A New Upper Bound for a Binary Additive Noisy Multiple Access Channel with Feedback

Ravi Tandon Sennur Ulukus
Department of Electrical and Computer Engineering
University of Maryland, College Park, MD 20742
ravit@umd.edu *ulukus@umd.edu*

***Abstract*— We use the idea of dependence balance [1] to obtain the first improvement over the cut-set bound for the discrete memoryless multiple access channel with noiseless feedback (MAC-FB). More specifically, we consider a binary additive noisy MAC-FB whose capacity does not coincide with the Cover-Leung achievable rate region [2]. Evaluating the dependence balance bound is difficult due to an involved auxiliary random variable. We overcome this difficulty by using functional analysis to explicitly evaluate our upper bound for the binary additive noisy MAC-FB and show that it is strictly less than the cut-set bound for the symmetric-rate point on the capacity region.**

I. Introduction

Noiseless feedback can increase the capacity region of the discrete memoryless MAC, unlike for the single user discrete memoryless channel. This was shown by Gaarder and Wolf in [3] for the binary erasure MAC, which is defined as $Y = X_1 + X_2$. Cover and Leung obtained an achievable rate region for the general MAC-FB based on block Markov superposition coding in [2]. This region is in general larger than the capacity region of the MAC without feedback. For a specific class of MAC-FB, Willems [4] developed an outer bound that equals the Cover-Leung region. For this class of MAC-FB, each channel input (say X_1) should be a deterministic function of the other channel input (X_2) and the channel output (Y). The channel considered by Gaarder and Wolf, where $Y = X_1 + X_2$, falls into this class of channels. Therefore, Cover-Leung region is the capacity region for this channel. However, for general MAC-FB, the best known outer bound is the cut-set bound, which, in general, is loose. An intuitive reason for the cut-set bound to be loose for the general MAC-FB is its permissibility of arbitrary input distributions, some of which yielding rates which may not be achievable. For instance, even though Cover-Leung achievability scheme [2] does introduce correlation between X_1 and X_2, it is a limited form of correlation, as channel inputs are conditionally independent given an auxiliary random variable.

The idea of dependence balance was introduced by Hekstra and Willems in [1] to obtain an outer bound on the capacity region of the single output two-way channel. The basic idea behind this outer bound is to restrict the set of allowable input distributions. The authors in [1] also developed a parallel channel extension for the dependence balance bound. The parallel channel extension can be interpreted as follows: the parallel channel output can be considered as a genie aided information which is made available at both transmitters and it also effects the set of allowable input distributions through the dependence balance bound. Depending on the choice of the parallel channel, there is an inherent tradeoff between the set of allowable input distributions and the excessive mutual information rate terms which appear in the rate expressions as a consequence of the genie aided output. We will exploit this tradeoff provided by the parallel channel extension of the dependence balance bound to obtain a strict improvement over the cut-set bound for a particular MAC whose feedback capacity is not known. It should be noted here that the idea of using parallel channel extension for MAC-FB was also suggested by Kramer in [5].

To motivate the choice of our MAC, consider the binary erasure MAC used by Gaarder and Wolf given by $Y = X_1 + X_2$. If we introduce binary additive noise at the channel output, then the channel becomes $Y = X_1 + X_2 + N$, where all X_1, X_2 and N are binary and N has a uniform distribution. This is a non-deterministic noisy MAC which does not fall into any class for which the feedback capacity is known. We will consider the symmetric-rate point[1] on the capacity region of this channel.

The symmetric-capacity of this channel without feedback is 0.40564 bits/transmission. Cover-Leung's achievable symmetric-rate for this channel was obtained in [5] as 0.43621 bits/transmission. In [5], Kramer obtained an improved symmetric-rate inner bound as 0.43879 bits/transmission by using superposition coding and binning with code trees. The cut-set upper bound on the symmetric-rate was obtained in [5] as 0.45915 bits/transmission. We use the parallel channel extension of the dependence balance bound to obtain a symmetric-rate upper bound of 0.45330 bits/transmission which strictly improves upon the cut-set bound. Even though the improvement from the cut-set bound is modest, our main contribution is to illustrate the usefulness of dependence balance and to explicitly evaluate the dependence balance bound which involves an auxiliary random variable. It should be remarked that the capacity region of the two user Gaussian MAC-FB was established by Ozarow in [6] where it was shown that the cut-set bound was tight. The channel considered

This work was supported by NSF Grants CCF 04-47613, CCF 05-14846, CNS 07-16311 and CCF 07-29127.

