On the Capacity Region of the Gaussian Z-channel

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***Abstract*— We investigate the capacity region of the Gaussian Z-channel with a small crossover link gain, i.e., $\alpha \leq 1$. For the case of $\alpha < 1$, we provide an achievable region, and the converse for most of the achievable region. We also derive lower and upper bounds for the part of the region where the capacity boundary is unclear. For the case of $\alpha = 1$, we determine the capacity region exactly.**

I. INTRODUCTION

With recent advances in the theory, technology and applications of wireless communications, the network structures of interest are moving away from traditional networks such as the multiple access channel (MAC) and the broadcast channel (BC) which model cellular systems, and towards arbitrary network structures which model ad-hoc wireless networks. One important such network is the interference channel [1]–[6]. An interference channel is a simple two-transmitter two-receiver network, where each transmitter has a message for only one of the receivers. A more general network structure is the X-channel [7], where the channel is the same as the interference channel except that both transmitters have messages for both receivers. Unfortunately, most problems in network information theory, including the simple interference channel, have been open for quite a number of years.

Recently, [7] has proposed a new multiuser model, called the Z-channel; see Figure 1. The Z-channel is a special case of the X-channel in that there is only one crossover link and as a consequence, the transmitter that does not have a crossover link has only one message to send. In [7], an achievable region for the Gaussian Z-channel is provided for the case of $\alpha > 1+P_1$. In this paper, we focus on the model of the Gaussian Z-channel where the crossover link is weak, more specifically, $\alpha < 1$. We derive an achievable region and claim that this region is almost equal to the capacity region by proving most of the converse. We also derive some lower and upper bounds on the capacity region. Finally, for the special case of $\alpha = 1$, we determine the capacity region exactly.

II. SYSTEM MODEL

The Gaussian Z-channel has two transmitters and two receivers as shown in Figure 1. The received signals at receivers R1 and R2 are given as,

$$Y_1 = X_1 + \sqrt{\alpha}X_2 + Z_1 \quad (1)$$
$$Y_2 = X_2 + Z_2 \quad (2)$$

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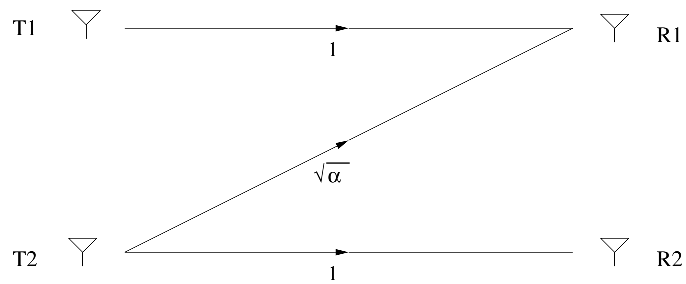

Fig. 1. The Z-channel.

where X_1 and X_2 are the signals transmitted by transmitters T1 and T2, and Z_1, Z_2 are independent Gaussian random variables with zero mean and unit variance. The transmitters T1 and T2 are subject to power constraints P_1 and P_2, respectively. The received signals in (1) and (2) can equivalently be written as,

$$Y_1 = \frac{X_1}{\sqrt{\alpha}} + X_2 + \frac{Z_1}{\sqrt{\alpha}} \quad (3)$$
$$Y_2 = X_2 + Z_2 \quad (4)$$

since scaling does not affect the capacity region. For the rest of this paper, we will be working with the channel model in (3) and (4).

Three independent messages are transmitted in a Z-channel: the message from transmitter T1 to receiver R1, denoted as W_{11}, the message from transmitter T2 to receiver R1, denoted as W_{21}, and the message from transmitter T2 to receiver R2, denoted as W_{22}. The capacity region of the Z-channel is a three dimensional volume, with axes R_{11}, R_{21} and R_{22} corresponding to the rates of messages W_{11}, W_{21} and W_{22}.

In this paper, we mainly study the case of $\alpha < 1$. Reference [7] studied the case of $\alpha > 1 + P_1$. These two cases correspond to two different kinds of "degradedness" conditions on the channels from transmitter T2 to both receivers. In the absence of the link between transmitter T1 and receiver R1, the channels from transmitter T2 to both receivers constitute a traditional BC [1]. Given the existence of the link from transmitter T1 to receiver R1, the condition, $\alpha > 1 + P_1$ assumed in [7], corresponds to the case that the signal of transmitter T2 received at receiver R2 is a "degraded" version of the same signal received at receiver R1 (for Gaussian inputs). The condition, $\alpha < 1$, that we assume in this paper, corresponds to the case that the signal of transmitter T2 received at receiver R1 is a "degraded" version of the same signal at receiver R2. The "degradedness" condition we have here is stronger than the one in [7], in that, it is valid for all distributions of X_1.

