

On the Capacity Region of the Gaussian Multiple Access Channel with Noisy Feedback

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Abstract—We provide a new outer bound on the capacity region of the two-user Gaussian multiple access channel (MAC) with AWGN-corrupted feedback. Our outer bound is based on the idea of dependence balance due to Hekstra and Willems [1]. Evaluating our outer bound is non-trivial as it involves taking a union over joint densities of three random variables, one of which is an auxiliary random variable. We resolve this difficulty by proving that it is sufficient to consider jointly Gaussian random variables when evaluating our outer bound. As the feedback noise variances become large, our outer bound collapses to the capacity region of the Gaussian MAC without feedback, thereby yielding the first non-trivial result for a Gaussian MAC with noisy feedback. Furthermore, as the feedback noise variances tend to zero, our outer bound collapses to the capacity region of the Gaussian MAC with noiseless feedback, which was established by Ozarow [2]. For all non-zero, finite values of the feedback noise variances, our outer bound strictly improves upon the cut-set outer bound.

I. INTRODUCTION

The capacity region of the two-user Gaussian multiple access channel (MAC) with noiseless feedback was characterized by Ozarow in [2]. An achievable rate region based on the classical work of Schalkwijk and Kailath [3] was shown to be the same as the cut-set outer bound, thereby yielding the feedback capacity region. Until now, the Gaussian MAC with noiseless feedback remains the only multiple access channel, for which, the feedback capacity region has been explicitly characterized in a single-letter form.

In this paper, we study the Gaussian MAC with noisy feedback, where transmitter j , $j = 1, 2$, receives a feedback, Y_{F_j} , which is the sum of the channel output Y and independent Gaussian noise Z_j , i.e., $Y_{F_j} = Y + Z_j$. This channel is a special case of the MAC with generalized feedback, which has been previously studied by Carleial [4], Willems et al. [5], where achievable rate regions have been proposed. As far as outer bounds on the capacity region are concerned, the only outer bound for Gaussian MAC with noisy feedback is the cut-set outer bound. Unfortunately, the cut-set bound is insensitive to the feedback noises Z_1 and Z_2 . Specifically, as the variances of Z_1 and Z_2 , i.e., $\sigma_{Z_1}^2$ and $\sigma_{Z_2}^2$ become large, one would expect the feedback to be useless. In other words, one would expect the capacity region of the Gaussian MAC with noisy feedback to collapse to the capacity region of the Gaussian MAC without feedback. This fact is not accounted for by the cut-set outer bound.

In this paper, we propose and explicitly evaluate an outer bound on the capacity region of the Gaussian MAC with noisy feedback. This outer bound is based on the idea of dependence balance [1], which was used to obtain outer bounds for the single output two-way channel. We should mention here that applying the idea of dependence balance to obtain improved outer bounds for Gaussian MAC with noisy feedback was proposed by Gastpar and Kramer in [6]. The outer bound in this paper, which involves an auxiliary random variable can also be obtained by exactly the same arguments as in [6]. Explicit evaluation of the outer bound is non-trivial as it involves taking a union over joint densities of three random variables, one of which is an auxiliary random variable. Our contribution is to explicitly evaluate this outer bound by proving that it suffices to consider jointly Gaussian random variables when evaluating our outer bound.

From our explicit evaluation, we also demonstrate that as the feedback noise variances become large, i.e., $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \rightarrow \infty$, our outer bound collapses to the capacity region of the Gaussian MAC with no-feedback, thereby yielding a capacity result. Furthermore, as $\sigma_{Z_1}^2, \sigma_{Z_2}^2 \rightarrow 0$, our outer bound collapses to the capacity region of the Gaussian MAC with noiseless feedback [2]. Moreover, for all non-zero values of $\sigma_{Z_1}^2$ and $\sigma_{Z_2}^2$, our outer bound strictly improves upon the cut-set outer bound, thereby providing a continuum of bounds as a function of the feedback noise variances, $(\sigma_{Z_1}^2, \sigma_{Z_2}^2)$.