[1]By symmetric-rate point, we refer to the maximum rate R such that the rate pair (R, R) lies in the capacity region of MAC-FB.

in this paper can be thought of as the discrete version of the channel considered by Ozarow. Interestingly, our result shows that the cut-set bound is not tight for a discrete binary version of the Gaussian MAC-FB.

II. System Model

A discrete memoryless two-user MAC-FB (see Figure 1) is defined by two input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, an output alphabet $\mathcal{Y}$, and the channel probability transition function $p(y|x_1,x_2)$ for $(x_1,x_2,y) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}$. A (n, M_1, M_2, P_e) code for the MAC-FB consists of two sets of encoding functions $f_{1i} : \mathcal{M}_1 \times \mathcal{Y}^{i-1} \rightarrow \mathcal{X}_1$, $f_{2i} : \mathcal{M}_2 \times \mathcal{Y}^{i-1} \rightarrow \mathcal{X}_2$ for $i = 1,\ldots,n$ and a decoding function $g : \mathcal{Y}^n \rightarrow \mathcal{M}_1 \times \mathcal{M}_2$. The two transmitters produce independent and uniformly distributed messages $W_1 \in \{1,\ldots,M_1\}$ and $W_2 \in \{1,\ldots,M_2\}$, respectively, and transmit them through n channel uses. The average error probability is $P_e = Pr(g(Y^n) \neq (W_1,W_2))$. A rate pair (R_1,R_2) is said to be achievable for MAC-FB if for any $\epsilon \geq 0$, there exists a pair of n encoding functions $\{f_{1i}\}_{i=1}^n$, $\{f_{2i}\}_{i=1}^n$, and a decoding function $g : \mathcal{Y}^n \rightarrow \mathcal{M}_1 \times \mathcal{M}_2$ such that $R_1 \leq \log(M_1)/n$, $R_2 \leq \log(M_2)/n$ and $P_e \leq \epsilon$ for sufficiently large n. The capacity region of MAC-FB (C_{MAC}^{FB}) is the closure of the set of all achievable rate pairs (R_1,R_2).

III. Cut-Set Outer Bound for MAC-FB

By applying Theorem 14.10.1 in [7] a cut-set outer bound on the capacity region of MAC-FB can be obtained as:

$$\mathcal{CS} = \Big\{(R_1,R_2) : R_1 \leq I(X_1;Y|X_2) \quad (1)$$
$$R_2 \leq I(X_2;Y|X_1) \quad (2)$$
$$R_1 + R_2 \leq I(X_1,X_2;Y)\Big\} \quad (3)$$

where the random variables (X_1,X_2,Y) have the joint distribution

$$p(x_1,x_2,y) = p(x_1,x_2)p(y|x_1,x_2) \quad (4)$$

The cut-set bound allows all input distributions $p(x_1,x_2)$, which makes it seemingly loose since an achievable scheme might not achieve arbitrary correlation and rates given by the cut-set bound. Our aim is to restrict the set of allowable input distributions by using a dependence balance approach.

IV. Dependence Balance Outer Bound for MAC-FB

The capacity region of MAC-FB (C_{MAC}^{FB}) is contained within $\mathcal{DB}$, where

$$\mathcal{DB} = \Big\{(R_1,R_2) : R_1 \leq I(X_1;Y|X_2,T) \quad (5)$$
$$R_2 \leq I(X_2;Y|X_1,T) \quad (6)$$
$$R_1 + R_2 \leq I(X_1,X_2;Y)\Big\} \quad (7)$$