In this paper, we consider only deterministic encoders, which incur no loss in performance [8]. All logarithms are defined with respect to base e.

III. Achievable Region

Let us define four quantities:

$$c_{11}(\beta) = \frac{1}{2}\log\left(1 + \frac{P_1}{\alpha\beta P_2 + 1}\right) \tag{5}$$

$$c_{21}(\beta) = \frac{1}{2}\log\left(1 + \frac{\alpha(1-\beta)P_2}{\alpha\beta P_2 + 1}\right) \tag{6}$$

$$c_{22}(\beta) = \frac{1}{2}\log\left(1 + \beta P_2\right) \tag{7}$$

$$c_{1}(\beta) = \frac{1}{2}\log\left(1 + \frac{P_1 + \alpha(1-\beta)P_2}{\alpha\beta P_2 + 1}\right) \tag{8}$$

The following theorem states an achievable region for the Gaussian Z-channel when $\alpha < 1$.

Theorem 1 *If $\alpha < 1$, the following region is achievable in the Gaussian Z-channel:*

$$R_{11} \leq c_{11}(\beta) \tag{9}$$

$$R_{21} \leq c_{21}(\beta) \tag{10}$$

$$R_{22} \leq c_{22}(\beta) \tag{11}$$

$$R_{11} + R_{21} \leq c_{1}(\beta) \tag{12}$$

for any $0 \leq \beta \leq 1$.

Before providing the proof of Theorem 1, we show an example of the achievable region in Figure 2, where $P_1 = 1$, $P_2 = 5$ and $\alpha = 0.5$. The boundary of the capacity region is traced as we change β from 0 to 1, and interpret inequalities in (9)-(12) as equalities. Each fixed β determines a pentagon on a plane parallel to the R_{11}-R_{21} plane as defined by inequalities (9), (10) and (12), and also a rate R_{22} as defined by inequality (11). Therefore, the achievable region is a concatenation of pentagons of varying sizes along the R_{22} axis.

Proof (Theorem 1): For simplicity, we will not present probability of error calculations, but rather, we will describe a scheme the transmitters and receivers may use to achieve the region given in (9) to (12).

Fix a β between 0 and 1, it suffices to show that the two rate triplets: $(R_{11}, R_{21}, R_{22}) = (c_{11}(\beta), c_1(\beta) - c_{11}(\beta), c_{22}(\beta))$ and $(R_{11}, R_{21}, R_{22}) = (c_1(\beta) - c_{21}(\beta), c_{21}(\beta), c_{22}(\beta))$ are achievable. This is because, if these two triplets are achievable, then all other points of the region can be achieved by the usual time-sharing technique.

First, we will show that $(R_{11}, R_{21}, R_{22}) = (c_{11}(\beta), c_1(\beta) - c_{11}(\beta), c_{22}(\beta))$ can be achieved. Transmitter T2 dedicates βP_2 power for transmitting message W_{22} using codebook C_{22}, and $(1-\beta)P_2$ power for transmitting message W_{21} using codebook C_{21}. It transmits the sum of the two codewords. Transmitter T1 uses all its power P_1 for transmitting message W_{11} using codebook C_{11}.

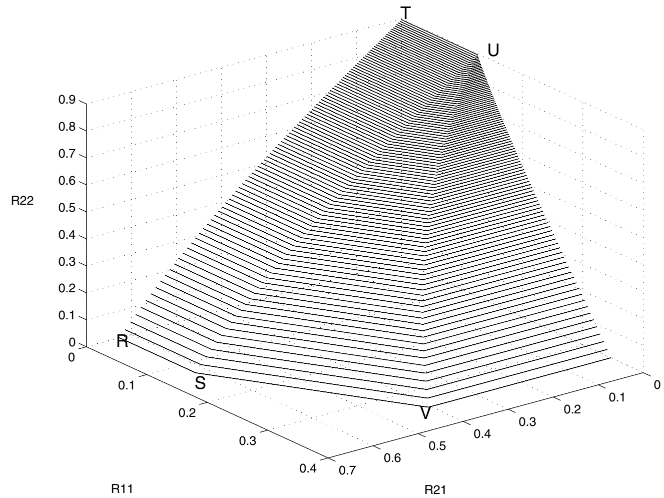

Fig. 2. The achievable region.

