Cyclic Scheduling for Age of Information Minimization with Generate at Will Status Updates

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***Abstract*—We study the scheduling problem in a status update system composed of multiple information sources with different service time distributions and weights, for the purpose of minimizing the weighted sum age of information (AoI). In particular, we study *open-loop* schedulers which rely only on the statistics (specifically, only on the first two moments) of the source service times, in contrast to closed-loop schedulers that also make use of the actual realizations of the service times and the AoI processes in making scheduling decisions. We consider the generate-at-will (GAW) model, and develop an analytical method to calculate the exact AoI for probabilistic and cyclic open-loop schedulers. In both cases, the server initiates the sampling of a source and the ensuing transmission of the update packet from the source to the server in an open-loop manner; either based on a certain probability (probabilistic scheme) or according to a deterministic cyclic pattern (cyclic scheme). We derive the optimum open-loop cyclic scheduling policy in closed form for the specific case of $N = 2$ sources and propose well-performing heuristic cyclic schedulers for general number of sources, i.e., $N > 2$. Numerical examples are provided to validate the existing methods.**

I. Introduction

Timely delivery of time-sensitive updates plays a vital role in networked control and remote monitoring systems. In multi-source status update systems, for each information source-n, $n = 1, 2, \dots, N$, an associated random process is sampled with the sample values written into information packets. These packets are subsequently collected by a server to be forwarded towards the destination(s), e.g., a base station (BS), of the information sources [1], [2]. A widely studied instrument to quantify timeliness is the age of information (AoI) process for source-n which is defined as the random process $\Delta_n(t) = t - u_n(t)$ where $u_n(t)$ is the generation time of the last status update packet received at the destination from source-n. Sample paths of the AoI process $\Delta_n(t)$ increase in time with unit slope with abrupt drops at packet reception instances. In the generate-at-will (GAW) model, the server decides when a source is sampled along with the transmission of its status update packet.

In this paper, we study the scheduling of the transmission of time-sensitive information packets generated by source-n, $n = 1, 2, \dots, N$, towards the BS of a cellular network for the GAW scenario in a continuous-time setting; see Fig. 1. The information sources use different modulation and coding schemes depending on their distance from the BS, and therefore, have heterogeneous (or asymmetric) packet service times denoted by random variable S_n, with mean s_n and second moment q_n. We also assume that the deployed modulation, coding, and ARQ schemes ensure that packet transmission errors are negligible.

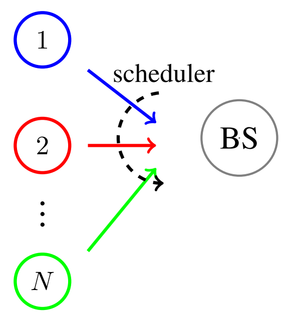

Fig. 1: Information packets from N information sources are collected by the base station (BS).

In order to provide system level freshness, the goal of the scheduler is to minimize the weighted sum AoI, or system AoI, which is defined as the weighted sum of the mean values of the individual AoI processes. We note that the general service time model we assume in this paper is different than the widely used deterministic service time model used for slotted systems, which are studied extensively in the existing AoI literature with heterogeneous packet error probabilities; see [1]–[3]. We note that the latter model becomes a sub-case of the former one when the same packet is to be re-transmitted at each instant the packet is errored.

The scheduling problem with an age minimization goal is known to be an NP-hard problem in general heterogeneous multi-source settings [4]. However, some special cases are shown to be tractable in [4] along with their optimality conditions. Two main types of schedulers stand out in the literature, namely, closed-loop and open-loop schedulers. Closed-loop schedulers make use of the actual realizations of the service times and/or of the resultant individual AoI processes in making scheduling decisions. For example, age-aware schedulers

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such as max-weight or Whittle-index policies (see [3], [5]) fall into this category. On the other hand, open-loop schedulers are age-agnostic and they rely only on the apriori statistical information related to the per-source service times in addition to source weights. Open-loop schedulers can be probabilistic or cyclic, depending on whether the scheduler chooses to serve information sources with certain probabilities [6], or according to a fixed pattern of transmissions which repeats itself [7]. Open-loop scheduling policies can be developed offline and their implementation complexities are much lower than their closed-loop counterparts. The focus of this paper is on open-loop scheduling for system AoI minimization.

Our work is most closely related to [7] where a general design framework (Eywa) is proposed to build high-performance cyclic schedulers for AoI-related optimization and decision problems. However, [7] is based on a discrete-time setting with deterministic service times (one slot duration) whereas in our work, we study generally distributed service times in a continuous-time setting. Our model may be more suitable for physical layer networks for which users can choose an appropriate modulation, coding, and ARQ scheme depending on their distance from the BS.

