Outer Bounds for User Cooperation

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***Abstract*—We obtain a dependence balance based outer bound on the capacity region of the two-user multiple access channel with generalized feedback (MAC-GF). We investigate a Gaussian MAC with user-cooperation (MAC-UC), where each transmitter receives an additive white Gaussian noise corrupted version of the channel input of the other transmitter. For all non-zero values of cooperation noise variances, our outer bound strictly improves upon the cut-set outer bound. Moreover, as the variances of the cooperation noises become large, our outer bound collapses to the capacity region of the Gaussian MAC without cooperation.**

I. Introduction

The multiple access channel with generalized feedback (MAC-GF) was first introduced by Carleial [1]. The model therein allows for different feedback signals at the two transmitters. For this channel model, Carleial [1] obtained an achievable rate region using block Markov superposition encoding and windowed decoding. An improvement over this achievable rate region was obtained by Willems et. al. in [2] by using block Markov superposition encoding combined with backwards decoding.

As far as the converse is concerned for MAC-GF, a well known outer bound is the cut-set outer bound. The cut-set bound allows all input distributions, thereby permitting arbitrary correlation between the channel inputs and hence is seemingly loose. In this paper, we use the idea of dependence balance [3] to obtain a new outer bound for the capacity region of the MAC-GF.

To illustrate the usefulness of our outer bound, we investigate the Gaussian MAC with user cooperation (MAC-UC). Sendonaris, Erkip and Aazhang [4] studied a model where each transmitter receives a version of the other transmitter's current channel input corrupted with additive white Gaussian noise. They named this model as *user cooperation* model. This model is particularly suitable for a wireless setting since the transmitters can potentially overhear each other. An achievable rate region for the user cooperation model was given in [4] using the result of [2] and was shown to strictly exceed the rate region if the transmitters ignore the overheard signals.

We provide an explicit evaluation for our outer bound for the Gaussian MAC-UC. For this channel model, the cut-set outer bound is sensitive to cooperation noise variances, but not sensitive enough. Intuitively speaking, as the backward noise variances become large, one would expect the cut-set bound to collapse to the capacity region of the MAC without cooperation. Instead, the cut-set bound converges to the capacity region of the Gaussian MAC with noiseless output feedback [5]. On the other hand, in the limit when cooperation noise variances become too large, our outer bound converges to the capacity region of the Gaussian MAC with no cooperation, thereby yielding a capacity result. For all non-zero and finite values of cooperation noise variances, our outer bound strictly improves upon the cut-set outer bound. Our dependence balance based outer bound coincides with the cut-set bound only when the backward noise variances are identically zero, in which case both outer bounds collapse to the total cooperation line.

This work was supported by NSF Grants CCF 04-47613, CCF 05-14846, CNS 07-16311 and CCF 07-29127.

II. MAC with Generalized Feedback

A discrete memoryless two-user multiple access channel with generalized feedback (MAC-GF) (see Figure 1) is defined by: two input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, an output alphabet for the receiver $\mathcal{Y}$, feedback output alphabets $\mathcal{Y}_{F_1}$ and $\mathcal{Y}_{F_2}$ at transmitters 1 and 2, respectively, and a probability transition function $p(y, y_{F_1}, y_{F_2}|x_1, x_2)$, defined for all triples $(y, y_{F_1}, y_{F_2}) \in \mathcal{Y} \times \mathcal{Y}_{F_1} \times \mathcal{Y}_{F_2}$, for every pair $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$.