II. SYSTEM MODEL

We consider the two-user Gaussian MAC with AWGN-corrupted feedback (see Figure 1) which is defined by the following:

$$Y = X_1 + X_2 + Z \quad (1)$$

$$Y_{F_1} = Y + Z_1 \quad (2)$$

$$Y_{F_2} = Y + Z_2 \quad (3)$$

where Y_{F_j} is the noisy feedback available to the j^{th} transmitter, where $j = 1, 2$. A (n, M_1, M_2, P_e) code for this channel consists of two sets of encoding functions f_{1i}, f_{2i} for $i = 1, \dots, n$ and a decoding function g

$$f_{1i} : \mathcal{M}_1 \times \mathbb{R}^{i-1} \rightarrow \mathbb{R}, \quad i = 1, \dots, n$$

$$f_{2i} : \mathcal{M}_2 \times \mathbb{R}^{i-1} \rightarrow \mathbb{R}, \quad i = 1, \dots, n$$

$$g : \mathbb{R}^n \rightarrow \mathcal{M}_1 \times \mathcal{M}_2$$

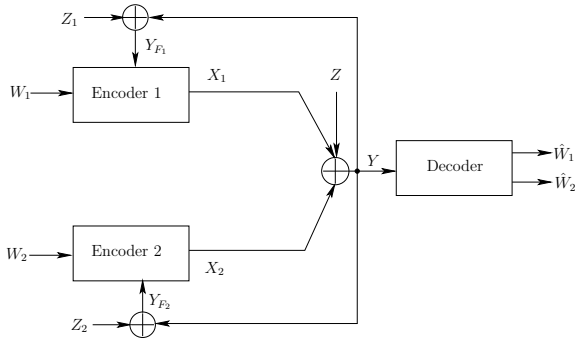


Fig. 1. The Gaussian MAC with noisy feedback.

The two transmitters produce independent and uniformly distributed messages $W_1 \in \{1, \dots, M_1\}$ and $W_2 \in \{1, \dots, M_2\}$, respectively, and transmit them through n channel uses. Furthermore, the transmitter codewords are assumed to satisfy average power constraints,

$$\frac{1}{n} \sum_{i=1}^n f_{1i}^2(W_1, Y_{F_1}^{i-1}) \leq P_1 \quad (4)$$

$$\frac{1}{n} \sum_{i=1}^n f_{2i}^2(W_2, Y_{F_2}^{i-1}) \leq P_2 \quad (5)$$

The average error probability is defined as $P_e = Pr[g(Y^n) \neq (W_1, W_2)]$. A rate pair (R_1, R_2) is said to be achievable for the Gaussian MAC with noisy feedback if for any $\epsilon \geq 0$, there exists a pair of n encoding functions $\{f_{1i}\}_{i=1}^n$, $\{f_{2i}\}_{i=1}^n$, and a decoding function g such that $R_1 \leq \log(M_1)/n$, $R_2 \leq \log(M_2)/n$ and $P_e \leq \epsilon$ for sufficiently large n . The capacity region of the Gaussian MAC with noisy feedback is the closure of the set of all achievable rate pairs (R_1, R_2) .

III. OUTER BOUND BASED ON DEPENDENCE BALANCE

The following outer bound for the Gaussian MAC with degraded feedback was proposed by Gastpar and Kramer [6]. This outer bound is primarily based on the idea of dependence balance. The outer bound consists of the set of non-negative rates (R_1, R_2) satisfying

$$R_1 \leq I(X_1; Y|X_2, T) \quad (6)$$

$$R_2 \leq I(X_2; Y|X_1, T) \quad (7)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y|T) \quad (8)$$

where the involved random variables satisfy the Markov chain

$$T \rightarrow (X_1, X_2) \rightarrow Y \rightarrow (Y_{F_1}, Y_{F_2}) \quad (9)$$

and only such joint distributions are permitted which satisfy the following dependence balance constraint

$$I(X_1; X_2|T) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T) \quad (10)$$

In what follows, we explicitly evaluate the above outer bound and show that it is sufficient to consider jointly Gaussian (T, X_1, X_2) which satisfy the dependence balance bound in (10).

