,0)\in\mathcal{C}} R_0 + \mu_1 R_1 \right\}\end{aligned} \tag{3}$$

The proofs of these lemmas as well as the proofs of all upcoming results are omitted due to the space limitations here. For the part of the boundary of the capacity region where the common message rate is strictly positive, we have the following outer bound.

Theorem 1 *The optimization problem in (1) is bounded as*

$$\begin{aligned}&\max_{(R_0,R_1,R_2)\in\mathcal{C}} R_0 + \mu_1 R_1 + \mu_2 R_2 \leq \\ &\max \min_{j=1,2} I(U; Y_j) + \mu_1 I(V; Y_1|U) + \mu_2 I(X; Y_2|V)\end{aligned} \tag{4}$$

for $1 \geq \mu_1 \geq \mu_2$, *where the second maximization is over all random variable triples* (U, V, X) *satisfying* $U \to V \to X \to (Y_1, Y_2)$. *The outer bound for the case* $1 \geq \mu_2 \geq \mu_1$ *can be obtained from (4) by swapping the subscripts 1 and 2.*

This outer bound is reminiscent of the Korner-Marton outer bound [8], which is for the case where there is no common message. Our outer bound can be viewed as a generalization of the Korner-Marton outer bound to the case where both common and private messages are present. Theorem 1 can be proved by following the proof of the Korner-Marton outer bound and using Lemma 1. Alternatively, the outer bound in Theorem 1 can be obtained from the outer bound in [5] by loosening it. This alternative derivation also reveals that the outer bound in Theorem 1 is potentially looser than not only the outer bound in [5] but also the one in [6] due to the equivalence of the latter two bounds. We also expect the outer bound in Theorem 1 to be potentially looser than the outer bound in [7] because when we set $R_0 = 0$, our outer bound reduces to the Korner-Marton outer bound [8] which strictly includes the one obtained from [7]'s outer bound by setting $R_0 = 0$. Although our outer bound seems to be potentially looser than the existing outer bounds [5]–[7], it is easier to evaluate due the Markov chain among (U, V, X), and provides greater insight for the capacity region of the Gaussian MIMO broadcast channel with common and private messages. In particular, while we have not been able to prove it, if jointly Gaussian (U, V, X) are sufficient to evaluate our outer bound for the Gaussian MIMO broadcast channel with common and private messages, then it is tight, implying that all of the existing outer bounds in [5]–[7] would be tight for the Gaussian MIMO channel with common and private messages.

III. Gaussian MIMO Broadcast Channel with Common and Private Messages

The Gaussian MIMO broadcast channel is defined by

$$\mathbf{Y}_1 = \mathbf{H}_1 \mathbf{X} + \mathbf{N}_1 \tag{5}$$

$$\mathbf{Y}_2 = \mathbf{H}_2 \mathbf{X} + \mathbf{N}_2 \tag{6}$$

where the channel input $\mathbf{X}$ is a $t \times 1$ vector, $\mathbf{H}_j$ is the jth user's channel gain matrix of size $r_j \times t$, $\mathbf{Y}_j$ is the channel output of the jth user which is an $r_j \times 1$ vector, and the Gaussian random vector $\mathbf{N}_j$ is of size $r_j \times 1$ with a covariance matrix $\mathbf{\Sigma}_j$ which is assumed to be strictly positive-definite, i.e., $\mathbf{\Sigma}_j \succ \mathbf{0}$. We consider two different constraints on the channel input. The first one is a covariance constraint, i.e.,

$$E\left[\mathbf{X}\mathbf{X}^\top\right] \preceq \mathbf{S} \tag{7}$$

where $\mathbf{S}$ is a positive semi-definite matrix, i.e., $\mathbf{S} \succeq \mathbf{0}$. The second constraint on the channel input is a total power constraint, i.e.,

$$E\left[\mathbf{X}^\top\mathbf{X}\right] = \text{tr}\left(E\left[\mathbf{X}\mathbf{X}^\top\right]\right) \leq P \tag{8}$$