A (n, M_1, M_2, P_e) code for the MAC-GF consists of two sets of encoding functions $f_{1i} : \mathcal{M}_1 \times \mathcal{Y}_{F_1}^{i-1} \rightarrow \mathcal{X}_1$, $f_{2i} : \mathcal{M}_2 \times \mathcal{Y}_{F_2}^{i-1} \rightarrow \mathcal{X}_2$ for $i = 1, \ldots, n$ and a decoding function $g : \mathcal{Y}^n \rightarrow \mathcal{M}_1 \times \mathcal{M}_2$. The two transmitters produce independent and uniformly distributed messages $W_1 \in \{1, \ldots, M_1\}$ and $W_2 \in \{1, \ldots, M_2\}$, respectively, and transmit them through n channel uses. The average error probability is defined as, $P_e = \Pr[(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)]$. A rate pair (R_1, R_2) is said to be achievable for MAC-GF if for any $\epsilon \geq 0$, there exists a pair of n encoding functions $\{f_{1i}\}_{i=1}^n$, $\{f_{2i}\}_{i=1}^n$, and a decoding function g such that $R_1 \leq \log(M_1)/n$, $R_2 \leq \log(M_2)/n$ and $P_e \leq \epsilon$ for sufficiently large n. The capacity region of MAC-GF is the closure of the set of all achievable rate pairs (R_1, R_2).

III. Cut-set Outer Bound

A general outer bound on the capacity region of a multi-terminal network is the cut-set outer bound [6]. The cut-set outer bound for MAC-GF is given by

$$\mathcal{CS} = \big\{(R_1, R_2) : \; R_1 \leq I(X_1; Y, Y_{F_2}|X_2) \tag{1}$$

$$R_2 \leq I(X_2; Y, Y_{F_1}|X_1) \tag{2}$$

$$R_1 + R_2 \leq I(X_1, X_2; Y)\big\} \tag{3}$$

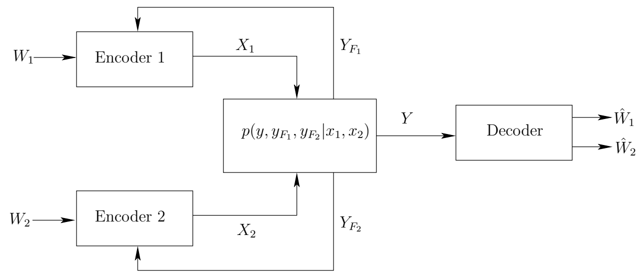

Fig. 1. The multiple access channel with generalized feedback (MAC-GF).

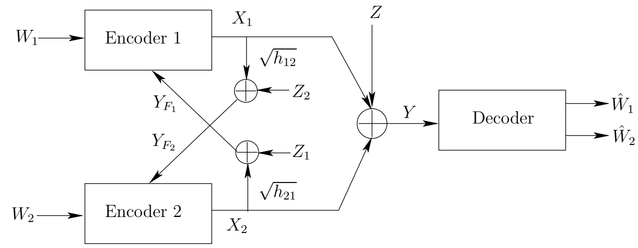

Fig. 2. The Gaussian MAC with user cooperation (MAC-UC).

where the random variables X_1, X_2 and (Y, Y_{F_1}, Y_{F_2}) have the joint distribution

$$p(x_1, x_2, y, y_{F_1}, y_{F_2}) = p(x_1, x_2)p(y, y_{F_1}, y_{F_2}|x_1, x_2) \quad (4)$$

The cut-set bound is seemingly loose since it allows arbitrary correlation among channel inputs by permitting arbitrary input distributions $p(x_1, x_2)$. On the other hand, our dependence balance based outer bound only permits those input distributions which satisfy a non-trivial dependence balance constraint.

IV. A New Outer Bound for MAC-GF

Theorem 1: The capacity region of MAC-GF is contained in the region

$$\begin{aligned} \mathcal{DB} = \big\{(R_1, R_2) :\ & R_1 \leq I(X_1; Y, Y_{F_2}|X_2, T_2) && (5)\\ & R_2 \leq I(X_2; Y, Y_{F_1}|X_1, T_1) && (6)\\ & R_1 + R_2 \leq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2}|T_1, T_2) && (7)\\ & R_1 + R_2 \leq I(X_1, X_2; Y)\big\} && (8) \end{aligned}$$

where the random variables $(T_1, T_2, X_1, X_2, Y, Y_{F_1}, Y_{F_2})$ have the joint distribution

$$p(t_1, t_2, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t_1, t_2, x_1, x_2). \\ p(y, y_{F_1}, y_{F_2}|x_1, x_2) \quad (9)$$

and also satisfy the following dependence balance bound

$$I(X_1; X_2|T_1, T_2) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T_1, T_2) \quad (10)$$

The proof of Theorem 1 can be found in [7].