Our main contributions are: 1) We propose a method to calculate the mean AoI of an information source for a probabilistic scheduler with given transmission probabilities, and for a cyclic scheduler with given transmission patterns, for GAW systems, using only the first two moments of the service times. 2) We also derive the optimum cyclic scheduler for the case of two sources. We show that the optimum schedule is in the form of transmitting a number of subsequent updates for one of the sources followed by transmitting a single update for the other source. We obtain the parameters of this policy in closed-form, and show that they depend only on the first and second moments of the source service times in addition to the source weights. 3) We also propose a heuristic search-based cyclic scheduler for general number of information sources.

II. System Model

We consider the status update system in Fig. 1 with N information sources indexed by $n = 1, 2, \ldots, N$ each of which generating samples of an associated random process with the sampled values carried inside the information packets to be sent to a BS which is the destination of the status update system. Service times of source-n packets, denoted by S_n, time required for transmission of source-n information packets to the BS, are i.i.d. and have a general distribution with moment generating function (MGF) denoted by $G_n(s) = \mathbb{E}[e^{sS_n}]$, mean $s_n = \mathbb{E}[S_n] = G'_n(0)$, second moment $q_n = \mathbb{E}[S_n^2] = G''_n(0)$, variance $v_n = q_n - s_n^2$, and squared coefficient of variation $c_n = \frac{v_n}{s_n^2}$. The per-source service times are independent of each other and there are no packet errors. We let Δ_n denote the steady-state random variable for the associated AoI process observed at the BS for source-n, $n = 1, \ldots, N$. We define the system AoI as the mean of the weighted age random variable $\Delta = \sum_{n=1}^{N} w_n \Delta_n$,

$$\mathbb{E}[\Delta] = \sum_{n=1}^{N} w_n \mathbb{E}[\Delta_n], \tag{1}$$

where the source weights w_ns reflect the relative urgency of the individual sources.

The probabilistic GAW (P-GAW) server chooses an information source for sampling and transmission with probability p_n with $\sum_{n=1}^{N} p_n = 1$, i.e., work-conserving server, at a scheduling instant. Once the transmission of a source-n packet is completed (which requires a duration of S_n), a new scheduling instant is initiated.

The cyclic GAW (C-GAW) server works with a given cycle length $K \geq N$ and a pattern row vector (or pattern in short) with $P = [P_0, P_1, \ldots, P_{K-1}]$ of size K such that $P_k \in \{1, 2, \ldots, N\}$. The resulting C-GAW scheduler initiates the sampling and transmission of source-P_k information packet at scheduling instants $k + iK$, $i \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of non-negative integers. Let $\alpha_P(n)$, $n = 1, \ldots, N$ denote the number of times that source-n appears in the pattern vector P. Note that $\alpha_P(n) > 0$, for all n, for the pattern to be feasible, i.e., yield a bounded weighted AoI. For convenience, we denote by s_P, v_P and q_P, the mean, variance, and the second moment, respectively, of the sum of the service times needed to transmit the entire pattern P,

$$s_P = \sum_{i=0}^{K-1} s_{P_i}, \quad v_P = \sum_{i=0}^{K-1} v_{P_i}, \quad q_P = v_P + s_P^2, \tag{2}$$

as individual service times within a pattern are independent.

III. Analysis of AoI with Open-loop Scheduling

In this section, we provide expressions for the mean AoI for source-$n, \mathbb{E}[\Delta_n]$, for P-GAW and C-GAW open-loop schedulers.

Let $S_{n,k}$ denote the service time of the kth transmission of source-n and let $\tilde{S}_{n,k}$ denote the time duration between the end of the kth transmission and the beginning of the $(k+1)$st transmission of source-n. Note that $\tilde{S}_{n,k}$ is the sum of the service times of all information packets generated from sources other than source-n between two successive transmissions of source-n. Common to both types of schedulers, the kth cycle of the AoI process $\Delta_n(t)$ starts from the value $S_{n,k}$ and increases with unit slope for a duration of $\tilde{S}_{n,k} + S_{n,k+1}$ until it hits the value $S_{n,k} + \tilde{S}_{n,k} + S_{n,k+1}$ at which point a new cycle is initiated which will start from the value $S_{n,k+1}$. The duration of, and the area under $\Delta_n(t)$ throughout the kth cycle, denoted by $T_{n,k}$ and $A_{n,k}$, respectively, are

$$T_{n,k} = \tilde{S}_{n,k} + S_{n,k+1}, \tag{3}$$

$$A_{n,k} = S_{n,k}(\tilde{S}_{n,k} + S_{n,k+1}) + \frac{1}{2}(\tilde{S}_{n,k} + S_{n,k+1})^2. \tag{4}$$

Note that the asymptotically stationary discrete-time random process $T_{n,k}$ has a limiting marginal distribution, i.e., $T_{n,k} \sim T_n$, i.e., $\lim_{k\to\infty} \mathbb{P}[T_{n,k} \leq x] = \mathbb{P}[T_n \leq x]$, for all $x \geq 0$. Similarly, $A_{n,k} \sim A_n$ and $\tilde{S}_{n,k} \sim \tilde{S}_n$, for the random variables A_n and $\tilde{S}_n$, respectively. Let random variable $\tilde{S}_n$ have the MGF $\tilde{G}_n(s) = \mathbb{E}[e^{s\tilde{S}_n}]$, mean $\tilde{s}_n = \mathbb{E}[\tilde{S}_n]$, and second moment $\tilde{q}_n = \mathbb{E}[\tilde{S}_n^2]$.