We now define a sub-class of Gaussian MIMO broadcast channels called the aligned Gaussian MIMO broadcast channel, which is obtained from (5)-(6) by setting $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{I}$:

$$\mathbf{Y}_1 = \mathbf{X} + \mathbf{N}_1 \tag{9}$$

$$\mathbf{Y}_2 = \mathbf{X} + \mathbf{N}_2 \tag{10}$$

In this work, we first study the aligned channel in (9)-(10), and prove our results for this channel. Then, we raise these results to the general case in (5)-(6) by using the analysis in [1].

We conclude this section by presenting the achievable rate region, hereafter called the DPC region, in [4]. In the achievable scheme in [4], the common message is encoded by a standard Gaussian codebook, and the private messages are encoded by DPC. Each user decodes the common message by treating the signals carrying the private messages as noise. Next, users decode their private messages. Since a DPC scheme is used to encode the private messages, one of the users observes an interference-free link depending on the encoding order at the transmitter. We next define

$$R_{0j}(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_c)=\frac{1}{2}\log\frac{|\mathbf{H}_j(\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_c)\mathbf{H}_j^\top+\mathbf{\Sigma}_j|}{|\mathbf{H}_j(\mathbf{K}_1+\mathbf{K}_2)\mathbf{H}_j^\top+\mathbf{\Sigma}_j|},\quad j=1,2 \quad (11)$$

$$R_1(\mathbf{K}_1,\mathbf{K}_2)=\frac{1}{2}\log\frac{|\mathbf{H}_1(\mathbf{K}_1+\mathbf{K}_2)\mathbf{H}_1^\top+\mathbf{\Sigma}_1|}{|\mathbf{H}_1\mathbf{K}_2\mathbf{H}_1^\top+\mathbf{\Sigma}_1|} \quad (12)$$

$$R_2(\mathbf{K}_2)=\frac{1}{2}\log\frac{|\mathbf{H}_2\mathbf{K}_2\mathbf{H}_2+\mathbf{\Sigma}_2|}{|\mathbf{\Sigma}_2|} \quad (13)$$

where $\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_c$ denote the covariance matrices allotted for the first user's private message, the second user's private message, and the common message, respectively. The DPC region is stated in the following theorem.

Theorem 2 ([4]) *The rate triples (R_0,R_1,R_2) lying in the following region*

$$\mathcal{R}^{\rm DPC}(\mathbf{S})=\operatorname{conv}\left(\mathcal{R}_1^{\rm DPC}(\mathbf{S})\cup\mathcal{R}_2^{\rm DPC}(\mathbf{S})\right) \quad (14)$$

are achievable, where conv *is the convex hull operator, $\mathcal{R}_1^{\rm DPC}(\mathbf{S})$ consists of rate triples (R_0,R_1,R_2) satisfying*

$$R_0\le R_{0j}(\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_c),\quad j=1,2 \quad (15)$$

$$R_1\le R_1(\mathbf{K}_1,\mathbf{K}_2) \quad (16)$$

$$R_2\le R_2(\mathbf{K}_2) \quad (17)$$

for some positive semi-definite matrices $\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_c$ such that $\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_c\preceq\mathbf{S}$, and $\mathcal{R}_2^{\rm DPC}(\mathbf{S})$ can be obtained from $\mathcal{R}_1^{\rm DPC}(\mathbf{S})$ by swapping the subscripts 1 and 2.