Receiver R1 looks at codebook C_{21} only, treating everything else as noise, and therefore obtains a rate of

$$R_{21} = \frac{1}{2}\log\left(1 + \frac{(1-\beta)P_2}{\frac{P_1}{\alpha} + \beta P_2 + \frac{1}{\alpha}}\right) = c_1(\beta) - c_{11}(\beta) \tag{13}$$

Then, it subtracts the effect of W_{21} off, looks at codebook C_{11}, treating everything else as noise, and obtains a rate of

$$R_{11} = \frac{1}{2}\log\left(1 + \frac{\frac{P_1}{\alpha}}{\beta P_2 + \frac{1}{\alpha}}\right) = c_{11}(\beta) \tag{14}$$

Together, this is a rate of $R_{11} + R_{21} = c_1(\beta)$.

Receiver R2, looks at codebook C_{21} only, treating everything else as noise, since

$$R_{21} = \frac{1}{2}\log\left(1 + \frac{(1-\beta)P_2}{\frac{P_1}{\alpha} + \beta P_2 + \frac{1}{\alpha}}\right) \tag{15}$$

$$\leq \frac{1}{2}\log\left(1 + \frac{(1-\beta)P_2}{\beta P_2 + 1}\right) \tag{16}$$

it can decode W_{21} without error. Subtracting the effect of W_{21} off, looking at codebook C_{22}, receiver R2 gets a rate of

$$R_{22} = \frac{1}{2}\log(1 + \beta P_2) = c_{22}(\beta) \tag{17}$$

Thus, rate triplet $(c_{11}(\beta), c_1(\beta) - c_{11}(\beta), c_{22}(\beta))$ is achieved.

When both transmitters and receiver R2 operate in exactly the same way as explained above, and receiver R1 performs the successive decoding in the reverse order (i.e., it decodes W_{11} first and then W_{21}), the rate triplet $(R_{11}, R_{21}, R_{22}) = (c_1(\beta) - c_{21}(\beta), c_{21}(\beta), c_{22}(\beta))$ is achieved. ■

We have established the achievability of the region defined by (9)-(12). Next, we will investigate the converse of this achievable region. We will show that, in most places, the achievable region is actually tight, i.e., it is equal to the capacity region of the channel.

IV. The Converse

Theorem 2 *The achievable rate triplets* (R_{11}, R_{21}, R_{22}) *have to satisfy*

$$R_{21} \leq c_{21}(\beta) \quad (18)$$

$$R_{22} \leq c_{22}(\beta) \quad (19)$$

$$R_{11} + R_{21} \leq c_1(\beta) \quad (20)$$

for some $0 \leq \beta \leq 1$.

Referring back to Figure 2, this theorem states that, of the three surfaces that make up the achievability region, two of them, the surface defined by $TRSU$ and the surface defined by USV, are actually tight.

Proof (Theorem 2): We will prove this by using ideas similar to El Gamal's alternative proof [9] to Bergmans' proof [10].

Since there is no cooperation between the two receivers, the capacity region of this channel depends on the joint distribution $p(y_1, y_2|x_1, x_2)$ only through the two marginals $p(y_1|x_1, x_2)$ and $p(y_2|x_1, x_2)$ [4]. Thus, we will concentrate on the following channel which will yield the same capacity region as our original channel (3)-(4),

$$Y_1 = \frac{X_1}{\sqrt{\alpha}} + Y_2 + \tilde{Z} \quad (21)$$

$$Y_2 = X_2 + Z_2 \quad (22)$$

where $\tilde{Z}$ and Z_2 are Gaussian random variables with zero mean and variance $\frac{1}{\alpha} - 1$ and 1, respectively. Let rate triplets (R_{11}, R_{21}, R_{22}) be achievable. Then by Fano's inequality [1], there exists an ϵ_n such that

$$H(W_{22}|Y_2^n) \leq n\epsilon_n \quad (23)$$

$$H(W_{21}, W_{11}|Y_1^n) \leq n\epsilon_n \quad (24)$$

and as $n \to \infty$, $\epsilon_n \to 0$.