The mean AoI for source-n, $\mathbb{E}[\Delta_n]$ is then written as the following ratio, see e.g., [1],

$$\mathbb{E}[\Delta_n] = \frac{\mathbb{E}[A_n]}{\mathbb{E}[T_n]}, \tag{5}$$

which then can be written in terms of the parameters s_n, q_n, $\tilde{s}_n$, and $\tilde{q}_n$ as

$$\mathbb{E}[\Delta_n] = \frac{2s_n^2 + 4s_n\tilde{s}_n + q_n + \tilde{q}_n}{2(s_n + \tilde{s}_n)}. \tag{6}$$

In order to find $\mathbb{E}[\Delta_n]$, we need to calculate the first two moments of $\tilde{S}_n$, which depend on how the scheduling actions are made. In the next two sub-sections, we will specialize the general eqn. (6) for P-GAW and C-GAW scheduling schemes.

A. P-GAW Scheduling

We first focus on P-GAW and source-n. Between every two successive source-n transmissions, we have $K_n \geq 0$ transmissions from other sources where K_n has a delayed geometric distribution with parameter p_n. Hence, $H_n(z) = \mathbb{E}[z^{K_n}] = \frac{p_n}{1-(1-p_n)z}$, which is the probability generating function (PGF) of K_n. Note that $\tilde{S}_n$ corresponds to the sum of the service times of K_n transmissions from all the sources other than source-n. Each of the K_n transmissions belongs to source-m with probability $p_m/(1-p_n)$, $m \neq n$. Let V_n denote the service time of any one of these transmissions. Therefore, using the method of collective marks, we write the MGF of $\tilde{S}_n$ as follows,

$$\tilde{G}_n(s) = \frac{p_n}{1-(1-p_n)\mathbb{E}[e^{sV_n}]} = \frac{p_n}{1-\sum_{m\neq n} p_m G_m(s)}. \tag{7}$$

We differentiate $\tilde{G}_n(s)$ twice to obtain $\tilde{s}_n$ and $\tilde{q}_n$,

$$\tilde{s}_n = \tilde{G}'_n(0) = \frac{\sum_{m\neq n} p_m s_m}{p_n} \tag{8}$$

$$\tilde{q}_n = \tilde{G}''_n(0) = \frac{\sum_{m\neq n} p_m q_m}{p_n} + \frac{2(\sum_{m\neq n} p_m s_m)^2}{p_n^2}. \tag{9}$$

Substituting the expressions (8) and (9) into (6), we obtain the mean AoI for any of the sources for P-GAW.

B. C-GAW Scheduling

Recall that for a given pattern P with pattern length K, source-n appears $\alpha_P(n)$ times in the frame. Let $\tilde{P}(n,k)$, $k = 0, 1, \ldots, \alpha_P(n)-1$, denote the sub-pattern obtained by deleting all entries in the original pattern except for the entries between the kth and $(k+1)$st (modulo $\alpha_P(n)$) appearances of source-n, excluding the end points. As an example, consider source-1 for a scenario with $N = 3$ and

$$K = 7, \quad P = \begin{bmatrix} 3 & 1 & 2 & 3 & 1 & 3 & 2 \end{bmatrix}. \tag{10}$$

In this case, $\alpha_P(1) = 2$, and we have two sub-patterns for source-1,

$$\tilde{P}(1,0) = \begin{bmatrix} 2 & 3 \end{bmatrix}, \quad \tilde{P}(1,1) = \begin{bmatrix} 3 & 2 & 3 \end{bmatrix}. \tag{11}$$

Notice that sub-patterns are not feasible. Recalling the definitions in (2) (which also apply to non-feasible patterns), it is not difficult to write

$$\tilde{s}_n = \frac{1}{\alpha_P(n)} \sum_{k=1}^{\alpha_P(n)} s_{\tilde{P}(n,k-1)}, \quad \tilde{q}_n = \frac{1}{\alpha_P(n)} \sum_{k=1}^{\alpha_P(n)} q_{\tilde{P}(n,k-1)}. \tag{12}$$

Similar to P-GAW, we obtain $\mathbb{E}[\Delta_n]$ for C-GAW by substituting the expressions (12) into (6).