This DPC region is for channels subject to the covariance constraint in (7). The DPC region for channels with the total power constraint $\mathcal{R}^{\rm DPC}(P)$ can be obtained from Theorem 2 by changing the constraint $\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_c\preceq\mathbf{S}$ with the trace constraint $\operatorname{tr}(\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_c)\le P$. For aligned channels, we set $\mathbf{K}_c=\mathbf{S}-\mathbf{K}_1-\mathbf{K}_2$, and the expressions in (11)-(12) will be denoted by $R_{0j}^{\rm AL}(\mathbf{K}_1,\mathbf{K}_2),R_1^{\rm AL}(\mathbf{K}_1,\mathbf{K}_2),R_2^{\rm AL}(\mathbf{K}_2)$. The corresponding achievable rate regions for the aligned channel will be denoted by $\mathcal{R}_{12}^{\rm DPC-AL}(\mathbf{S}),\mathcal{R}_{21}^{\rm DPC-AL}(\mathbf{S}),\mathcal{R}^{\rm DPC-AL}(\mathbf{S})$, and the capacity region for the aligned channel by $\mathcal{C}^{\rm AL}(\mathbf{S}),\mathcal{C}^{\rm AL}(P)$.

The DPC region is tight in several cases. The first one is the case where each receiver gets only a private message, i.e., $R_0=0$ [1]. The other case is the degraded message sets scenario in which we have either $R_1=0$ or $R_2=0$ [2]. In both of these cases, there are only two messages to be sent. The case when both private messages and a common message are present is investigated in [2], [3]. In [2], [3], outer bounds on the capacity region with private and common messages are given and these outer bounds are shown to match with the DPC region in certain regions. Furthermore, [2], [3] show that for a given common message rate R_0, the DPC region achieves the private message sum rate capacity, i.e., the maximum of R_1+R_2. Finally, [2], [3] show that if the common message rate is beyond a certain threshold, the DPC region matches the capacity region.

IV. Aligned Case

Here, we consider the aligned Gaussian MIMO broadcast channel. We first present an observation that can be obtained by comparing the outer bound in Theorem 1 with the DPC region. To this end, we assume that jointly Gaussian $(U,V,\mathbf{X})$ are sufficient to evaluate the outer bound in Theorem 1. Without loss of generality, consider the case $1\ge\mu_1\ge\mu_2$,

$$\max_{(R_0,R_1,R_2)\in\mathcal{C}^{\rm AL}(\mathbf{S})} R_0+\mu_1R_1+\mu_2R_2$$
$$\le\max\ \min_{j=1,2} I(U;\mathbf{Y}_j)+\mu_1I(V;\mathbf{Y}_1|U)+\mu_2I(\mathbf{X};\mathbf{Y}_2|V) \quad (18)$$
$$=\max\ \min_{j=1,2} R_{0j}^{\rm AL}(\mathbf{K}_1,\mathbf{K}_2)+\mu_1R_1^{\rm AL}(\mathbf{K}_1,\mathbf{K}_2)+\mu_2R_2^{\rm AL}(\mathbf{K}_2) \quad (19)$$
$$=\max_{(R_0,R_1,R_2)\in\mathcal{R}^{\rm DPC-AL}(\mathbf{S})}\mu_0R_0+\mu_1R_1+\mu_2R_2 \quad (20)$$

where (18) is due to Theorem 1, (19) comes from our assumption that jointly Gaussian $(U,V,\mathbf{X})$ are sufficient to evaluate the outer bound in Theorem 1, and (20) is due to Theorem 2. Equation (20) implies the following theorem.

Theorem 3 *If jointly Gaussian $(U,V,\mathbf{X})$ are sufficient to evaluate the outer bound in Theorem 1 for the aligned Gaussian MIMO broadcast channel, we have $\mathcal{C}^{\rm AL}(\mathbf{S})=\mathcal{R}^{\rm DPC-AL}(\mathbf{S})$.*

Unfortunately, we are not able to show that jointly Gaussian $(U,V,\mathbf{X})$ are sufficient to evaluate the outer bound in Theorem 1. However, we show that jointly Gaussian $(U,V,\mathbf{X})$ are the maximizer of the following expression for $j=1,2$

$$I(U;\mathbf{Y}_j)+\mu_1I(V;\mathbf{Y}_1|U)+\mu_2I(\mathbf{X};\mathbf{Y}_2|V) \quad (21)$$

for $1\ge\mu_1\ge\mu_2$, which correspond to a looser version of the outer bound in Theorem 1. This result is stated next.