V. Gaussian MAC with User Cooperation

In this section, we consider the Gaussian MAC with user cooperation [4], where each transmitter receives a noisy version of the other transmitter's channel input. The user cooperation model (see Figure 2) is a special instance of a MAC-GF, where the channel outputs are described as,

$$Y = \sqrt{h_{10}}X_1 + \sqrt{h_{20}}X_2 + Z \quad (11)$$

$$Y_{F_1} = \sqrt{h_{21}}X_2 + Z_1 \quad (12)$$

$$Y_{F_2} = \sqrt{h_{12}}X_1 + Z_2 \quad (13)$$

where Z,Z_1 and Z_2 are independent, zero-mean, Gaussian random variables with variances σ_Z^2, $\sigma_{Z_1}^2$ and $\sigma_{Z_2}^2$, respectively. The channel gains h_{10}, h_{20}, h_{12} and h_{21} are assumed to be fixed and known at all terminals. Moreover, the channel inputs are subject to average power constraints, $E[X_1^2] \leq P_1$ and $E[X_2^2] \leq P_2$. Note that the channel model described above has a special probability structure, namely,

$$p(y, y_{F_1}, y_{F_2}|x_1, x_2) = p(y|x_1, x_2)p(y_{F_1}|x_2)p(y_{F_2}|x_1) \quad (14)$$

For any MAC-GF with a transition probability as in (14), we have the following strengthened version of Theorem 1.

Theorem 2: The capacity region of any MAC-GF with a transition probability in the form of (14), is contained in the region

$$\begin{aligned} \mathcal{DB}_{UC} = \big\{(R_1, R_2) :\ & R_1 \leq I(X_1; Y, Y_{F_2}|X_2, T) && (15)\\ & R_2 \leq I(X_2; Y, Y_{F_1}|X_1, T) && (16)\\ & R_1 + R_2 \leq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2}|T) && (17)\\ & R_1 + R_2 \leq I(X_1, X_2; Y)\big\} && (18) \end{aligned}$$

where the random variables $(T, X_1, X_2, Y, Y_{F_1}, Y_{F_2})$ have the joint distribution

$$p(t, x_1, x_2, y, y_{F_1}, y_{F_2}) = p(t, x_1, x_2)p(y|x_1, x_2). \\ p(y_{F_1}|x_2)p(y_{F_2}|x_1) \quad (19)$$

and also satisfy the following dependence balance bound

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T) \quad (20)$$

where the random variable T is subject to a cardinality constraint $|\mathcal{T}| \leq |\mathcal{X}_1||\mathcal{X}_2| + 3$.

The proof of Theorem 2 can be found in [7]. The main idea behind the proof of Theorem 2 is to use (14) so that the resulting outer bound is expressed in terms of one auxiliary random variable T as opposed to two auxiliary random variables (T_1, T_2) appearing in Theorem 1. It can be shown using standard methods [8] that this outer bound also holds for continuous alphabets with average input power constraints.

In the next section, we will show that it suffices to consider jointly Gaussian (T, X_1, X_2) satisfying (20) when evaluating Theorem 2 for the Gaussian MAC with user cooperation described in (11)-(13). We should also remark here that dependence balance approach was first applied by Gastpar and Kramer for the Gaussian MAC with noisy feedback in [9] and the Gaussian interference channel with noisy feedback in [10].

VI. Outline for Evaluating $\mathcal{DB}_{UC}$

The main difficulty in evaluating our outer bound, $\mathcal{DB}_{UC}$ for the Gaussian MAC-UC is to identify the optimal selection of joint densities of (T, X_1, X_2). Our aim will be to prove that it is sufficient to consider jointly Gaussian (T, X_1, X_2) satisfying (20) while evaluating the outer bound.