We develop a series of bounds on R_{22},

$$\begin{aligned} nR_{22} &= H(W_{22}) && (25)\\ &= H(W_{22}|Y_2^n) + I(W_{22}; Y_2^n) && (26)\\ &\leq H(W_{22}|Y_2^n) + I(W_{22}; Y_2^n|W_{21}) && (27)\\ &= H(W_{22}|Y_2^n) + h(Y_2^n|W_{21}) - h(Y_2^n|W_{21}, W_{22}) && (28)\\ &= H(W_{22}|Y_2^n) + h(Y_2^n|W_{21}) - h(Z_2^n) && (29)\\ &\leq n\epsilon_n + h(Y_2^n|W_{21}) - \frac{n}{2}\log(2\pi e) && (30) \end{aligned}$$

where (27) is obtained from (26) using the independence of messages W_{21} and W_{22}, (29) is obtained from (28) because we consider deterministic encoders, thus given W_{21} and W_{22}, we know X_2^n, and therefore the only remaining randomness is in Z_2^n. Finally, (30) follows from (23) and the fact that Z_2^n is an i.i.d. Gaussian sequence with unit variance.

Next, we develop a bound for R_{21},

$$\begin{aligned} nR_{21} &= H(W_{21}) && (31)\\ &= H(W_{21}|Y_1^n) + I(W_{21}; Y_1^n) && (32)\\ &\leq H(W_{11}, W_{21}|Y_1^n) + I(W_{21}; Y_1^n|W_{11}) && (33)\\ &= H(W_{11}, W_{21}|Y_1^n) + h(Y_1^n|W_{11}) \\ &\quad - h(Y_1^n|W_{11}, W_{21}) && (34)\\ &\leq n\epsilon_n + h(Y_1^n|W_{11}) - h(Y_1^n|W_{11}, W_{21}) && (35)\\ &\leq n\epsilon_n + \frac{n}{2}\log(2\pi e)\left(P_2 + \frac{1}{\alpha}\right) - h(Y_1^n|W_{11}, W_{21}) && (36) \end{aligned}$$

Finally, we develop a bound for $R_{11} + R_{21}$,

$$\begin{aligned} n(R_{11} + R_{21}) &= H(W_{11}, W_{21}) && (37)\\ &= H(W_{11}, W_{21}|Y_1^n) + I(W_{11}, W_{21}; Y_1^n) && (38)\\ &= H(W_{11}, W_{21}|Y_1^n) + h(Y_1^n) \\ &\quad - h(Y_1^n|W_{11}, W_{21}) && (39)\\ &\leq n\epsilon_n + h(Y_1^n) - h(Y_1^n|W_{11}, W_{21}) && (40)\\ &\leq n\epsilon_n + \frac{n}{2}\log(2\pi e)\left(\frac{P_1}{\alpha} + P_2 + \frac{1}{\alpha}\right) \\ &\quad - h(Y_1^n|W_{11}, W_{21}) && (41) \end{aligned}$$

where (36) and (41) follow from [11, Lemma 2].

Consider the following series of inequalities,

$$\begin{aligned} \frac{n}{2}\log(2\pi e)\left(\frac{1}{\alpha}\right) &= h(Y_1^n|W_{11}, W_{21}, W_{22}) && (42)\\ &\leq h(Y_1^n|W_{11}, W_{21}) && (43)\\ &\leq h(Y_1^n|W_{11}) && (44)\\ &\leq \frac{n}{2}\log(2\pi e)\left(P_2 + \frac{1}{\alpha}\right) && (45) \end{aligned}$$

Thus, there exists a $\beta \in [0, 1]$, such that

$$h(Y_1^n|W_{11}, W_{21}) = \frac{n}{2}\log(2\pi e)\left(\beta P_2 + \frac{1}{\alpha}\right) \quad (46)$$

From (36), (41) and (46), we see that there exists a $\beta \in [0, 1]$ such that

$$nR_{21} \leq n\epsilon_n + nc_{21}(\beta) \quad (47)$$

$$n(R_{11} + R_{21}) \leq n\epsilon_n + nc_1(\beta) \quad (48)$$

Finally, for R_{22}, we argue as follows,

$$\begin{aligned} h(Y_1^n|W_{11}, W_{21}) &= h\left(\frac{X_1^n}{\sqrt{\alpha}} + Y_2^n + \tilde{Z}^n|W_{11}, W_{21}\right) && (49)\\ &= h(Y_2^n + \tilde{Z}^n|W_{11}, W_{21}) && (50)\\ &= h(Y_2^n + \tilde{Z}^n|W_{21}) && (51) \end{aligned}$$

where (50) follows because X_1^n is a deterministic function of W_{11}, and (51) follows because Y_2^n and $\tilde{Z}^n$ are independent of W_{11}.