Lemma 3 *Let $(U,V,\mathbf{X})$ be an arbitrary random vector satisfying the Markov chain $U\to V\to\mathbf{X}$. $\mathbf{X}$ is a covariance-constrained random vector such that $E\left[\mathbf{X}\mathbf{X}^\top\right]\preceq\mathbf{S}$, where $\mathbf{S}\succ\mathbf{0}$. For $1\ge\mu_1\ge\mu_2$, we have, for $j=1,2$,*

$$\max\ I(U;\mathbf{Y}_j)+\mu_1I(V;\mathbf{Y}_1|U)+\mu_2I(\mathbf{X};\mathbf{Y}_2|V)$$
$$=\max\ R_{0j}^{\rm AL}(\mathbf{K}_1,\mathbf{K}_2)+\mu_1R_1^{\rm AL}(\mathbf{K}_1,\mathbf{K}_2)+\mu_2R_2^{\rm AL}(\mathbf{K}_2) \quad (22)$$

We use an extremal inequality from [9] and channel enhancement [1] in the proof of Lemma 3. Using this result, we obtain the following theorem.

Theorem 4 *The capacity region of the aligned Gaussian MIMO broadcast channel with common and private messages satisfies the following inclusions*

$$\mathcal{R}^{\rm DPC-AL}(\mathbf{S})\subseteq\mathcal{C}^{\rm AL}(\mathbf{S})\subseteq\tilde{\mathcal{R}}^{\rm DPC-AL}(\mathbf{S}) \quad (23)$$

where $\tilde{\mathcal{R}}^{\rm DPC-AL}(\mathbf{S})$ *is given by the union of the rate triples* (R_0, R_1, R_2) *such that*

$$\Big(R_0 - \min\{g_{1,AL}^+(\mathbf{S}), g_{2,AL}^+(\mathbf{S})\}, R_1, R_2\Big) \in \mathcal{R}^{\rm DPC-AL}(\mathbf{S}) \tag{24}$$

where $x^+ = \max(0,x)$*, and* $g_{1,AL}(\mathbf{S})$ *is*

$$g_{1,AL}(\mathbf{S}) = \max_{\mathbf{0}\preceq\mathbf{K}\preceq\mathbf{S}} \frac{1}{2}\log\frac{|\mathbf{K}+\mathbf{\Sigma}_1|}{|\mathbf{\Sigma}_1|} - \frac{1}{2}\log\frac{|\mathbf{K}+\mathbf{\Sigma}_2|}{|\mathbf{\Sigma}_2|} \tag{25}$$

and, $g_{2,AL}(\mathbf{S})$ *can be obtained from* $g_{1,AL}(\mathbf{S})$ *by swapping the subscripts 1 and 2.*

Theorem 4 states that extending the DPC region in the R_0 direction by a fixed amount $\min\{g_{1,AL}^+(\mathbf{S}), g_{2,AL}^+(\mathbf{S})\}$ gives us an outer bound for the capacity region of the aligned Gaussian MIMO broadcast channel with common and private messages. We note that this gap is finite irrespective of the value of $\mathbf{S}$ since we have $\mathbf{\Sigma}_j \succ \mathbf{0},\ j=1,2$, which, in turn, implies $g_{j,AL}(\mathbf{S}) < \infty,\ j=1,2$. Furthermore, we note that $\min\{g_{1,AL}^+(\mathbf{S}), g_{2,AL}^+(\mathbf{S})\}$ is zero when either $g_{1,AL}(\mathbf{S}) = 0$ or $g_{2,AL}(\mathbf{S}) = 0$ which occurs if and only if we have either $\mathbf{\Sigma}_2 \preceq \mathbf{\Sigma}_1$ or $\mathbf{\Sigma}_1 \preceq \mathbf{\Sigma}_2$. However, if one of these orders holds, the channel is degraded and its capacity region with common and private messages is known, and is equivalent to $\mathcal{R}^{\rm DPC-AL}(\mathbf{S})$. Thus, the outer bound in Theorem 4 is tight for this particular case.