IV. EVALUATION OF THE OUTER BOUND

We begin by considering the set of all distributions of three random variables (T, X_1, X_2) which satisfy the power

constraints, $E[X_1^2] \leq P_1$ and $E[X_2^2] \leq P_2$. Let us formally define this set of input distributions as follows,

$$\mathcal{P} = \{p(t, x_1, x_2) : E[X_1^2] \leq P_1, E[X_2^2] \leq P_2\}$$

For simplicity, we abbreviate jointly Gaussian distributions as \mathcal{JG} and distributions which are not jointly Gaussian as \mathcal{NG} .

We first partition this set into two disjoint subsets.

$$\mathcal{P}_G = \{p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } \mathcal{JG}\}$$

$$\mathcal{P}_{NG} = \{p(t, x_1, x_2) \in \mathcal{P} : (T, X_1, X_2) \text{ are } \mathcal{NG}\}$$

We further individually partition the two aforementioned sets \mathcal{P}_G and \mathcal{P}_{NG} , respectively, as follows,

$$\mathcal{P}_G^{DB} = \{p(t, x_1, x_2) \in \mathcal{P}_G : (T, X_1, X_2) \text{ satisfy (10)}\}$$

$$\mathcal{P}_G^{\overline{DB}} = \{p(t, x_1, x_2) \in \mathcal{P}_G : (T, X_1, X_2) \text{ do not satisfy (10)}\}$$

and

$$\mathcal{P}_{NG}^{DB} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG} : (T, X_1, X_2) \text{ satisfy (10)}\}$$

$$\mathcal{P}_{NG}^{\overline{DB}} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG} : (T, X_1, X_2) \text{ do not satisfy (10)}\}$$

Finally, we partition the set \mathcal{P}_{NG}^{DB} into two disjoint sets as follows with $\mathcal{P}_{NG}^{DB} = \mathcal{P}_{NG}^{DB(a)} \cup \mathcal{P}_{NG}^{DB(b)}$,

$$\mathcal{P}_{NG}^{DB(a)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{covariance matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there exists a } \mathcal{JG} (T_G, X_{1G}, X_{2G}) \text{ with covariance matrix } Q \text{ satisfying (10)}\}$$

$$\mathcal{P}_{NG}^{DB(b)} = \{p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB} : \text{covariance matrix of } p(t, x_1, x_2) \text{ is } Q \text{ and there does not exist a } \mathcal{JG} (T_G, X_{1G}, X_{2G}) \text{ with covariance matrix } Q \text{ satisfying (10)}\}$$

So far, we have partitioned the set of input distributions into 5 disjoint sets: \mathcal{P}_G^{DB} , $\mathcal{P}_G^{\overline{DB}}$, $\mathcal{P}_{NG}^{DB(a)}$, $\mathcal{P}_{NG}^{DB(b)}$ and $\mathcal{P}_{NG}^{\overline{DB}}$. To visualize this partition of the set of input distributions, see Figure 2. It is clear that the input distributions which fall into the sets $\mathcal{P}_G^{\overline{DB}}$ and $\mathcal{P}_{NG}^{\overline{DB}}$ need not be considered since they do not satisfy the constraint (10) and do not have any consequence when evaluating the outer bound. Therefore, we only need to restrict our attention on the three remaining sets \mathcal{P}_G^{DB} , $\mathcal{P}_{NG}^{DB(a)}$, and $\mathcal{P}_{NG}^{DB(b)}$ i.e., those input distributions which satisfy the dependence balance bound. Before proceeding, we first briefly outline the method to explicitly evaluate our outer bound in the following three steps:

1. In the first step, we will explicitly characterize the set of rate pairs for any input distribution belonging to the set \mathcal{P}_G^{DB} . Subsequently, we will show that for any input distribution belonging to the set $\mathcal{P}_{NG}^{DB(a)}$, there exists an input distribution in the set \mathcal{P}_G^{DB} which yields a set of larger rate pairs. This would lead to the conclusion that we only need to carefully deal with the set $\mathcal{P}_{NG}^{DB(b)}$.