IV. Optimum C-GAW Scheduling for Two Sources

In this section, we have $N = 2$ sources. Let us start with an arbitrary pattern P having cycle length $K \geq 2$ composed of $K_1 \geq 1$ and $K_2 \geq 1$ updates, for source-1 and source-2, respectively, with $K = K_1 + K_2$. We define a placement vector r (resp. z) of size K_1 (resp. K_2) such that r_k, $k = 0, 1, \ldots, K_1 - 1$ (resp. z_k, $k = 0, 1, \ldots, K_2 - 1$) is equal to the number of source-2 (resp. source-1) transmissions that take place between the kth and $(k+1)$st modulo K_1 (resp. modulo K_2) appearances of source-1 (resp. source-2) in the pattern P. Note that r_k (resp. z_k) corresponds to the length of the sub-pattern $\tilde{P}(1,k)$ (resp. $\tilde{P}(2,k)$).

Subsequently, the C-GAW scheduling policy for $N = 2$ sources is characterized with the triple $\pi = (K_1, K_2, r) \in \Pi$, $K_1 \geq 1$, $K_2 \geq 1$, and $\sum_{k=1}^{K_1} r_{k-1} = K_2$, where Π denotes all feasible scheduling policies. When a policy π is employed at the C-GAW server, the system AoI obtained with this policy is defined as $\mathbb{E}[\Delta^\pi] = w_1\mathbb{E}[\Delta_1^\pi] + w_2\mathbb{E}[\Delta_2^\pi]$, where the term $\mathbb{E}[\Delta_n^\pi]$ is the mean AoI for source-n attained under policy π. An optimum update policy, denoted by π^*, is one that minimizes the system AoI among all feasible policies $\pi \in \Pi$, and is not necessarily unique.

The next lemma is crucial for obtaining the optimum policy.

Lemma 1. *For a two-source C-GAW server, and for a given scheduling pattern P with cycle length K and number of source-1 and source-2 updates K_1 and K_2, such that $K = K_1 + K_2$, and $\gamma = K/K_1$, the optimum placement vector for source-1 $r^* = [r_0^*, \ldots, r_{K_1-1}^*]$ which jointly minimizes the mean AoI for both sources is given as follows. When $\gamma \in \mathbb{Z}$,*

$$r_{k-1}^* = \gamma - 1, \quad 1 \leq k \leq K_1, \tag{13}$$

and when $\gamma \notin \mathbb{Z}$,

$$r_{k-1}^* = \begin{cases} \lfloor\gamma\rceil - 1, & 1 \leq k \leq K_1\lceil\gamma\rceil - K, \\ \lfloor\gamma\rfloor, & K_1\lceil\gamma\rceil - K < k \leq K_1. \end{cases} \tag{14}$$

Proof. Without loss of generality, let us assume $K_1 \leq K_2$. In order to find $\mathbb{E}[\Delta_1]$, we employ (6), and also (12) together with (2), for the special case of $N = 2$, to first write

$$\tilde{s}_1 = \frac{K_2 s_2}{K_1}, \quad \tilde{q}_1 = \frac{K_2 v_2}{K_1} + \frac{s_2^2}{K_1}\sum_{k=1}^{K_1} r_{k-1}^2, \tag{15}$$

in order to write $\mathbb{E}[\Delta_1] =$

$$\frac{K_1(2s_1^2+q_1) + K_2(4s_1s_2+q_2-s_2^2) + s_2^2 \sum_{k=1}^{K_1} r_{k-1}^2}{2(K_1s_1 + K_2s_2)} \quad (16)$$

To minimize (16), it is sufficient to minimize the expression $\sum_{k=1}^{K_1} r_{k-1}^2$ under the constraint $\sum_{k=1}^{K_1} r_{k-1} = K_2$, one solution to which is given in (13)-(14). Similarly, it can be shown that $\mathbb{E}[\Delta_2]$ is minimized if the entries in its placement vector are either 0 or 1 since $K/K_2 \leq 2$ and the number of 1s should be K_1. Obviously, such selection of placement vector z for source-2 is already achieved with the selection of placement vector r for source-1 as given in (13)-(14). Thus, the selection of placement vector for source-1 in (13)-(14) simultaneously minimizes expected ages of both sources, completing the proof. □

From Lemma 1, we focus only on policies $\pi = (K_1, K_2) \in \Pi', K_n \geq 1, n = 1, 2,$ that use the optimum placement vector, which is equivalent to the policy $\pi = (K_1, K_2, r)$ where the placement vector r is employed according to Lemma 1. The next lemma further narrows down the candidate set of optimum policies.

Lemma 2. *For a two-source C-GAW server, the optimum policy is either in the form $\pi = (K_1, 1)$ or $(1, K_2)$.*

Proof. Let us be given a policy $\pi = (K_1, K_2) \in \Pi'$ with $K = K_1 + K_2$, $K_1 \leq K_2$ and $\gamma = K/K_1$ with the placement vector r obtained according to Lemma 1. When $\gamma \in \mathbb{Z}$, then the policy (K_1, K_2) is already equivalent to the policy $(1, \gamma - 1)$.