V. General Case

Using the analysis in Section V.B of [1], Theorems 3 and 4 can be raised to the most general form of the Gaussian MIMO broadcast channel in (5)-(6). In particular, Theorem 3 implies that if jointly Gaussian $(U, V, \mathbf{X})$ are sufficient to evaluate the outer bound in Theorem 1 for the aligned Gaussian MIMO broadcast channel, the capacity region of the (general) Gaussian MIMO broadcast channel is equal to the DPC region. We next state the extension of Theorem 4 to the most general channel model in (5)-(6) for the total power constraint in (8).

Theorem 5 *The capacity region of the Gaussian MIMO broadcast channel with common and private messages satisfies the following inclusions*

$$\mathcal{R}^{\rm DPC}(P) \subseteq \mathcal{C}(P) \subseteq \tilde{\mathcal{R}}^{\rm DPC}(P) \tag{26}$$

where $\tilde{\mathcal{R}}^{\rm DPC}(P)$ *is defined as the union of rate triples* (R_0, R_1, R_2) *satisfying*

$$(R_0 - \min\{g_1^+(P), g_2^+(P)\}, R_1, R_2) \in \mathcal{R}^{\rm DPC}(P) \tag{27}$$

where $x^+ = \max(0,x)$*, and* $g_1(P)$ *is*

$$g_1(P) = \max_{\mathbf{0}\preceq\mathbf{K},\, {\rm tr}(\mathbf{K})\leq P} \frac{1}{2}\log\frac{|\mathbf{H}_1\mathbf{K}\mathbf{H}_1^\top+\mathbf{\Sigma}_1|}{|\mathbf{\Sigma}_1|} - \frac{1}{2}\log\frac{|\mathbf{H}_2\mathbf{K}\mathbf{H}_2^\top+\mathbf{\Sigma}_2|}{|\mathbf{\Sigma}_2|} \tag{28}$$

and, $g_2(P)$ *can be obtained from* $g_1(P)$ *by swapping the subscripts* 1 *and* 2*.*

Theorem 5 states that extending $\mathcal{R}^{DPC}(P)$ in the R_0 direction by a fixed amount $\min\{g_1^+(P), g_2^+(P)\}$ provides an outer bound for the capacity region of the general Gaussian MIMO broadcast channel with common and private messages. We note that, contrary to the aligned case, this gap is not always finite for all values of P depending on the channel gain matrices $\mathbf{H}_1, \mathbf{H}_2$. However, the gap between the inner and outer bounds might still be finite irrespective of the value of P, for which the necessary and sufficient conditions are stated in the following theorem.

Theorem 6 *The gap given in Theorem 5 is always finite, i.e.,*

$$\lim_{P\to\infty} \min\{g_1^+(P), g_2^+(P)\} < \infty \tag{29}$$

iff either ${\rm null}(\mathbf{H}_1) \subseteq {\rm null}(\mathbf{H}_2)$ *or* ${\rm null}(\mathbf{H}_2) \subseteq {\rm null}(\mathbf{H}_1)$ *holds.*

The necessary and sufficient conditions in Theorem 6 are weaker than the degradedness condition which also results in a finite gap. The degradedness condition is as follows [9]: The first user is said to be degraded with respect to the second user if there exits a matrix $\mathbf{D}$ such that $\mathbf{D}\mathbf{D}^\top \preceq \mathbf{I}$ and $\mathbf{D}\mathbf{H}_2 = \mathbf{H}_1$. This condition implies ${\rm null}(\mathbf{H}_2) \subseteq {\rm null}(\mathbf{H}_1)$. However, the latter condition does not always imply the existence of a matrix $\mathbf{D}$ with $\mathbf{D}\mathbf{D}^\top \preceq \mathbf{I}$ and $\mathbf{D}\mathbf{H}_2 = \mathbf{H}_1$, i.e., the degradedness.