2. Following this step, we will use the fact that any input distribution belonging to the set $\mathcal{P}_{NG}^{DB(b)}$ satisfies the constraint (10) along with a multivariate version of the entropy power inequality [7], [8] to obtain an improved upper bound on the

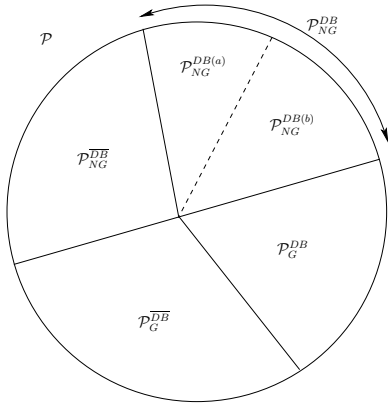


Fig. 2. A partition of the set of input distributions \mathcal{P} .

corresponding sum-rate $I(X_1, X_2; Y|T)$.

3. Finally, for any non-Gaussian input distribution $p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB(b)}$, we will construct a jointly Gaussian input distribution which would satisfy (10) and yield a set of larger rate pairs than the set of rate pairs of the fixed non-Gaussian input distribution.

We first characterize the set of jointly Gaussian input distributions, i.e., \mathcal{P}_G^{DB} , which satisfy the constraint (10). For this purpose, we will consider the set of valid 3×3 covariance matrices of three random variables (T, X_1, X_2) . Let us define \mathcal{Q} as the set of all valid covariance matrices. A typical element Q in the set \mathcal{Q} takes the following form,

$$Q = E[(X_1 \ X_2 \ T)(X_1 \ X_2 \ T)^T] = \begin{pmatrix} P_1 & \rho_{12}\sqrt{P_1P_2} & \rho_{1T}\sqrt{P_1P_T} \\ \rho_{12}\sqrt{P_1P_2} & P_2 & \rho_{2T}\sqrt{P_2P_T} \\ \rho_{1T}\sqrt{P_1P_T} & \rho_{2T}\sqrt{P_2P_T} & P_T \end{pmatrix} \quad (11)$$

A necessary condition for Q to be a valid covariance matrix is that it is positive semi-definite, i.e., $\det(Q) \geq 0$. This is equivalent to saying that,

$$\det(Q) = P_1P_2P_T\Delta \geq 0 \quad (12)$$

where we have defined for simplicity,

$$\Delta = (1 - \rho_{12}^2 - \rho_{1T}^2 - \rho_{2T}^2 + 2\rho_{1T}\rho_{2T}\rho_{12}) \quad (13)$$

Consider a jointly Gaussian triple (T_G, X_{1G}, X_{2G}) with a covariance matrix Q . Let us first characterize the set of rate constraints for this triple. It is straightforward to evaluate the three rate constraints appearing in (6)-(8) for a jointly Gaussian triple with covariance matrix Q ,

$$I(X_{1G}; Y|X_{2G}, T_G) = \frac{1}{2} \log \left(1 + \frac{f_1(Q)}{\sigma_Z^2} \right) \quad (14)$$

$$I(X_{2G}; Y|X_{1G}, T_G) = \frac{1}{2} \log \left(1 + \frac{f_2(Q)}{\sigma_Z^2} \right) \quad (15)$$

$$I(X_{1G}, X_{2G}; Y|T_G) = \frac{1}{2} \log \left(1 + \frac{f_3(Q)}{\sigma_Z^2} \right) \quad (16)$$

where we have defined

$$f_1(Q) = \text{Var}(X_{1G}|X_{2G}, T_G) = \frac{\Delta P_1}{(1 - \rho_{2T}^2)} \quad (17)$$

$$f_2(Q) = \text{Var}(X_{2G}|X_{1G}, T_G) = \frac{\Delta P_2}{(1 - \rho_{1T}^2)} \quad (18)$$

and,

$$f_3(Q) = \text{Var}(X_{1G}|T_G) + \text{Var}(X_{2G}|T_G) + 2\text{Cov}(X_{1G}, X_{2G}|T_G) = (1 - \rho_{1T}^2)P_1 + (1 - \rho_{2T}^2)P_2 + 2(\rho_{12} - \rho_{1T}\rho_{2T})\sqrt{P_1P_2} \quad (19)$$