Now, consider the case $\gamma \notin \mathbb{Z}$. Let us define two auxiliary policies $\pi_a = (1, \lfloor \gamma \rfloor - 1)$ and $\pi_b = (1, \lfloor \gamma \rfloor)$. The system AoI obtained with the policy π can be written as,

$$\mathbb{E}[\Delta^\pi] = \frac{t_a \mathbb{E}[\Delta^{\pi_a}] + t_b \mathbb{E}[\Delta^{\pi_b}]}{t_a + t_b}, \quad (17)$$

where

$$t_a = (s_2(\lfloor \gamma \rfloor - 1) + s_1)\,(K_1 \lceil \gamma \rceil - K)\,, \quad (18)$$

$$t_b = (s_2(\lceil \gamma \rceil - 1) + s_1)\,(K_1 - K_1 \lceil \gamma \rceil + K)\,. \quad (19)$$

Note that (17) immediately reveals that either π_a or π_b should give a lesser system AoI than the original policy $\pi = (K_1, K_2)$ from which we conclude that the optimum policy must be in the form $(1, K_2)$. When $K_1 > K_2$, one can swap the sources and follow the same lines of the above proof to show that the optimum policy can alternatively be in the form $(K_1, 1)$. □

Let us first focus only on the policies of the form $\pi = (K_1, 1)$ and apply convex optimization to find the best value of K_1, denoted by K_1^*, that minimizes the system AoI. For this purpose, the mean AoI for each of the two sources obtained with policy $(K_1, 1)$ are first written in terms of the parameter K_1 using (6) and (16) as,

$$\mathbb{E}[\Delta_1] = \frac{K_1(2s_1^2 + q_1) + 4s_1s_2 + q_2}{2(K_1s_1 + s_2)}, \quad (20)$$

$$\mathbb{E}[\Delta_2] = \frac{K_1^2s_1^2 + K_1(4s_1s_2 + q_1 - s_1^2) + 2s_2^2 + q_2}{2(K_1s_1 + s_2)}. \quad (21)$$

Relaxing the integer nature of K_1, we write the first and second derivatives of the system AoI as a function of $x \in \mathbb{R}$ by replacing each occurrence of K_1 in the system AoI expression by the real-valued variable x,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}[\Delta] = \frac{w_2s_1}{2} - \frac{w_2s_1\psi_1}{2(s_2 + xs_1)^2}, \quad (22)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathbb{E}[\Delta] = \frac{w_2s_1^2\psi_1}{(s_2 + xs_1)^3}, \quad (23)$$

where

$$\psi_1 = \frac{1}{s_1w_2}(s_1q_2 - q_1s_2 + (w_1 + 1)s_1^2s_2 - w_2s_2^2s_1). \quad (24)$$

We note that if $\psi_1 < 0$, then from (22), the function $\mathbb{E}[\Delta]$ is a monotonically increasing function of x in which case $K_1^* = 1$. On the other hand, if ψ_1 is positive, then from (23), the function $\mathbb{E}[\Delta]$ is a convex function of x, and using the KKT optimality conditions [8], we find the value of the parameter x, denoted by x^*, for which the expression in (22) when evaluated at x^* becomes zero. Consequently, we obtain

$$x^* = \frac{\sqrt{\psi_1} - s_2}{s_1}. \quad (25)$$

If $x^* \leq 1$ which occurs when $\psi_1 \leq (s_1 + s_2)^2$, then $K_1^* = 1$. Otherwise, if $x^* \in \mathbb{Z}$, then $K_1^* = x^*$ and if $x^* \notin \mathbb{Z}$ then K_1^* is either $\lfloor x^* \rfloor$ or $\lceil x^* \rceil$ depending on which of the two choices yields a lower system AoI.

We now repeat the same analysis for policies in the form of $\pi = (1, K_2)$ and obtain K_2^*, that minimizes the system AoI for all such policies. For this purpose, we write the mean AoI for the two sources obtained with the policy $(1, K_2)$ using (6) and (16) as,

$$\mathbb{E}[\Delta_1] = \frac{K_2^2s_2^2 + K_2(4s_1s_2 + q_2 - s_2^2) + 2s_1^2 + q_1}{2(K_2s_2 + s_1)}, \quad (26)$$

$$\mathbb{E}[\Delta_2] = \frac{K_2(2s_2^2 + q_2) + 4s_1s_2 + q_1}{2(K_2s_2 + s_1)}. \quad (27)$$

Relaxing the integer nature of K_2, we write the first and second derivatives of the system AoI as a function of $y \in \mathbb{R}$ upon replacing each occurrence of K_2 in the system AoI expression by the real-valued variable y,