For channels satisfying the condition in Theorem 6, the degrees of freedom region can be obtained due to the finiteness of the gap. To this end, we need to introduce some definitions. A degrees of freedom triple (d_0, d_1, d_2) is said to be achievable if there exists a rate triple $(R_0, R_1, R_2) \in \mathcal{C}(P)$ such that

$$d_j = \lim_{P\to\infty} \frac{R_j}{\frac{1}{2}\log P}, \quad j = 0,1,2 \tag{30}$$

The degrees of freedom region $\mathcal{D}$ is defined as the convex closure of all achievable degrees of freedom triples (d_0, d_1, d_2). The degrees of freedom region for channels satisfying the condition in Theorem 6 is stated in the following theorem.

Theorem 7 *If* ${\rm null}(\mathbf{H}_2) \subseteq {\rm null}(\mathbf{H}_1)$*, the degrees of freedom region* $\mathcal{D}$ *of the Gaussian MIMO broadcast channel with common and private messages is given by the union of the degrees of freedom triples* (d_0, d_1, d_2) *satisfying*

$$d_0 \leq {\rm rank}\left(\mathbf{H}_1^\top\mathbf{H}_1\right) - \alpha \tag{31}$$

$$d_1 \leq \alpha - \beta \tag{32}$$

$$d_2 \leq \beta + {\rm rank}\left(\mathbf{H}_2^\top\mathbf{H}_2\right) - {\rm rank}\left(\mathbf{H}_1^\top\mathbf{H}_1\right) \tag{33}$$

for some α, β *such that* $0 \leq \beta \leq \alpha \leq {\rm rank}\left(\mathbf{H}_1^\top\mathbf{H}_1\right)$.

This theorem states that if P is sufficiently large, the Gaussian MIMO broadcast channel with ${\rm null}(\mathbf{H}_2) \subseteq {\rm null}(\mathbf{H}_1)$ behaves as if it is a parallel Gaussian broadcast channel with ${\rm rank}\left(\mathbf{H}_2^\top\mathbf{H}_2\right)$ sub-channels. The first user has access to the ${\rm rank}\left(\mathbf{H}_1^\top\mathbf{H}_1\right)$ of total ${\rm rank}\left(\mathbf{H}_2^\top\mathbf{H}_2\right)$ sub-channels. For a fixed α, β, ${\rm rank}\left(\mathbf{H}_1^\top\mathbf{H}_1\right) - \alpha$ of the sub-channels that the first user can access need to be used for the transmission of the common message, $\alpha - \beta$ of them need to be used for the transmission of the first user's private message, and the remaining

β of the sub-channels that the first user 1 can access need to be used for the transmission of the second user's private message in addition to the $\operatorname{rank}\left(\mathbf{H}_2^\top \mathbf{H}_2\right) - \operatorname{rank}\left(\mathbf{H}_1^\top \mathbf{H}_1\right)$ sub-channels that only the second user has access.

VI. Connections to the Outer Bound in Weingarten *et. al.*, 2006

We now revisit the outer bound in [2], [3] for the Gaussian MIMO broadcast channel with common and private messages, and establish the connections between our outer bound in Theorem 4 (hence the outer bound in Theorem 5) and the outer bound in [2], [3]. In other words, we present an alternative proof for Theorem 4 by using the outer bound in [2], [3], which is given by

$$\mathcal{C}^{\mathrm{AL}}(\mathbf{S}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \tag{34}$$

where $\mathcal{B}_1, \mathcal{B}_2$ are given by

$$\mathcal{B}_1 = \left\{(R_0, R_1, R_2) : (R_0 + R_1, R_2) \in \mathcal{C}^{\mathrm{PR-AL}}(\mathbf{S})\right\} \tag{35}$$

$$\mathcal{B}_2 = \left\{(R_0, R_1, R_2) : (R_1, R_0 + R_2) \in \mathcal{C}^{\mathrm{PR-AL}}(\mathbf{S})\right\} \tag{36}$$