Now, let us consider $h(Y_2^n + \tilde{Z}^n|W_{21})$. We know that

$$h(\tilde{Z}^n|W_{21}) = h(\tilde{Z}^n) = \frac{n}{2}\log(2\pi e)\left(\frac{1}{\alpha} - 1\right) \quad (52)$$

Applying entropy power inequality [10, Lemma II], we have

$$h(Y_2^n + \tilde{Z}^n|W_{21}) \geq \frac{n}{2}\log(2\pi e)\left(\frac{e^{\frac{2}{n}h(Y_2^n|W_{21})}}{2\pi e} + \frac{1}{\alpha} - 1\right) \quad (53)$$

Combining (53) with (46) and (51), we have

$$h(Y_2^n|W_{21}) \le \frac{n}{2}\log(2\pi e)(\beta P_2+1) \tag{54}$$

Thus, from (30), we have

$$nR_{22} \le n\epsilon_n + c_{22}(\beta) \tag{55}$$

Since $\epsilon_n \to 0$ as $n \to \infty$, using (47), (48) and (55), we obtain the inequalites (18), (19) and (20), proving the theorem. ■

The converse that is missing is the part that describes R_{11}, when R_{21} is so small that $R_{11}+R_{21} < c_1(\beta)$. This will be addressed in the discussion section next by developing some lower and upper bounds on the capacity region.

V. Discussion

As stated above, combining Theorems 1 and 2, we see that the achievable region in Theorem 1 for $R_{21}, R_{22}, R_{11}+R_{21}$ is in fact tight. The only unsureness comes from R_{11}.

As mentioned in [7], the Z-channel includes the MAC, the BC and the Z-interference channel as special cases. By setting $\beta = 0$ in the achievable region in Theorem 1, we get

$$R_{11} \le \frac{1}{2}\log(1+P_1) \tag{56}$$

$$R_{21} \le \frac{1}{2}\log(1+\alpha P_2) \tag{57}$$

$$R_{11}+R_{21} \le \frac{1}{2}\log(1+P_1+\alpha P_2) \tag{58}$$

$$R_{22} = 0 \tag{59}$$

which is exactly the capacity region for the Gaussian MAC with link gains 1 and $\sqrt{\alpha}$, and noise variance 1 [1]. By setting $P_1 = 0$ in the achievable region in Theorem 1, we get

$$R_{22} \le \frac{1}{2}\log(1+\beta P_2) \tag{60}$$

$$R_{21} \le \frac{1}{2}\log\left(1+\frac{\alpha(1-\beta)P_2}{\alpha\beta P_2+1}\right) \tag{61}$$

$$R_{11} = 0 \tag{62}$$

which is exactly the capacity region for the Gaussian BC with channel gains 1 and $\sqrt{\alpha}$, and noise variance 1 [1]. By setting $R_{21} = 0$ in the capacity region of the Gaussian Z-channel, we should get the capacity region of the Gaussian Z-interference channel [4], which is still an open problem.

A. Sum Capacity of the Gaussian Z-channel

Similar to the Z-interference channel case, the sum capacity of the Gaussian Z-channel is known for $\alpha < 1$ based on the achievable region of Theorem 1 and the converse theorem, Theorem 2. The sum capacity is

$$\max_{0\le\beta\le1} c_{22}(\beta)+c_1(\beta) \tag{63}$$

It can be easily verified that when $\beta = 1$, we attain the maximum and the sum capacity for the Gaussian Z-channel when $\alpha < 1$ is

$$\frac{1}{2}\log\left(\frac{(1+P_2)(1+P_1+\alpha P_2)}{1+\alpha P_2}\right) \tag{64}$$

The sum capacity is attained at point U in Figure 2.