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}[\Delta] = \frac{w_1s_2}{2} - \frac{w_1s_2\psi_2}{2(s_1 + ys_2)^2}, \quad (28)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\mathbb{E}[\Delta] = \frac{w_1s_2^2\psi_2}{(s_1 + ys_2)^3}, \quad (29)$$

where

$$\psi_2 = \frac{1}{s_2w_1}(s_2q_1 - q_2s_1 + (w_2 + 1)s_2^2s_1 - w_1s_1^2s_2). \quad (30)$$

Similar to the previous analysis, if $\psi_2 < 0$, from (28), the function $\mathbb{E}[\Delta]$ is a monotonically increasing function of y in which case $K_2^* = 1$. If ψ_2 is positive, then from (29), the function $\mathbb{E}[\Delta]$ is a convex function of y. Consequently, we obtain the value of the parameter y, denoted by y^*, for which the expression (28) when evaluated at y^* becomes zero,

$$y^* = \frac{\sqrt{\psi_2} - s_2}{s_1}. \qquad (31)$$

If $y^* \leq 1$ which occurs when $\psi_2 \leq (s_1 + s_2)^2$, then $K_2^* = 1$. Otherwise, if $y^* \in \mathbb{Z}$, then $K_2^* = y^*$ and when $y^* \notin \mathbb{Z}$, then K_2^* is either $\lfloor y^* \rfloor$ or $\lceil y^* \rceil$ depending on which of the two choices yields a lower system AoI. Finally, the optimum policy is either $(K_1^*, 1)$ or $(1, K_2^*)$, depending on which one gives rise to a lower system AoI.

Theorem 1 below provides a closed-form expression for the optimum C-GAW server for the case of two sources.

Theorem 1. *Consider the system AoI minimization problem for the two-source C-GAW server with the expressions for the parameters ψ_1, x^*, ψ_2, and y^* given in equations* (24)*,* (25)*,* (30)*, and* (31)*, respectively. The inequality $\psi_n > (s_1 + s_2)^2$ cannot hold for the two sources simultaneously. If $\psi_n \leq (s_1 + s_2)^2$, $n = 1, 2$, then the round robin policy* $(1, 1)$ *is the optimum policy. If $\psi_1 > (s_1 + s_2)^2$, then $(1, \lfloor x^* \rfloor)$ or $(1, \lceil x^* \rceil)$ is the optimum policy depending on which one results in a lower system AoI using* (20) *and* (21)*. Similarly, if $\psi_2 > (s_1 + s_2)^2$, then $(\lfloor y^* \rfloor, 1)$ or $(\lceil y^* \rceil, 1)$ is the optimum policy depending on which one results in a lower system AoI using* (26) *and* (27)*.*

Proof. We first prove by contradiction that $\psi_n > (s_1 + s_2)^2$ cannot hold for both values of n. Let us first assume,

$$\psi_n = a_n (s_1 + s_2)^2, \qquad (32)$$

where $a_n > 1$ for $n = 1, 2$. It is not difficult to show using (24) and (30) that the following holds,

$$\psi_1 w_2 s_1 + \psi_2 w_1 s_2 = s_1 s_2 (s_1 + s_2), \qquad (33)$$

Substituting (32) in the expression in (33), we obtain

$$\frac{a_2(s_1 + s_2)}{s_1} w_1 + \frac{a_1(s_1 + s_2)}{s_2} w_2 = 1, \qquad (34)$$

which cannot hold since $w_1 + w_2 = 1$, and therefore, the left hand side of (34) must be strictly larger than 1 contradicting the assumption (32). The rest of the proof simply follows from the derivations from (20) to (31). □

V. Insertion Search (IS) Scheduling Algorithm for Multiple Sources

In the previous section, we found the optimum scheduler for $N = 2$ sources for C-GAW. Optimum scheduling for general $N > 2$ sources for C-GAW remains an open problem. In this section, we propose a heuristic search-based scheduler called insertion search (IS) for general numbers of sources. For the description of the IS algorithm, we first define the insertion operator $g(P, n, k)$, for $1 \leq n \leq N$, $0 \leq k < K$, which maps the original frame P with cycle length K, to a new pattern P' with cycle length $K + 1$, which is obtained by inserting a source-n transmission before the kth element of P for $k = 0, 1, \ldots, K - 1$. As an example, let P be given as in (10). Then, $g(P, 1, 6)$ is obtained by inserting a source-1 transmission just before the 6th entry of P,

$$g(P, 1, 6) = \begin{bmatrix} 3 & 1 & 2 & 3 & 1 & 3 & 1 & 2 \end{bmatrix}. \qquad (35)$$