where the random variables (X_1,X_2,Y,T) have the joint distribution

$$p(t,x_1,x_2,y) = p(t)p(x_1,x_2|t)p(y|x_1,x_2) \quad (8)$$

and also satisfy the following dependence balance bound,

$$I(X_1;X_2|T) \leq I(X_1;X_2|Y,T) \quad (9)$$

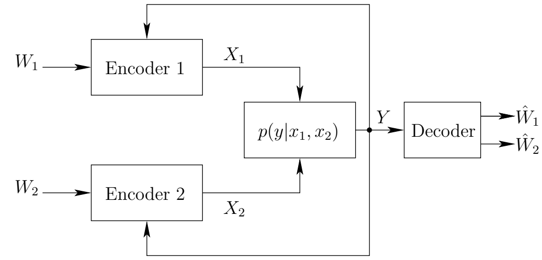

Figure 1: The multiple access channel with feedback (MAC-FB).

where T is subject to a cardinality constraint of $|\mathcal{T}| \leq |\mathcal{X}_1||\mathcal{X}_2| + 2$.

V. Adaptive Parallel Channel Extension of the Dependence Balance Bound

In [1], Hekstra and Willems also developed an adaptive parallel channel extension of the dependence balance bound which is given as follows: Let $\Delta(\mathcal{U})$ denote the set of all distributions of U and $\Delta(\mathcal{U}|\mathcal{V})$ denote the set of all conditional distributions of U given V, then for any mapping $F : \Delta(\mathcal{X}_1 \times \mathcal{X}_2) \rightarrow \Delta(\mathcal{Z}|\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y})$, we have $C_{MAC}^{FB} \subset \mathcal{DB}_{PC}$ where

$$\mathcal{DB}_{PC} = \Big\{(R_1,R_2) : R_1 \leq I(X_1;Y,Z|X_2,T) \quad (10)$$
$$R_2 \leq I(X_2;Y,Z|X_1,T) \quad (11)$$
$$R_1 \leq I(X_1;Y|X_2) \quad (12)$$
$$R_2 \leq I(X_2;Y|X_1) \quad (13)$$
$$R_1 + R_2 \leq I(X_1,X_2;Y) \quad (14)$$
$$R_1 + R_2 \leq I(X_1,X_2;Y,Z|T)\Big\} \quad (15)$$

where the random variables (T,X_1,X_2,Y,Z) have the joint distribution

$$p(t,x_1,x_2,y,z) = p(t,x_1,x_2)p(y|x_1x_2)p^+(z|x_1,x_2,y,t) \quad (16)$$

such that for all t,

$$p^+(z|x_1,x_2,y,t) = F(p_{X_1,X_2}(x_1,x_2|t)) \quad (17)$$

and such that

$$I(X_1;X_2|T) \leq I(X_1;X_2|Y,Z,T) \quad (18)$$

where T is subject to a cardinality bound of $|\mathcal{T}| \leq |\mathcal{X}_1||\mathcal{X}_2| + 3$.

We should remark that the parallel channel (defined by $p^+(z|x_1,x_2,y,t)$) is selected apriori and for every choice of the parallel channel, one obtains an outer bound on C_{MAC}^{FB}, which is in general tighter than the cut-set bound. The set of allowable input distributions $p(t,x_1,x_2)$ are those which satisfy the constraint in (18). Also note that only the right hand side of (18), i.e., only $I(X_1;X_2|Y,Z,T)$ depends on the choice of the parallel channel. By carefully selecting $p^+(z|x_1,x_2,y,t)$, one can reduce $I(X_1;X_2|Y,Z,T)$, thereby making the constraint (18) more stringent, consequently reducing the set of allowable input distributions.

To motivate the choice of our parallel channel, first consider a trivial choice of Z: $Z = \phi$ (a constant). For this choice of