Finally, evaluating the constraint (10), this triple satisfies the constraint (10) iff,

$$f_3(Q) \leq f_1(Q) + f_2(Q) + \frac{f_1(Q)f_2(Q)}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2\sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)}\right)} \quad (20)$$

To summarize, the set of rate pairs permissible by a jointly Gaussian triple (T_G, X_{1G}, X_{2G}) with a given covariance matrix Q which satisfies (20), are characterized by the following inequalities,

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{f_1(Q)}{\sigma_Z^2} \right) \quad (21)$$

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{f_2(Q)}{\sigma_Z^2} \right) \quad (22)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{f_3(Q)}{\sigma_Z^2} \right) \quad (23)$$

We now return to the set of non-Gaussian distributions which satisfy the constraint (10), i.e., the set \mathcal{P}_{NG}^{DB} of distributions. We first consider any non-Gaussian distribution $p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB(a)}$ with a covariance matrix Q . For such an input distribution, we know by the maximum entropy theorem [9], that the set of rates given by a jointly Gaussian triple with the same covariance matrix Q are always at least as large as the set of rates of a non-Gaussian distribution. Therefore, for the set of distributions in this set, we can find a jointly Gaussian (T_G, X_{1G}, X_{2G}) satisfying (10) and yielding a set of larger rate pairs than the non-Gaussian distribution. Hence, for evaluating our outer bound, we can ignore the set $\mathcal{P}_{NG}^{DB(a)}$.

We now arrive at the second step of our evaluation. Consider any input distribution $p(t, x_1, x_2) \in \mathcal{P}_{NG}^{DB(b)}$ with a covariance matrix Q . By the definition of the set $\mathcal{P}_{NG}^{DB(b)}$, we know that Q does not satisfy (10), which implies the following,

$$f_3(Q) > f_1(Q) + f_2(Q) + \frac{f_1(Q)f_2(Q)}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2\sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)}\right)} \quad (24)$$

We also note the following fact,

$$I(X_1; Y|X_2, T) \leq \frac{1}{2} \log \left(1 + \frac{f_1(Q)}{\sigma_Z^2} \right) \quad (25)$$

$$I(X_2; Y|X_1, T) \leq \frac{1}{2} \log \left(1 + \frac{f_2(Q)}{\sigma_Z^2} \right) \quad (26)$$

$$I(X_1, X_2; Y|T) \leq \frac{1}{2} \log \left(1 + \frac{f_3(Q)}{\sigma_Z^2} \right) \quad (27)$$

which is a simple consequence of the maximum entropy theorem [9]. Note that so far, we have not used the fact that the given non-Gaussian input distribution satisfies the constraint

(10). We will now make use of this fact by rewriting (10) as follows,

$$0 \leq I(X_1; X_2 | Y_{F_1}, Y_{F_2}, T) - I(X_1; X_2 | T) \quad (28)$$

$$= I(X_1; Y_{F_1}, Y_{F_2} | X_2, T) - I(X_1; Y_{F_1}, Y_{F_2} | T) \quad (29)$$

$$= h(Y_{F_1}, Y_{F_2} | X_1, T) + h(Y_{F_1}, Y_{F_2} | X_2, T) \\ - h(Y_{F_1}, Y_{F_2} | T) - h(Y_{F_1}, Y_{F_2} | X_1, X_2, T) \quad (30)$$