We begin by considering the set of all distributions of three random variables (T, X_1, X_2) which satisfy the power constraints, $E\big[X_1^2\big] \leq P_1$ and $E\big[X_2^2\big] \leq P_2$. Let us formally define this set of input distributions as

$$\mathcal{P} = \{p(t, x_1, x_2) : E\big[X_1^2\big] \leq P_1, E\big[X_2^2\big] \leq P_2\}$$

For simplicity, we abbreviate jointly Gaussian distributions as $\mathcal{JG}$ and distributions which are not jointly Gaussian as $\mathcal{NG}$. We first partition $\mathcal{P}$ into two disjoint subsets,

$$\mathcal{P}_G = \{p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } \mathcal{JG}\}$$
$$\mathcal{P}_{NG} = \{p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } \mathcal{NG}\}$$

We further individually partition the sets $\mathcal{P}_G$ and $\mathcal{P}_{NG}$, respectively, as

$$\mathcal{P}_G^{DB} = \{p(t, x_1, x_2) \in \mathcal{P}_G : (T, X_1, X_2) \text{ satisfy (20)}\}$$
$$\mathcal{P}_G^{\overline{DB}} = \{p(t, x_1, x_2) \in \mathcal{P}_G : (T, X_1, X_2) \text{ do not satisfy (20)}\}$$
$$\mathcal{P}_{NG}^{DB} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG} : (T, X_1, X_2) \text{ satisfy (20)}\}$$
$$\mathcal{P}_{NG}^{\overline{DB}} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG} : (T, X_1, X_2) \text{ do not satisfy (20)}\}$$

Finally, we partition the set $\mathcal{P}_{NG}^{DB}$ into two disjoint sets $\mathcal{P}_{NG}^{DB(a)}$ and $\mathcal{P}_{NG}^{DB(b)}$ with $\mathcal{P}_{NG}^{DB} = \mathcal{P}_{NG}^{DB(a)} \bigcup \mathcal{P}_{NG}^{DB(b)}$, as

$$\mathcal{P}_{NG}^{DB(a)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{ covariance matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there exists a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with covariance matrix } Q \text{ satisfying (20)}\}$$

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{ covariance matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } \mathcal{JG}\ (T_G, X_{1G}, X_{2G}) \text{ with covariance matrix } Q \text{ satisfying (20)}\}$$

So far, we have partitioned the set of input distributions into five disjoint sets: $\mathcal{P}_G^{DB}$, $\mathcal{P}_G^{\overline{DB}}$, $\mathcal{P}_{NG}^{DB(a)}$, $\mathcal{P}_{NG}^{DB(b)}$ and $\mathcal{P}_{NG}^{\overline{DB}}$ (see Figure 3). It is clear that the input distributions which fall into the sets $\mathcal{P}_G^{\overline{DB}}$ and $\mathcal{P}_{NG}^{\overline{DB}}$ need not be considered since they do not satisfy the constraint (20). Therefore, we only need to restrict our attention on the three remaining sets $\mathcal{P}_G^{DB}$, $\mathcal{P}_{NG}^{DB(a)}$, and $\mathcal{P}_{NG}^{DB(b)}$, i.e., those input distributions which satisfy the dependence balance bound.

We explicitly evaluate our outer bound in the following three steps:

1. We first explicitly characterize the region of rate pairs provided by our outer bound for the probability distributions in the set $\mathcal{P}_G^{DB}$.

2. In the second step, we will show that for any input distribution belonging to the set $\mathcal{P}_{NG}^{DB(a)}$, there exists an input distribution in the set $\mathcal{P}_G^{DB}$ which yields a set of larger rate pairs. Therefore, we do not need to consider the input distributions in the set $\mathcal{P}_{NG}^{DB(a)}$ in evaluating our outer bound.

3. We next focus on the set $\mathcal{P}_{NG}^{DB(b)}$ and show that for any non-Gaussian input distribution $p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB(b)}$, we can construct a jointly Gaussian input distribution satisfying (20),

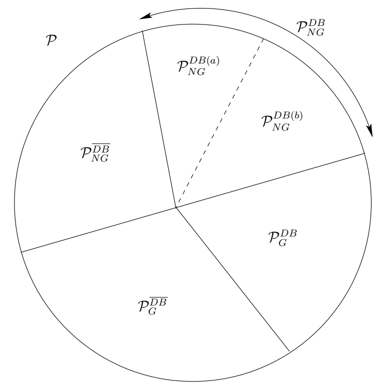

Fig. 3. A partition of the set of input distributions $\mathcal{P}$.

i.e., we can find a corresponding input distribution in $\mathcal{P}_G^{DB}$, which yields a set of rates which includes the set of rates of the fixed non-Gaussian input distribution $p(t, x_1, x_2)$. Therefore, we do not need to consider the input distributions in the set $\mathcal{P}_{NG}^{DB(b)}$ either in evaluating our outer bound.