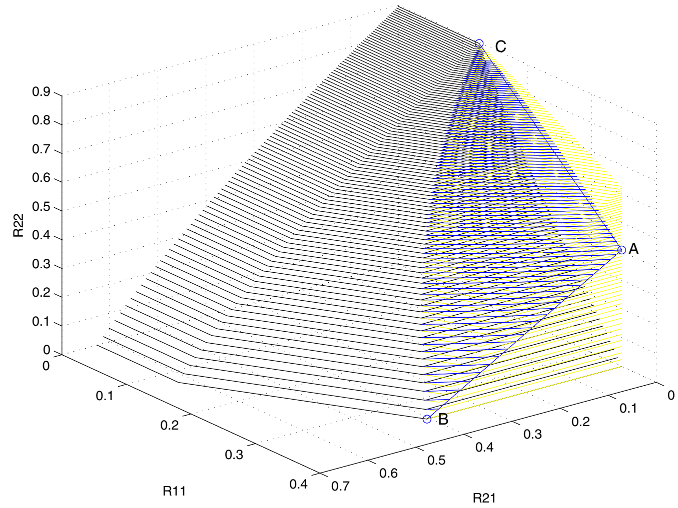

Fig. 3. All achievable regions and the upper bound in (65).

B. Lower and Upper Bounds for the Capacity Region

Next, we will derive lower and upper bounds for the capacity region for the portion where a converse is missing. An obvious upper bound for the capacity region is obtained by combining

$$R_{11} \le \frac{1}{2}\log(1+P_1) \tag{65}$$

with (10), (11) and (12) for any $0 \le \beta \le 1$. In Figure 3, the achievable region in Theorem 1 is shown in black and this upper bound is shown in yellow.

Two other achievable regions can be derived to close the gap between the lower and upper bounds on the capacity region.

Larger Achievable Region 1: It is clear that the following three points given as triplets of (R_{11}, R_{21}, R_{22}) are achievable.

$$\text{Point } A\text{: } \left(\frac{1}{2}\log(1+P_1), 0, \frac{1}{2}\log\left(1+\frac{\alpha P_2}{P_1+1}\right)\right) \tag{66}$$

$$\text{Point } B\text{: } \left(\frac{1}{2}\log(1+P_1), \frac{1}{2}\log\left(1+\frac{\alpha P_2}{P_1+1}\right), 0\right) \tag{67}$$

$$\text{Point } C\text{: } \left(\frac{1}{2}\log\left(1+\frac{P_1}{\alpha P_2+1}\right), 0, \frac{1}{2}\log(1+P_2)\right) \tag{68}$$

These three points are shown in Figure 3. Joining the lines between points A and B and points A and C, and the curve connecting points B and C, we can obtain a plane which is achievable by time sharing. It is also worth noting that the line AB is optimal by the same argument as in [4, Theorem 1], i.e., line AB is achievable and no point above line AB on the plane of $R_{11} = \frac{1}{2}\log(1+P_1)$ is achievable.

Larger Achievable Region 2: Using the technique of successive decoding [3], we can split X_2 into three parts:

$$X_2 = X_{21} + X_{comm} + X_{priv} \tag{69}$$

where X_{21} is a function of message W_{21} and $X_{comm} + X_{priv}$ together carry message W_{22}. Let X_{comm} and X_{priv} be independent. X_{comm} is intended to be decoded by both receiver R1 and receiver R2, even though receiver R1 is not interested in decoding any part of message W_{22}. X_{priv} is decoded by receiver R2 only. Receiver R1 treats X_{priv} as noise. Transmitter T2 uses power $\bar{\gamma}P_2$ for X_{21}, power $\mu\gamma P_2$ for X_{comm}, and power $\bar{\mu}\gamma P_2$ for X_{priv}, where $\bar{\gamma} = 1 - \gamma$, $\bar{\mu} = 1 - \mu$ and γ and μ vary from 0 to 1.