We express the above constraint as follows,

$$h(Y_{F_1}, Y_{F_2} | T) + h(Y_{F_1}, Y_{F_2} | X_1, X_2, T) \\ \leq h(Y_{F_1}, Y_{F_2} | X_1, T) + h(Y_{F_1}, Y_{F_2} | X_2, T) \quad (31)$$

We start by obtaining a lower bound for the first term $h(Y_{F_1}, Y_{F_2} | T)$ in (31),

$$h(Y_{F_1}, Y_{F_2} | T) \\ = \int f_T(t) h(Y + Z_1, Y + Z_2 | T = t) dt \quad (32)$$

$$\geq \frac{1}{2} \log \left((2\pi e)^2 \sigma_{Z_1}^2 \sigma_{Z_2}^2 + 2\pi e (\sigma_{Z_1}^2 + \sigma_{Z_2}^2) e^{h(Y|T)} \right) \quad (33)$$

where (33) follows from a multivariate generalization of Costa's entropy power inequality [7], [8] and a subsequent application of Jensen's inequality [9]. A detailed derivation of this step can be found in [10]. We now obtain an upper bound for the right hand side of (31) by using the maximum entropy theorem [9] as,

$$h(Y_{F_1}, Y_{F_2} | X_1, T) + h(Y_{F_1}, Y_{F_2} | X_2, T) \\ \leq \frac{1}{2} \log \left((2\pi e)^4 (f_1(Q)(\sigma_{Z_1}^2 + \sigma_{Z_2}^2) + \eta) \right. \\ \left. (f_2(Q)(\sigma_{Z_1}^2 + \sigma_{Z_2}^2) + \eta) \right) \quad (34)$$

where we have defined

$$\eta = \sigma_{Z_1}^2 \sigma_{Z_2}^2 + \sigma_Z^2 (\sigma_{Z_1}^2 + \sigma_{Z_2}^2) \quad (35)$$

Now, using (31), (33) and (34), we obtain an upper bound on $h(Y|T)$ as follows,

$$h(Y|T) \leq \frac{1}{2} \log \left((2\pi e) (\sigma_Z^2 + f(Q)) \right) \quad (36)$$

where we have defined for simplicity,

$$f(Q) = f_1(Q) + f_2(Q) + \frac{f_1(Q)f_2(Q)}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)} \right)} \quad (37)$$

Using (36), we obtain an upper bound on the sum-rate $I(X_1, X_2; Y|T)$ for any non-Gaussian distribution in $\mathcal{P}_{NG}^{DB(b)}$ as,

$$I(X_1, X_2; Y|T) \leq \frac{1}{2} \log \left(1 + \frac{f(Q)}{\sigma_Z^2} \right) \quad (38)$$

Comparing with (27) and using the fact that Q satisfies (24) we have the following set of inequalities,

$$I(X_1, X_2; Y|T) \leq \frac{1}{2} \log \left(1 + \frac{f(Q)}{\sigma_Z^2} \right) < \frac{1}{2} \log \left(1 + \frac{f_3(Q)}{\sigma_Z^2} \right) \quad (39)$$

This leads to the observation that a combined application of the

entropy power inequality and the dependence balance bound yields a strictly smaller upper bound for $I(X_1, X_2; Y|T)$ for any distribution in $\mathcal{P}_{NG}^{DB(b)}$ than the one suggested by the maximum entropy theorem.

We now arrive at the final step of the evaluation of our outer bound. In particular, we will show the existence of a valid covariance matrix S for which the following inequalities hold true,

$$f_1(Q) \leq f_1(S) \quad (40)$$

$$f_2(Q) \leq f_2(S) \quad (41)$$

$$f(Q) \leq f_3(S) \quad (42)$$

and

$$f_3(S) = f_1(S) + f_2(S) + \frac{f_1(S)f_2(S)}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)} \right)} \quad (43)$$

Before showing the existence of such an S , we first characterize the set of covariance matrices Q which satisfy (24). First recall that for any Q to be a valid covariance matrix, we had the condition $\det(Q) \geq 0$ which is equivalent to $\Delta \geq 0$, which amounts to

$$1 - \rho_{12}^2 - \rho_{1T}^2 - \rho_{2T}^2 + 2\rho_{1T}\rho_{2T}\rho_{12} \geq 0 \quad (44)$$