To set the stage for our evaluation in steps $1-3$, let us define $\mathcal{Q}$ as the set of all valid 3×3 covariance matrices of three random variables (T, X_1, X_2). A typical element Q in the set $\mathcal{Q}$ takes the following form,

$$Q = E\big[(X_1\ X_2\ T)(X_1\ X_2\ T)^T\big] = \begin{pmatrix} P_1 & \rho_{12}\sqrt{P_1P_2} & \rho_{1T}\sqrt{P_1P_T} \\ \rho_{12}\sqrt{P_1P_2} & P_2 & \rho_{2T}\sqrt{P_2P_T} \\ \rho_{1T}\sqrt{P_1P_T} & \rho_{2T}\sqrt{P_2P_T} & P_T \end{pmatrix} \quad (21)$$

A necessary condition for Q to be a valid covariance matrix is that it is positive semi-definite, i.e., $\det(Q) \geq 0$. This is equivalent to saying that,

$$\det(Q) = P_1P_2P_T\Delta \geq 0 \quad (22)$$

where we have defined for simplicity,

$$\Delta = 1 - \rho_{12}^2 - \rho_{1T}^2 - \rho_{2T}^2 + 2\rho_{1T}\rho_{2T}\rho_{12} \quad (23)$$

VII. Evaluation of $\mathcal{DB}_{UC}$

We start with step 1 and characterize the set of jointly Gaussian triples (T_G, X_{1G}, X_{2G}) in $\mathcal{P}_G^{DB}$. For this purpose, we first rewrite (20) as follows,

$$h(Y_{F_1}, Y_{F_2}|T) + h(Y_{F_1}, Y_{F_2}|X_1, X_2, T) \leq h(Y_{F_1}, Y_{F_2}|X_1, T) + h(Y_{F_1}, Y_{F_2}|X_2, T) \quad (24)$$

Making use of the following equalities,

$$h(Y_{F_1}, Y_{F_2}|X_1, X_2, T) = \frac{1}{2}\log\left((2\pi e)^2\sigma_{Z_1}^2\sigma_{Z_2}^2\right) \quad (25)$$

$$h(Y_{F_1}, Y_{F_2}|X_1, T) = \frac{1}{2}\log\left((2\pi e)\sigma_{Z_2}^2\right) + h(Y_{F_1}|X_1, T) \quad (26)$$

$$h(Y_{F_1}, Y_{F_2}|X_2, T) = \frac{1}{2}\log\left((2\pi e)\sigma_{Z_1}^2\right) + h(Y_{F_2}|X_2, T) \quad (27)$$

we obtain a simplified expression for (24) as,

$$h(Y_{F_1}, Y_{F_2}|T) \leq h(Y_{F_1}|X_1, T) + h(Y_{F_2}|X_2, T) \quad (28)$$

By using the Markov chain $Y_{F_1} \to X_2 \to (T, Y_{F_2})$, we first observe that the dependence balance constraint in (28) is equivalent to the following two equalities,

$$I(Y_{F_1}; X_1|T) = 0 \tag{29}$$

$$I(Y_{F_2}; X_2|Y_{F_1}, T) = 0 \tag{30}$$

Next, we show that if any jointly Gaussian triple (T, X_1, X_2) satisfies the constraints (29)-(30) then it satisfies the Markov chain $X_1 \to T \to X_2$. Conversely, we will show that if any jointly Gaussian triple (T, X_1, X_2) satisfies $X_1 \to T \to X_2$, then it satisfies (29)-(30). It is straightforward to check that for a jointly Gaussian (T_G, X_{1G}, X_{2G}), the constraints (29)-(30) are equivalent to $\text{Cov}(X_{1G}, X_{2G}|T_G) = 0$ which is in turn equivalent to

$$\rho_{12} = \rho_{1T}\rho_{2T} \tag{31}$$

This implies that a jointly Gaussian triple satisfies (29)-(30) iff $\rho_{12} = \rho_{1T}\rho_{2T}$.