Receiver R1 uses decoding order X_{21}, then X_{comm} and finally X_1, and receiver R2 uses decoding order X_{21}, then X_{comm} and finally X_{priv}. Let $\mathcal{A}1(\mu, \gamma)$ be the set of R_{11}, R_{21} and R_{22} that satisfies the following inequalities:

$$R_{21} \leq \frac{1}{2} \log \left(1 + \frac{\bar{\gamma}P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \gamma P_2}\right) \tag{70}$$

$$R_{comm} \leq \frac{1}{2} \log \left(1 + \frac{\mu\gamma P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \bar{\mu}\gamma P_2}\right) \tag{71}$$

$$R_{11} \leq \frac{1}{2} \log \left(1 + \frac{\frac{P_1}{\alpha}}{\frac{1}{\alpha} + \bar{\mu}\gamma P_2}\right) \tag{72}$$

$$R_{priv} \leq \frac{1}{2} \log \left(1 + \bar{\mu}\gamma P_2\right) \tag{73}$$

$$R_{22} = R_{comm} + R_{priv} \tag{74}$$

Receiver R1 uses decoding order X_{comm}, then X_{21} and finally X_1, and receiver R2 uses decoding order X_{comm}, then X_{21} and finally X_{priv}. Let $\mathcal{A}2(\mu, \gamma)$ be the set of R_{11}, R_{21} and R_{22} that satisfies the following inequalities:

$$R_{comm} \leq \frac{1}{2} \log \left(1 + \frac{\mu\gamma P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \bar{\mu}\gamma P_2 + \bar{\gamma}P_2}\right) \tag{75}$$

$$R_{21} \leq \frac{1}{2} \log \left(1 + \frac{\bar{\gamma}P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \bar{\mu}\gamma P_2}\right) \tag{76}$$

$$R_{11} \leq \frac{1}{2} \log \left(1 + \frac{\frac{P_1}{\alpha}}{\frac{1}{\alpha} + \bar{\mu}\gamma P_2}\right) \tag{77}$$

$$R_{priv} \leq \frac{1}{2} \log \left(1 + \bar{\mu}\gamma P_2\right) \tag{78}$$

$$R_{22} = R_{comm} + R_{priv} \tag{79}$$

Then, an achievable region for the part where the converse is missing is the convex hull of

$$\left(\bigcup_{0 \leq \mu, \gamma \leq 1} \mathcal{A}1(\mu, \gamma)\right) \bigcup \left(\bigcup_{0 \leq \mu, \gamma \leq 1} \mathcal{A}2(\mu, \gamma)\right) \tag{80}$$

Figure 3 shows *Larger Achievable Region 2* and the lines AB and AC defined in *Larger Achievable Region 1* in blue. As we can see, there is still a gap between the lower and upper bounds, and additional research is needed to find the exact capacity region. We would like to mention here that using a coding scheme similar to [12], we would get an even larger achievable region than *Larger Achievable Region 2*.

C. *The Capacity Region when $\alpha = 1$*

Finally, it is worth noting that the Gaussian Z-channel with $\alpha < 1$ has the same capacity region as the channel in Figure 4 where Z and Z_2 are zero-mean Gaussian random variables

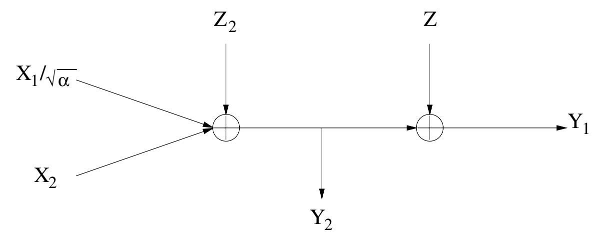

Fig. 4. The equivalent channel.

with variance $\frac{1}{\alpha} - 1$ and 1, respectively. This can be proved by the argument in [4, Appendix A]. Noting that the two channels have the same capacity is useful, since the capacity region of the channel in Figure 4 might be easier to determine in some cases. For example, for $\alpha = 1$, Y_1 and Y_2 are statistically equivalent, thus both receiver R1 and receiver R2 are able to decode all three messages, W_{11}, W_{12} and W_{22}, similar to a MAC. Thus, the capacity region of the Gaussian Z-channel with $\alpha = 1$ is

$$R_{11} \leq \frac{1}{2} \log(1 + P_1) \tag{81}$$

$$R_{21} + R_{22} \leq \frac{1}{2} \log(1 + P_2) \tag{82}$$

$$R_{11} + R_{21} + R_{22} \leq \frac{1}{2} \log(1 + P_1 + P_2) \tag{83}$$

VI. CONCLUSION

In this paper, we provided an achievable region for the recently proposed Gaussian Z-channel when $\alpha < 1$. We were able to prove most of the converse for this achievable region. We also provided an upper bound and two larger achievable regions to characterize the capacity region better. We determined the exact capacity region when $\alpha = 1$.

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