Given these definitions, we now describe the IS algorithm. For this purpose, we first start with the round robin (RR) policy (sources are served one after the other, in a repeating sequence) as our initial policy which results in an initial pattern P such that $P_i = i + 1, 0 \leq i \leq N - 1$ with cycle length N which gives the minimum system AoI among all cyclic schedulers with cycle length less than or equal to N. This stems from the observation that patterns with cycle length strictly less than N are not feasible, and the RR policy is the only feasible policy with cycle length N. Subsequently, at each iteration of our proposed algorithm, our goal is to find the lowest system AoI pattern $P' = g(P, n, k)$ (for some n, k) with cycle length $K + 1$, using the pattern P with cycle length K, through the use of insertion operations only. Finding the best such pattern P' in terms of system AoI is done by searching exhaustively through each of the candidate sources for insertion and their insertion position, followed with the computation of system AoI for each candidate pattern. Note that, this step requires the computation of system AoI of less than NK patterns, since the patterns $g(P, n, k)$ and $g(P, n, k + 1)$ (modulo K) are equivalent when $P_k = n$, and there is no need to calculate the system AoI for both. If we can reduce the system AoI by increasing the cycle length through insertion operations, we repeat the process. This process is terminated if the maximum cycle length K_{max} is reached, or system AoI can not be reduced by increasing the cycle length by one.

VI. Numerical Examples

In the first example, we focus on the case $N = 2$, for which we have obtained the optimum cyclic scheduler in Section IV. In particular, we consider the GAW model with exponentially distributed service times, i.e., $q_n = 2s_n^2$, $n = 1, 2$, for the case $s_1 = 5$, $w_1 = 0.8$ and $w_2 = 0.2$. C-GAW* refers to the optimum policy given in Theorem 1. For benchmarking purposes, we also study the RR and the P-GAW* policies, the latter obtained by solving the system AoI for P-GAW using the technique described in Section III and employing exhaustive search over the transmission probabilities to obtain the optimum probabilities p_n^*, $n = 1, 2$, which yields the minimum system AoI among all possible probabilistic schedulers.

The system AoI is depicted in Fig. 2(a) as a function of s_2 for the three cases: P-GAW*, RR, and C-GAW*. When C-GAW* deviates from RR, we let n' denote the source which requires $K_{n'}^* > 1$ transmissions for the C-GAW* server, whereas when they are equal, we fix $n' = 1$ and $K_{n'}^* = 1$. The parameter p_1^* for P-GAW* and the parameter $K_{n'}^*$ for C-GAW* are depicted in Fig. 2(b). Fig. 2(a) reveals that the system AoI obtained with RR is lower than that of the optimum probabilistic P-GAW* server for a wide range of s_2 values despite the fact that RR server does not take into consideration the individual source weights or service time

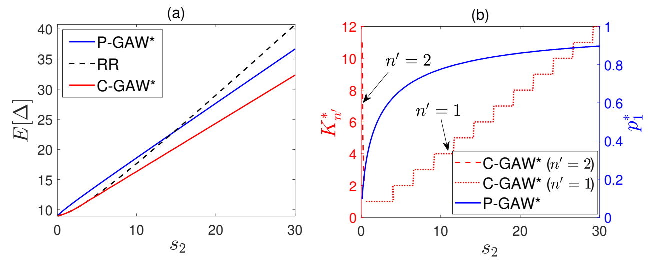

Fig. 2: (a) System AoI $\mathbb{E}[\Delta]$, (b) p_1^* for P-GAW* and $K_{n'}^*$ for C-GAW*, depicted as a function of s_2 for exponentially distributed service times, when $w_1 = 0.8$, $w_2 = 0.2$, $s_1 = 5$.

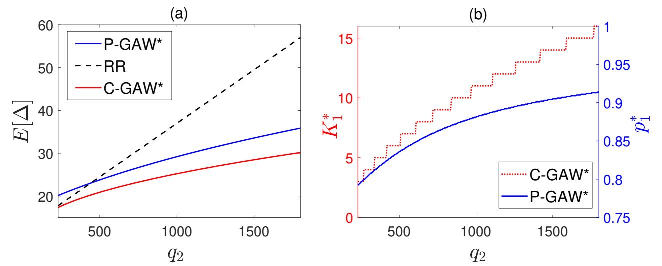

Fig. 3: (a) System AoI $\mathbb{E}[\Delta]$ (b) p_1^* for P-GAW* and K_1^* for C-GAW* ($n' = 1$ for this example in all cases), depicted as a function of $q_2 \geq 225$, when $w_1 = 0.8, w_2 = 0.2, s_1 = 5, s_2 = 15$.

statistics. Moreover, the system AoI obtained with the C-GAW* server is consistently lower than that of the other two servers and this performance gap also grows with increasing s_2. We conclude from this example that it is beneficial to use cyclic scheduling when compared to probabilistic and round robin scheduling for weighted sum AoI minimization.