On the other hand, consider any jointly Gaussian triple (T_G, X_{1G}, X_{2G}), with a covariance matrix Q which satisfies the Markov chain $X_{1G} \to T_G \to X_{2G}$. This is equivalent to $I(X_{1G}; X_{2G}|T_G) = 0$, which is equivalent to

$$\rho_{12} = \rho_{1T}\rho_{2T} \tag{32}$$

This implies that if a jointly Gaussian triple (T, X_1, X_2) satisfies the Markov chain $X_1 \to T \to X_2$, then it satisfies (32) and therefore it also satisfies (29)-(30) and vice versa. As a consequence, we have explicitly characterized the set $\mathcal{P}_G^{DB}$, i.e., it comprises of only such jointly Gaussian distributions, (T_G, X_{1G}, X_{2G}), for which $X_{1G} \to T_G \to X_{2G}$.

We can now write the set of rate pairs provided by our outer bound for a jointly Gaussian triple (T_G, X_{1G}, X_{2G}) in the set $\mathcal{P}_G^{DB}$ as

$$R_1 \leq I(X_{1G}; Y, Y_{F_2}|X_{2G}, T_G) \tag{33}$$

$$R_2 \leq I(X_{2G}; Y, Y_{F_1}|X_{1G}, T_G) \tag{34}$$

$$R_1 + R_2 \leq I(X_{1G}, X_{2G}; Y, Y_{F_1}, Y_{F_2}|T_G) \tag{35}$$

$$R_1 + R_2 \leq I(X_{1G}, X_{2G}; Y) \tag{36}$$

where (T_G, X_{1G}, X_{2G}) satisfies the Markov chain $X_{1G} \to T_G \to X_{2G}$.

In step 2, we consider any non-Gaussian input distribution $p(t, x_1, x_2)$ in $\mathcal{P}_{NG}^{DB(a)}$ with a covariance matrix Q. For such an input distribution, we know by the maximum entropy theorem [6], that the rates provided by a jointly Gaussian triple with the same covariance matrix Q are always at least as large as the rates provided by the chosen non-Gaussian distribution. Therefore, for any input distribution in $\mathcal{P}_{NG}^{DB(a)}$, there always exists an input distribution in $\mathcal{P}_G^{DB}$, satisfying (20), which yields larger rates. This means that we can ignore the set $\mathcal{P}_{NG}^{DB(a)}$ altogether while evaluating our outer bounds.

We now arrive at step 3 of the evaluation of our outer bound where we will show that for any non-Gaussian input distribution $p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB(b)}$, we can always find an input distribution in $\mathcal{P}_G^{DB}$, with a set of rate pairs which include the set of rate pairs of the fixed non-Gaussian input distribution $p(t, x_1, x_2)$. Consider any triple (T, X_1, X_2) with a non-Gaussian input distribution $p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB(b)}$, with a valid covariance matrix Q. By the definition of the set $\mathcal{P}_{NG}^{DB(b)}$, and as a consequence of (31), this covariance matrix has the property that $\rho_{12} \neq \rho_{1T}\rho_{2T}$. Moreover, this non-Gaussian distribution satisfies the dependence balance bound, i.e., it satisfies (29) and (30). For our purpose, we only need (29). Since $I(Y_{F_1}; X_1|T) = 0$, we make use of (11)-(13) to first arrive at

$$E[X_1X_2|T] = E[X_2|T]E[X_1|T] \tag{37}$$

We will now construct another triple (T', X_1, X_2) with a covariance matrix S by selecting

$$T' = E[X_1|T] \tag{38}$$

This particular selection is closely related to the recent work of Bross, Lapidoth and Wigger [11] where it was shown that jointly Gaussian distributions are sufficient to characterize the capacity region of Gaussian MAC with conferencing encoders. Although, we should also remark that when evaluating $\mathcal{DB}_{UC}$, we do not have a conditionally independent structure among (T, X_1, X_2) to start with. This structure arises from the dependence balance constraint (20), permitting us to use this approach.