and $\mathcal{C}^{\mathrm{PR-AL}}(\mathbf{S})$ denotes the capacity region of the two-user Gaussian MIMO broadcast channel when each user gets only a private message. We note that $\mathcal{C}^{\mathrm{PR-AL}}(\mathbf{S})$ can be obtained from the DPC region in Theorem 2 by setting $\mathbf{K}_c = \mathbf{0}, \mathbf{K}_1 + \mathbf{K}_2 = \mathbf{S}$ [1]. This outer bound is obtained by considering the scenario in which one of the users needs to decode both the common and its private message, while the other user needs to decode only its private message. For example, $\mathcal{B}_1$ is the capacity region of the Gaussian MIMO broadcast channel when the first user needs to decode (W_0, W_1) and the second user needs to decode only W_2.

We next find equivalent descriptions for $\mathcal{B}_1$ and $\mathcal{B}_2$. Consider the three-user Gaussian MIMO broadcast channel

$$\mathbf{Y}_{01} = \mathbf{X} + \mathbf{N}_1 \tag{37}$$

$$\mathbf{Y}_1 = \mathbf{X} + \mathbf{N}_1 \tag{38}$$

$$\mathbf{Y}_2 = \mathbf{X} + \mathbf{N}_2 \tag{39}$$

where the additional user's (the one with channel output $\mathbf{Y}_{01}$) observation is the same as the first user's observation. We consider the scenario where the additional user needs to decode W_0, the first user needs to decode W_1 and the second user needs to decode W_2 for the channel in (37)-(39). We note that the capacity region of this channel is known due to [1] and denoted by $\mathcal{C}_{31}^{\mathrm{PR-AL}}(\mathbf{S})$. Furthermore, since the additional user and the first user have the same observation, we have $\mathcal{B}_1 = \mathcal{C}_{31}^{\mathrm{PR-AL}}(\mathbf{S})$. Similarly, we can obtain $\mathcal{B}_2 = \mathcal{C}_{32}^{\mathrm{PR-AL}}(\mathbf{S})$ where $\mathcal{C}_{32}^{\mathrm{PR-AL}}(\mathbf{S})$ is the capacity region of the three-user Gaussian MIMO broadcast channel defined by $\mathbf{Y}_{02} = \mathbf{X} + \mathbf{N}_2$ and (38)-(39), where the additional user (the one with channel output $\mathbf{Y}_{02}$) needs to decode W_0, the first user needs to decode W_1 and the second user needs to decode W_2. Thus, the outer bound in (34) is equivalent to the following one

$$\mathcal{C}^{\mathrm{AL}}(\mathbf{S}) \subseteq \mathcal{C}_{31}^{\mathrm{PR-AL}}(\mathbf{S}) \cap \mathcal{C}_{32}^{\mathrm{PR-AL}}(\mathbf{S}) \tag{40}$$

Before showing that the outer bound given in Theorem 4 can also be obtained from the outer bound in (34), we note that the crucial step to prove Theorem 4 is to obtain

$$\begin{aligned} &\max_{(R_0,R_1,R_2)\in\mathcal{C}^{\mathrm{AL}}(\mathbf{S})} R_0 + \mu_1 R_1 + \mu_2 R_2 \\ &\leq \min_{j=1,2} \max_{\substack{\mathbf{0}\preceq\mathbf{K}_j,\ j=1,2 \\ \mathbf{K}_1+\mathbf{K}_2\preceq\mathbf{S}}} R_{0j}^{\mathrm{AL}}(\mathbf{K}_1,\mathbf{K}_2) + \mu_1 R_1^{\mathrm{AL}}(\mathbf{K}_1,\mathbf{K}_2) \\ &\qquad + \mu_2 R_2^{\mathrm{AL}}(\mathbf{K}_2) \end{aligned} \tag{41}$$