Z, (18) reduces to (9) and we are not restricting the set of allowable input distributions any more than the $\mathcal{DB}$ bound. Also note that the smallest value of $I(X_1;X_2|Y,Z,T)$ is zero. Thus, it follows that if we select a parallel channel such that $I(X_1;X_2|Y,Z,T)=0$ for every input distribution $p(t,x_1,x_2)$, then $I(X_1;X_2|T)=0$ by (18). Hence, the smallest set of input distributions permissible by $\mathcal{DB}_{PC}$ consists of those $p(t,x_1,x_2)$ for which X_1 and X_2 are conditionally independent given T, which is obtained by choosing a parallel channel such that for every $p(t,x_1,x_2)$, we have $I(X_1;X_2|Y,Z,T)=0$. Furthermore, for the class of parallel channels where $I(X_1;X_2|Y,Z,T)=0$, the bound in (15) is redundant. This can be seen from:

$$\begin{aligned} 0 &= I(X_1;X_2|T) - I(X_1;X_2|Y,Z,T) \\ &= I(X_1,X_2;Y,Z|T) - I(X_1;Y,Z|X_2,T) \\ &\quad - I(X_2;Y,Z|X_1,T) \end{aligned} \tag{19}$$

Using (19), it is clear that the sum of constraints (10) and (11) is at least as strong as the constraint (15).

VI. Binary Additive Noisy MAC with Feedback

To illustrate the usefulness of dependence balance, we will consider a binary input MAC given by $Y = X_1 + X_2 + N$ where N is binary, uniform over $\{0,1\}$ and is independent of X_1 and X_2. The channel output Y takes values from the set $\mathcal{Y} = \{0,1,2,3\}$. This channel does not fall into any class of MAC-FB for which the capacity region is known. This channel was also considered by Kramer in [8] where the first improvement over Cover-Leung achievable symmetric-rate was obtained.

To improve upon the symmetric-rate cut-set bound, we need to select a "good" parallel channel such that it restricts the input distributions to the smallest allowable set and yields small values of $I(X_1;Z|Y,X_2,T)$ and $I(X_2;Z|Y,X_1,T)$ at the same time. These two mutual information "leak" terms are the extra terms that appear in (10) and (11) relative to the rates appearing in (5) and (6), respectively. We select a parallel channel $p^+(z|x_1,x_2,y)$ such that $I(X_1;X_2|Y,Z,T)=0$. By (18), this will imply $I(X_1;X_2|T)=0$ and hence only distributions of the type $p(t,x_1,x_2)=p(t)p(x_1|t)p(x_2|t)$ will be allowed. By doing so, we restrict the set of allowable distributions to be the smallest permitted by $\mathcal{DB}_{PC}$, although we pay a penalty due to the positive "leak" terms $I(X_1;Z|Y,X_2,T)$ and $I(X_2;Z|Y,X_1,T)$.

Hence for a particular selection of the parallel channel such that $I(X_1;X_2|Y,Z,T)=0$, the overall problem is to find the maximum symmetric-rate such that

$$R_1 \le I(X_1;Y|X_2,T) + L_1 \tag{20}$$
$$R_2 \le I(X_2;Y|X_1,T) + L_2 \tag{21}$$
$$R_1 \le I(X_1;Y|X_2) \tag{22}$$
$$R_2 \le I(X_2;Y|X_1) \tag{23}$$
$$R_1 + R_2 \le I(X_1,X_2;Y) \tag{24}$$

over the set of distributions of the type $p(t,x_1,x_2) = p(t)p(x_1|t)p(x_1|t)$ where we have defined $L_1 = I(X_1;Z|Y,X_2,T)$ and $L_2 = I(X_2;Z|Y,X_1,T)$.

Two simple choices of Z for which $I(X_1;X_2|Y,Z,T)=0$ are $Z=X_1$ or $Z=X_2$. We will show that the cut-set bound can be improved upon by the choice $Z=X_1$. The case $Z=X_2$ will yield the same symmetric-rate upper bound as $Z=X_1$. The evaluation of the above bound for the symmetric-rate point is rather cumbersome because for binary inputs, the bound on $|\mathcal{T}|$ is $|\mathcal{T}| \le 7$. To the best of our knowledge, no one has been able to conduct an exhaustive search over an auxiliary random variable whose cardinality is larger than 4. In the next section, we show that a binary selection of T with uniform distribution is sufficient to obtain our symmetric-rate upper bound. We show that this upper bound is strictly less than the cut-set bound.