Returning to (38), we note that T' is a deterministic function of T and therefore, following is a valid Markov chain.

$$T' \to T \to (X_1, X_2) \to (Y, Y_{F_1}, Y_{F_2}) \tag{39}$$

We will now obtain the off diagonal elements of the covariance matrix S of the triple (T', X_1, X_2) by first noting the following equalities,

$$E[X_1T'] = \text{Var}(T') \tag{40}$$

$$E[X_2T'] = E_T[E[X_2|T]E[X_1|T]] \tag{41}$$

$$E[X_1X_2] = E_T[E[X_1|T]E[X_2|T]] \tag{42}$$

where (42) follows from (37). Using (40)-(42), we obtain that the covariance matrix S of the triple (T', X_1, X_2) satisfies

$$\rho_{12} = \rho_{1T'}\rho_{2T'} \tag{43}$$

Therefore, from (31) any jointly Gaussian (T'_G, X_{1G}, X_{2G}) triple with a covariance matrix S, with entries $(\rho_{12}, \rho_{1T'}, \rho_{2T'})$ satisfies (20).

We now arrive at the final step of the evaluation. In particular, we will show that the rates of this jointly Gaussian triple (T'_G, X_{1G}, X_{2G}) will include the rates of the given non-Gaussian triple (T, X_1, X_2). For the triple (T'_G, X_{1G}, X_{2G}), we have the following set of inequalities,

$$I(X_{1G}; Y, Y_{F_2}|X_{2G}, T'_G)$$

$$= h(Y, Y_{F_2}|X_{2G}, T'_G) - h(Y, Y_{F_2}|X_{1G}, X_{2G}, T'_G) \tag{44}$$

$$= h(\sqrt{h_{10}}X_{1G} + Z, \sqrt{h_{12}}X_{1G} + Z_2|X_{2G}, T'_G) - h(Y, Y_{F_2}|X_{1G}, X_{2G}, T'_G) \tag{45}$$

$$\geq h(\sqrt{h_{10}}X_1 + Z, \sqrt{h_{12}}X_1 + Z_2|X_2, T^{'}) - h(Y, Y_{F_2}|X_{1G}, X_{2G}, T^{'}_G) \quad (46)$$
$$\geq h(\sqrt{h_{10}}X_1 + Z, \sqrt{h_{12}}X_1 + Z_2|X_2, T^{'}, T) - h(Y, Y_{F_2}|X_1, X_2, T) \quad (47)$$
$$= h(\sqrt{h_{10}}X_1 + Z, \sqrt{h_{12}}X_1 + Z_2|X_2, T) - h(Y, Y_{F_2}|X_1, X_2, T) \quad (48)$$
$$= I(X_1; Y, Y_{F_2}|X_2, T) \quad (49)$$

where (46) follows from the fact that $(T^{'}, X_1, X_2)$ and $(T^{'}_G, X_{1G}, X_{2G})$ have the same covariance matrix S and by using the maximum entropy theorem [6]. Next, (47) follows from the fact that conditioning reduces differential entropy and finally (48) follows from the fact that $T^{'}$ is a deterministic function of T and by invoking the Markov chain in (39). Similarly, it can be shown that

$$I(X_{1G}, X_{2G}; Y) \geq I(X_1, X_2; Y) \quad (50)$$
$$I(X_{2G}; Y, Y_{F_1}|X_{1G}, T^{'}_G) \geq I(X_2; Y, Y_{F_1}|X_1, T) \quad (51)$$
$$I(X_{1G}, X_{2G}; Y, Y_{F_1}, Y_{F_2}|T^{'}_G) \geq I(X_1, X_2; Y, Y_{F_1}, Y_{F_2}|T) \quad (52)$$

Therefore, we conclude that for any non-Gaussian distribution $p(t, x_1, x_2) \in \mathcal{P}^{DB(b)}_{NG}$, there exists a jointly Gaussian distribution $p(t, x_1, x_2) \in \mathcal{P}^{DB}_G$ which satisfies the dependence balance bound (20) and yields a set of rates which include the set of rates given by the fixed non-Gaussian distribution. Hence, it suffices to consider jointly Gaussian distributions in $\mathcal{P}^{DB}_G$ to evaluate our outer bound.