In the second example, we use the same weights as before and exponentially distributed service times for source-1 with $s_1 = 5$. However, the service times for source-2 are now generally distributed with mean $s_2 = 15$. The system AoI is plotted in Fig. 3(a) for the three cases: P-GAW*, RR, C-GAW*, and the parameters p_1^* and K_1^* are plotted in Fig. 3(b), as a function of the second moment of the source-2 service time, q_2, for all values of which $n' = 1$. We observe that the C-GAW* server progressively favors source-1, i.e., K_1^* is increased, for increasing q_2 values. We also observe the adverse effect of the second moment of service times on system AoI and also its impact on the optimum policy. Moreover, C-GAW* server consistently outperforms the other two and the performance gap increases with increased q_2 values.

In the final example, we fix $N = 3$ and employ uniform weights, i.e., $w_n = 1/3$, $\forall n$. The parameters s_1 and s_2 are fixed to 2 and 5, respectively. Then, the mean service time s_3 is varied to obtain the system AoI using three policies, namely RR, P-GAW* and IS, with $K_{max} \to \infty$, which is depicted in Fig. 4 for three different scenarios: a) deterministic service times, i.e., $c_n = 0$ for all sources, b) exponentially distributed service times, i.e., $c_n = 1$ for all sources, c) $c_1 = 0$, $c_2 = 1$, $c_3 = 5$. In all the scenarios, IS outperforms the P-GAW* and

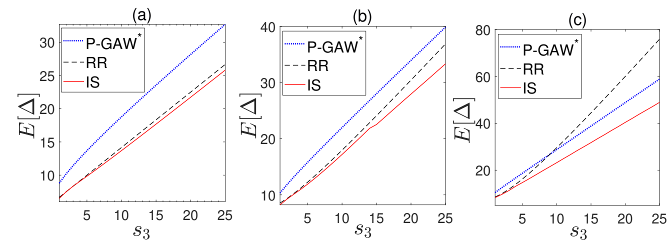

Fig. 4: System AoI as a function of s_3 obtained with P-GAW*, RR, and IS, when $s_1 = 2, s_2 = 5, w_n = 1/3, \forall n$ (a) $c_n = 0, \forall n$, (b) $c_n = 1, \forall n$, (c) $c_1 = 0, c_2 = 1, c_3 = 5$.

RR policies as in the $N = 2$ case when IS is equivalent to the optimum cyclic scheduler C-GAW*. The RR scheduler performs closely to IS in terms of system AoI for deterministic service times even when the mean service times are different for the information sources. However, the gap between IS and RR is larger when the service times are not deterministic but exponentially distributed. In Fig. 4(c), the variability in service times is different for each source and in this case, the system AoI is penalized severely when RR is used especially for larger values of s_3. In all the cases we have studied, IS outperformed P-GAW* consistently.

VII. Conclusions

An analytical method is proposed for finding the means of the individual AoI processes in a GAW system for both probabilistic and cyclic open-loop schedulers. Using the analytical results, we derive the optimum cyclic scheduler for the special case of two information sources. For the case of three or more information sources, a search-based heuristic cyclic scheduler is proposed whose effectiveness in system AoI optimization is shown through numerical examples involving three sources.

References

[1] R. D. Yates, Y. Sun, D. R. Brown, S. K. Kaul, E. Modiano, and S. Ulukus, "Age of information: An introduction and survey," *IEEE Journal on Selected Areas in Communications*, vol. 39, no. 5, pp. 1183–1210, May 2021.

[2] A. Kosta, N. Pappas, and V. Angelakis, "Age of information: A new concept, metric, and tool," *Foundations and Trends in Networking*, vol. 12, no. 3, pp. 162–259, 2017.

[3] I. Kadota, A. Sinha, E. Uysal-Biyikoglu, R. Singh, and E. Modiano, "Scheduling policies for minimizing age of information in broadcast wireless networks," *IEEE/ACM Transactions on Networking*, vol. 26, no. 6, pp. 2637–2650, 2018.

[4] Q. He, D. Yuan, and A. Ephremides, "Optimal link scheduling for age minimization in wireless systems," *IEEE Transactions on Information Theory*, vol. 64, no. 7, pp. 5381–5394, 2018.

[5] A. Maatouk, S. Kriouile, M. Assad, and A. Ephremides, "On the optimality of the Whittle's index policy for minimizing the age of information," *IEEE Transactions on Wireless Communications*, vol. 20, no. 2, pp. 1263–1277, 2021.

[6] N. Akar and E. O. Gamgam, "Distribution of age of information in status update systems with heterogeneous information sources: An absorbing Markov chain-based approach," *IEEE Communications Letters*, vol. 27, no. 8, pp. 2024–2028, 2023.

[7] C. Li, S. Li, Q. Liu, Y. T. Hou, W. Lou, and S. Kompella, "Eywa: A general approach for scheduler design in AoI optimization," in *IEEE Infocom*, May 2023.

[8] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.