VII. Evaluation of the Dependence Balance Bound

First note that for the binary additive noisy MAC in consideration, the following equalities hold for any distribution of the form $p(t,x_1,x_2)=p(t)p(x_1|t)p(x_2|t)$,

$$I(X_1;Y|X_2,T) = \frac{H(X_1|T)}{2} \tag{25}$$
$$I(X_2;Y|X_1,T) = \frac{H(X_2|T)}{2} \tag{26}$$

For the choice of $Z=X_1$, the information leaks are

$$L_1 = \frac{H(X_1|T)}{2} \tag{27}$$
$$L_2 = 0 \tag{28}$$

Substituting (25), (26), (27) and (28) in (20) to (24), we can restate the problem as finding the maximum symmetric-rate, R, subject to the following constraints,

$$R \le H(X_1|T) \tag{29}$$
$$R \le \frac{H(X_2|T)}{2} \tag{30}$$
$$R \le I(X_1;Y|X_2) \tag{31}$$
$$2R \le I(X_1,X_2;Y) \tag{32}$$

over all input distributions of the form $p(t,x_1,x_2) = p(t)p(x_1|t)p(x_2|t)$. Note that since $L_2=0$, among the two constraints (21) and (23), the constraint (23) is redundant, due to the fact that conditioning reduces entropy. By ignoring any of the constraints (29) to (32), one obtains an albeit looser upper bound on R. Ignoring the constraint on R in (31), we obtain

$$R \le H(X_1|T) \tag{33}$$
$$R \le \frac{H(X_2|T)}{2} \tag{34}$$
$$2R \le I(X_1,X_2;Y) \tag{35}$$

In what follows, we will characterize the maximum R over all allowable conditionally independent distributions $p(t)p(x_1|t)p(x_2|t)$ subject to the constraints in (33)-(35). Throughout the paper, $h(s)$ will refer to the binary entropy function. To characterize our upper bound, we will use the following function

$$\phi(s) = \begin{cases} \frac{1-\sqrt{1-2s}}{2}, & \text{for } 0 \le s \le 1/2, \\ \frac{1-\sqrt{2s-1}}{2}, & \text{for } 1/2 < s \le 1. \end{cases} \quad (36)$$

It was shown in [9] that the composite function $h(\phi(s))$ is symmetric around $s = 1/2$ and concave in s for $0 \le s \le 1$. The functions $\phi(s)$ and $h(\phi(s))$ are illustrated in Figure 2. From the definition of $\phi(s)$ in (36) it is clear that for any $s \in [0, 1]$, the function $\phi(s)$ satisfies the following property

$$\phi(2s(1-s)) = \min(s, 1-s) \quad (37)$$

As a consequence, the following holds as well,

$$h(\phi(2s(1-s))) = h(s) \quad (38)$$

Let the cardinality of the auxiliary random variable T be fixed and arbitrary, say $|\mathcal{T}|$. Then the joint distribution $p(t)p(x_1|t)p(x_2|t)$ can be described by the following: $q_{jt} = \Pr(X_j = 0|T = t)$ for $j = 1, 2$ and $p_t = \Pr(T = t)$ for $t = 1, 2, \ldots, |\mathcal{T}|$. We will characterize our symmetric-rate upper bound in terms of two variables u_1 and u_2, which are functions of $p(t, x_1, x_2)$, and are defined as,

$$u_1 = \sum_t p_t q_{1t}(1-q_{1t}) = \sum_t p_t u_{1t} \quad (39)$$

$$u_2 = \sum_t p_t q_{2t}(1-q_{2t}) = \sum_t p_t u_{2t} \quad (40)$$

where we have defined

$$u_{jt} = q_{jt}(1-q_{jt}), \quad j = 1, 2. \quad (41)$$

It should be noted that since $0 \le q_{jt} \le 1$ for $j = 1, 2$, the variables u_1, u_2, u_{1t} and u_{2t} all lie in the range $[0, \frac{1}{4}]$. Now consider the constraint (33),

$$R \le H(X_1|T) = \sum_t p_t h(q_{1t}) \quad (42)$$

$$= \sum_t p_t h(\phi(2q_{1t}(1-q_{1t}))) \quad (43)$$

$$= \sum_t p_t h(\phi(2u_{1t})) \quad (44)$$

$$\le h(\phi(2u_1)) \quad (45)$$

where (43) follows due to (38), (44) follows by (41), and (45) follows due to the fact that $h(\phi(s))$ is concave in s and application of Jensen's inequality [7]. Using a similar set of inequalities for the constraint in (34), we obtain