In particular, it is easy to verify that for any given pair $(\rho_{1T}, \rho_{2T}) \in [-1, 1] \times [-1, 1]$, the set of ρ_{12} which yield a valid Q are such that,

$$\rho_{1T}\rho_{2T} - \lambda \leq \rho_{12} \leq \rho_{1T}\rho_{2T} + \lambda \quad (45)$$

where we have defined

$$\lambda = \sqrt{(1 - \rho_{1T}^2)(1 - \rho_{2T}^2)} \quad (46)$$

We now consider two cases which can arise for a given covariance matrix Q .

Case 1. Q is such that $\rho_{12} = \rho_{1T}\rho_{2T} - \alpha$, for some $\alpha \in [0, \lambda]$: This case is rather trivial and the following simple choice of S works,

$$\rho_{1T}^{(S)} = \rho_{1T}, \quad \rho_{2T}^{(S)} = \rho_{2T} \quad (47)$$

$$\rho_{12}^{(S)} = \rho_{1T}\rho_{2T} \quad (48)$$

Clearly, this S satisfies the dependence balance bound. Moreover, the following inequalities hold as well,

$$f_1(Q) \leq f_1(S) = (1 - \rho_{1T}^2)P_1 \quad (49)$$

$$f_2(Q) \leq f_2(S) = (1 - \rho_{2T}^2)P_2 \quad (50)$$

$$f_3(Q) = (1 - \rho_{1T}^2)P_1 + (1 - \rho_{2T}^2)P_2 - 2\alpha\sqrt{P_1P_2} \\ \leq (1 - \rho_{1T}^2)P_1 + (1 - \rho_{2T}^2)P_2 \\ = f_3(S) \quad (51)$$

Case 2. Q is such that $\rho_{12} = \rho_{1T}\rho_{2T} + \alpha_0$, for some $\alpha_0 \in (0, \lambda]$ and Q satisfies (24): For this case, we will construct a valid covariance matrix S as follows,

$$\rho_{1T}^{(S)} = \rho_{1T}, \quad \rho_{2T}^{(S)} = \rho_{2T} \quad (52)$$

$$\rho_{12}^{(S)} = \rho_{1T}\rho_{2T} + \alpha^*, \quad \text{for some } 0 < \alpha^* < \alpha_0 \quad (53)$$

We define a parameterized covariance matrix $Q(\alpha)$ with entries as follows,

$$\rho_{1T}(\alpha) = \rho_{1T}, \quad \rho_{2T}(\alpha) = \rho_{2T} \quad (54)$$

$$\rho_{12}(\alpha) = \rho_{1T}\rho_{2T} + \alpha \quad (55)$$

where $0 \leq \alpha \leq \alpha_0$. We now define a function of the parameter α of a valid covariance matrix $Q(\alpha)$ as,

$$g(\alpha) = f_1(Q(\alpha)) + f_2(Q(\alpha)) + \frac{f_1(Q(\alpha))f_2(Q(\alpha))}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)}\right)} - f_3(Q(\alpha)) \quad (56)$$

We now note the fact that $g(0) > 0$, moreover Q satisfies (24) for some α_0 , which implies that, $g(\alpha_0) < 0$. Now, it is straightforward to check that $\partial g(\alpha)/\partial \alpha \leq 0$ which implies that $g(\alpha)$ is monotonically decreasing in α . This implies that there exists an $\alpha^* \in (0, \alpha_0)$ such that $g(\alpha^*) = 0$. We use this α^* to construct our new covariance matrix S as mentioned above. It now remains to check whether S satisfies the four conditions (40)-(43). The condition (43) is met with equality, since we have $g(\alpha^*) = 0$. Moreover, $f_1(Q) = f_1(Q(\alpha_0)) \leq f_1(Q(\alpha^*)) = f_1(S)$ since $f_1(Q(\alpha))$ is monotonically decreasing in α for $\alpha \in [0, \lambda]$. Similarly, we also have $f_2(Q) \leq f_2(S)$. Finally,

$$f(Q) = f_1(Q) + f_2(Q) + \frac{f_1(Q)f_2(Q)}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)}\right)} \quad (57)$$

$$\leq f_1(S) + f_2(S) + \frac{f_1(S)f_2(S)}{\left(\sigma_Z^2 + \frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{(\sigma_{Z_1}^2 + \sigma_{Z_2}^2)}\right)} \quad (58)$$