for $1 \geq \mu_1 \geq \mu_2$. In the original proof of Theorem 4, we use a loosened version of the outer bound in Theorem 1 and Lemma 3 to show (41). We now use the outer bound in (34) and the results of [1] to prove Theorem 4, i.e., to show (41):

$$\begin{aligned} &\max_{(R_0,R_1,R_2)\in\mathcal{C}^{\mathrm{AL}}(\mathbf{S})} R_0 + \mu_1 R_1 + \mu_2 R_2 \\ &\leq \max_{(R_0,R_1,R_2)\in\mathcal{C}_{31}^{\mathrm{PR-AL}}(\mathbf{S})\cap\mathcal{C}_{32}^{\mathrm{PR-AL}}(\mathbf{S})} R_0 + \mu_1 R_1 + \mu_2 R_2 && (42) \\ &\leq \min_{j=1,2} \max_{(R_0,R_1,R_2)\in\mathcal{C}_{3j}^{\mathrm{PR-AL}}(\mathbf{S})} R_0 + \mu_1 R_1 + \mu_2 R_2 && (43) \\ &= \min_{j=1,2} \max_{\substack{\mathbf{0}\preceq\mathbf{K}_j,\ j=1,2 \\ \mathbf{K}_1+\mathbf{K}_2\preceq\mathbf{S}}} R_{0j}^{\mathrm{AL}}(\mathbf{K}_1,\mathbf{K}_2) + \mu_1 R_1^{\mathrm{AL}}(\mathbf{K}_1,\mathbf{K}_2) \\ &\qquad + \mu_2 R_2^{\mathrm{AL}}(\mathbf{K}_2) && (44) \end{aligned}$$

where (42) is due to (40), and (44) comes from Theorem 4 in [1]. Thus, we show that the outer bounds in Theorems 4 and 5 are implicitly present in the outer bound provided in [2], [3].

VII. Conclusions

We study the Gaussian MIMO broadcast channel with common and private messages and provide an outer bound for the capacity region. This outer bound states that extending the DPC region in the common message rate direction by a fixed amount yields an outer bound for the capacity region. However, this fixed amount of gap is not finite for all Gaussian MIMO broadcast channels. We derive necessary and sufficient conditions for this gap to be finite.

References

[1] H. Weingarten, Y. Steinberg, and S. Shamai (Shitz). The capacity region of the Gaussian multiple-input multiple-output broadcast channel. *IEEE Trans. Inf. Theory*, 52(9):3936–3964, Sep. 2006.

[2] H. Weingarten, Y. Steinberg, and S. Shamai (Shitz). On the capacity region of the multi-antenna broadcast channel with common messages. In *IEEE ISIT*, Jul. 2006.

[3] H. Weingarten. *Multiple-input multiple-output broadcast systems*. PhD thesis, Technion, Haifa, Israel, 2007.

[4] N. Jindal and A. Goldsmith. Optimal power allocation for parallel broadcast channels with independent and common information. In *IEEE Intl. Symp. Inf. Theory*, page 215, Jun. 2004.

[5] Y. Liang, G. Kramer, and S. Shamai (Shitz). Capacity outer bounds for broadcast channels. In *Information Theory Workshop*, May 2008.

[6] C. Nair. An outer bound for 2-receiver discrete memoryless broadcast channels. Jul. 2008. Unpublished manuscript. Available at [arXiv: 0807.3593].

[7] C. Nair and A. El Gamal. An outer bound to the capacity region of the broadcast channel. *IEEE Trans. Inf. Theory*, 53(1):350–355, Jan. 2007.

[8] K. Marton. A coding theorem for the discrete memoryless channels. *IEEE Trans. Inf. Theory*, 25(1):306–311, May 1979.

[9] H. Weingarten, T. Liu, , S. Shamai (Shitz), Y. Steinberg, and P. Viswanath. The capacity region of the degraded multiple-input multiple-output compound broadcast channel. *IEEE Trans. Inf. Theory*, 55(11):5011–5023, Nov. 2009.