$$R \le \frac{h(\phi(2u_2))}{2} \quad (46)$$

We will now obtain an upper bound on $I(X_1, X_2; Y)$. First note that

$$I(X_1, X_2; Y) = H(Y) - H(Y|X_1, X_2) \quad (47)$$

$$= h^{(4)}(P_Y(0), P_Y(1), P_Y(2), P_Y(3)) - 1 \quad (48)$$

where

$$P_Y(0) = \sum_t p_t q_{1t} q_{2t}/2 \quad (49)$$

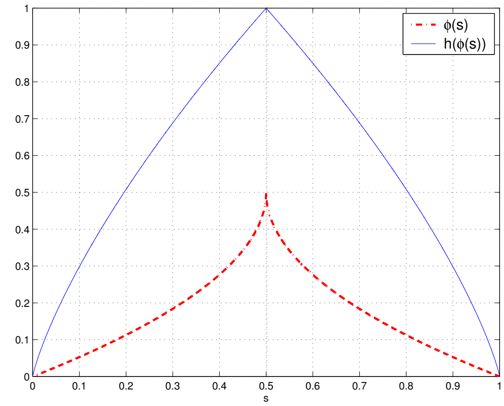

Figure 2: The functions $\phi(s)$ and $h(\phi(s))$.

$$P_Y(1) = \sum_t p_t \Big[q_{1t} + q_{2t} - q_{1t}q_{2t}\Big]/2 \quad (50)$$

$$P_Y(2) = \sum_t p_t \Big[1 - q_{1t}q_{2t}\Big]/2 \quad (51)$$

$$P_Y(3) = \sum_t p_t (1-q_{1t})(1-q_{2t})/2 \quad (52)$$

and $h^{(4)}(s_1, s_2, s_3, s_4) = -\sum_i s_i \log(s_i)$, for $s_i \ge 0$ and $\sum_i s_i = 1$. Note that,

$$h^{(4)}(a, b, c, d) = \frac{1}{2}h^{(4)}(a, b, c, d) + \frac{1}{2}h^{(4)}(d, c, b, a) \quad (53)$$

$$\le h^{(4)}\left(\frac{a+d}{2}, \frac{b+c}{2}, \frac{b+c}{2}, \frac{a+d}{2}\right) \quad (54)$$

$$= h(a+d) + h\left(\frac{1}{2}\right) \quad (55)$$

$$= h(1-(b+c)) + 1 \quad (56)$$

where (54) follows by the concavity of the entropy function and application of Jensen's inequality [7]. Now starting from (48), and using (56), we obtain an upper bound on $I(X_1, X_2; Y)$ as follows,

$$I(X_1, X_2; Y) \le h(1-(P_Y(1)+P_Y(2))) + h\left(\frac{1}{2}\right) - 1 \quad (57)$$

$$= h\left(\frac{1-u}{2}\right) \quad (58)$$

where we have defined

$$u = \sum_t p_t (q_{1t} + q_{2t} - 2q_{1t}q_{2t}) \quad (59)$$

$$= \sum_t p_t u_t \quad (60)$$

where $u_t = q_{1t} + q_{2t} - 2q_{1t}q_{2t}$. To obtain an upper bound on R in terms of u_1 and u_2 using (58), we will first obtain a lower bound on the variable u in terms of u_1 and u_2. For this purpose, define a function $f(x, y)$ for any $x \in [0, \frac{1}{2}]$ and $y \in [0, \frac{1}{2}]$ as follows,

$$f(x, y) \triangleq \phi(x) + \phi(y) - 2\phi(x)\phi(y) \quad (61)$$

$$= \frac{1-\sqrt{(1-2x)(1-2y)}}{2}$$

We will now state two lemmas which are necessary in obtaining an upper bound on $I(X_1, X_2; Y)$ in terms of u_1 and u_2.