This shows the existence of a valid covariance matrix S which satisfies (43) and yields a set of larger rate pairs than the rates of a non-Gaussian distribution with a covariance matrix Q .

Above two cases show that for any non-Gaussian distribution $p(t, x_1, x_2)$ in the set $\mathcal{P}_{NG}^{DB(b)}$, we can always find a jointly Gaussian triple (T_G, X_{1G}, X_{2G}) with a covariance matrix S which satisfies (10) and yields a set of larger rate pairs than the set of rate pairs of the given non-Gaussian distribution. This consequently completes the proof of the statement that it is sufficient to consider jointly Gaussian (T, X_1, X_2) when evaluating our outer bound.

Figure 3 shows our outer bound along with the cut-set outer bound and the capacity region of MAC without feedback for the case when $P_1 = P_2 = \sigma_Z^2 = 1$ and for backward noise variances $\sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 2 = 5 = 10$. For all positive values of $(\sigma_{Z_1}^2, \sigma_{Z_2}^2)$, our outer bound yields strict improvement over the cut-set bound. Figure 4 illustrates an achievable rate region based on superposition encoding and backward decoding [5], our outer bound, the cut-set bound and the capacity region of MAC without feedback for the case when $P_1 = P_2 = \sigma_Z^2 = 1$ and backward noise variances $\sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 0.3$.

V. CONCLUSION

We obtained a new outer bound for the capacity region of the Gaussian MAC with noisy feedback. Our outer bound is

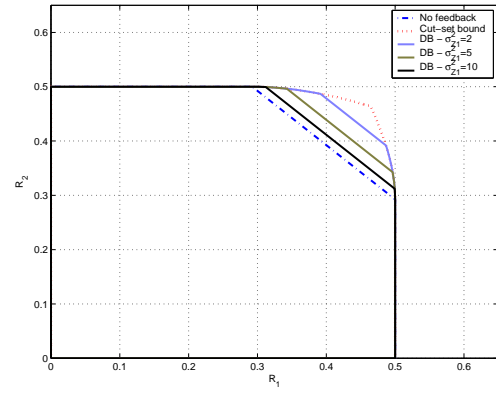


Fig. 3. Illustration of our outer bound for the case when $P_1 = P_2 = \sigma_Z^2 = 1$.

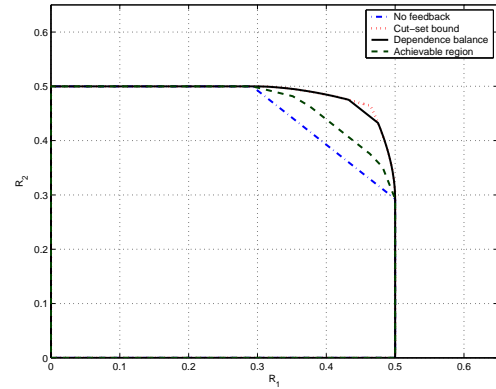


Fig. 4. Illustration of our outer bound and achievable rates by superposition coding for the case when $P_1 = P_2 = \sigma_Z^2 = 1$ and $\sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 0.3$.

tight for two limiting cases, namely when the feedback noise variances are zero, which corresponds to noiseless feedback and for the case when the feedback noise variances are large, which corresponds to no-feedback. For all non-zero, finite values of $(\sigma_{Z_1}^2, \sigma_{Z_2}^2)$, our outer bound improves upon the cut-set bound.

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