The explicit expressions for our outer bound and the cut-set bound for the Gaussian MAC-UC are rather tedious and we do not present them here. They can be found in [7]. Here, we plot and compare these outer bounds. Figure 4 illustrates the outer bounds and achievable rate region [4] for the case when $P_1 = P_2 = 5, \sigma^2_Z = 2$ and $\sigma^2_{Z_1} = \sigma^2_{Z_2} = 1$ and $h_{10} = h_{20} = h_{12} = h_{21} = 1$. Figure 5 illustrates these bounds and an achievable rate region [4] for one sided cooperation where $P_1 = P_2 = \sigma^2_Z = 1$ and $\sigma^2_{Z_1} = \sigma^2_{Z_2} = 1$ and $h_{10} = h_{20} = 1$, $h_{12} = 2, h_{21} = 0$.

VIII. Conclusions

We obtained a new outer bound for the capacity region of the two-user MAC with generalized feedback. We explicitly evaluated this outer bound for the Gaussian MAC with user cooperation. Our outer bound strictly improves upon the cut-set bound for all non-zero values of cooperation noise variances. Moreover, as the cooperation noise variances become large, our outer bound collapses to the capacity region of the Gaussian MAC without cooperation.

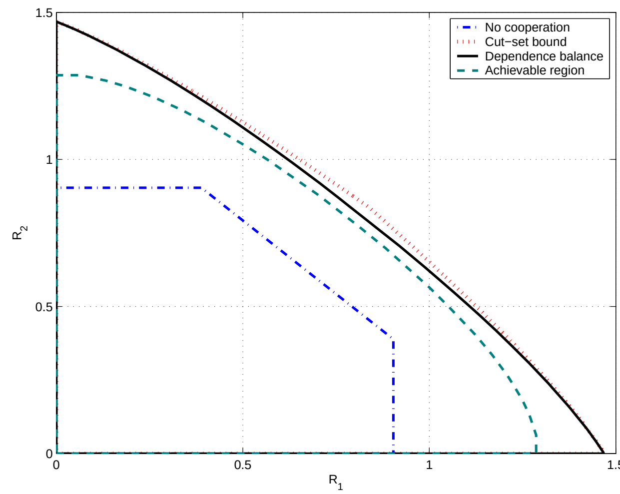

Fig. 4. Illustration of bounds for $P_1 = P_2 = 5, \sigma^2_Z = 2$, $\sigma^2_{Z_1} = \sigma^2_{Z_2} = 1$ and $h_{10} = h_{20} = h_{12} = h_{21} = 1$.

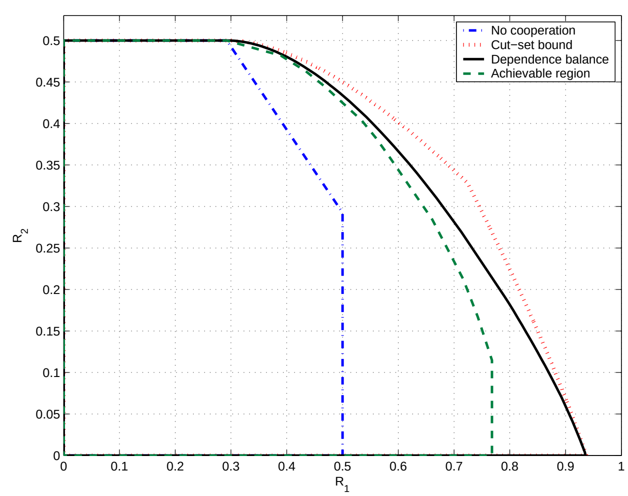

Fig. 5. Illustration of bounds for $P_1 = P_2 = \sigma^2_Z = 1$, $\sigma^2_{Z_1} = \sigma^2_{Z_2} = 1$ and $h_{10} = h_{20} = 1$, $h_{12} = 2, h_{21} = 0$.

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