Lemma 1: The variable

$$u_t = q_{1t} + q_{2t} - 2q_{1t}q_{2t} \tag{62}$$

is always lower bounded by $f(2u_{1t}, 2u_{2t})$ *for any* $q_{1t} \in [0,1], q_{2t} \in [0,1]$.

Lemma 2: The function

$$f(x,y) = \phi(x) + \phi(y) - 2\phi(x)\phi(y) \tag{63}$$

is jointly convex in (x,y) *for* $0 \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}$.

We omit the proofs of Lemmas 1 and 2 here due to space limitations; they can be found in [10]. Returning to (60), we now obtain a lower bound on u as follows,

$$\begin{aligned} u &= \sum_t p_t u_t && (64)\\ &\geq \sum_t p_t f(2u_{1t}, 2u_{2t}) && (65)\\ &\geq f\left(2\sum_t p_t u_{1t}, 2\sum_t p_t u_{2t}\right) && (66)\\ &= f(2u_1, 2u_2) && (67)\end{aligned}$$

where (65) follows from Lemma 1 and (66) follows by Lemma 2 and application of Jensen's inequality [7]. Having lower bounded u in terms of u_1 and u_2, we now obtain a third constraint on R in terms of u_1 and u_2, by continuing from (58),

$$\begin{aligned} 2R &\leq I(X_1, X_2; Y) && (68)\\ &\leq h\left(\frac{1-u}{2}\right) && (69)\\ &\leq h\left(\frac{1-f(2u_1, 2u_2)}{2}\right) && (70)\end{aligned}$$

where (70) follows by (67) and using the fact that the binary entropy $h(s)$ is monotonically increasing in s for $s \in [0, \frac{1}{2}]$.

To summarize our results, we have three bounds on R from (45), (46) and (70) which when combined yield an upper bound on the symmetric-rate as follows,

$$R \leq \max_{u_1 \in [0,\frac{1}{4}], u_2 \in [0,\frac{1}{4}]} \min\left\{ h(\phi(2u_1)), \frac{1}{2}h(\phi(2u_2)), \frac{1}{2}h\left(\frac{1-f(2u_1,2u_2)}{2}\right)\right\} \tag{71}$$

This yields an upper bound on the symmetric-rate R as 0.45330 bits/transmission. The optimal pair (u_1^*, u_2^*) which attains this symmetric-rate upper bound is,

$$(u_1^*, u_2^*) = (0.086063, 0.218333) \tag{72}$$

An input distribution $p(t, x_1, x_2)$ which attains this upper bound is specified as follows,

$$P(T=0) = P(T=1) = \frac{1}{2}$$
$$q_{10} = 1 - q_{11} = \phi(2u_1^*) = 0.095109$$
$$q_{20} = 1 - q_{21} = \phi(2u_2^*) = 0.322050 \tag{73}$$

The input distribution in (73) yields a symmetric-rate of 0.45330 bits/transmission. This shows that the bound obtained in (71) can be attained by a binary auxiliary random variable T with uniform distribution over $\{0, 1\}$.

VIII. Conclusion

In this paper, we used the parallel channel extension of the dependence balance bound ($\mathcal{DB}_{PC}$) to obtain an improvement over the symmetric-rate cut-set bound for a simple multiple access channel whose feedback capacity is not known. To be consistent with literature, we chose a binary additive noisy MAC, which was extensively studied in [8]. We used composite functions and their properties to obtain an upper bound of 0.45330 bits/transmission on the symmetric-rate, which is strictly less than the symmetric-rate cut-set bound 0.45915 bits/transmission. The improvement over the cut-set bound for the symmetric-rate is about 0.006 bits/transmission. Although this improvement is modest, the main contribution of this work is to show the usefulness and explicit evaluation of the dependence balance bound which improves upon the cut-set bound for the binary additive noisy MAC-FB.

Recently, Kramer and Gastpar have used the idea of dependence balance to obtain first improvement over the cut-set bound for a K-user Gaussian MAC with feedback and the Gaussian interference channel with noisy feedback [11], [12]. It would be interesting to generalize dependence balance bounds and their corresponding parallel channel extensions to Gaussian interference networks with generalized feedback.

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