Optimum Power Allocation in Fading MIMO Multiple Access Channels with Partial CSI at the Transmitters

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***Abstract*— We consider both the single-user and the multi-user power allocation problems in MIMO systems, where the receiver side has the perfect channel state information (CSI), and the transmitter side has partial CSI, which is in the form of covariance feedback. In a single-user MIMO system, we consider an iterative algorithm that solves for the eigenvalues of the optimum transmit covariance matrix that maximizes the rate. The algorithm is based on enforcing the Karush-Kuhn-Tucker (KKT) optimality conditions of the optimization problem at each iteration. We prove that this algorithm converges to the unique global optimum power allocation when initiated at an arbitrary point. We, then, consider the multi-user generalization of the problem, which is to find the eigenvalues of the optimum transmit covariance matrices of all users that maximize the sum rate of the MIMO multiple access channel (MIMO-MAC). For this problem, we propose an algorithm that finds the unique optimum power allocation policies of all users. At a given iteration, the multi-user algorithm updates the power allocation of one user, given the power allocations of the rest of the users, and iterates over all users in a round-robin fashion. Finally, we make several suggestions that significantly improve the convergence rate of the proposed algorithms.**

I. Introduction

In Gaussian MIMO multiple access systems, when the receiver side has the perfect CSI, the calculation of the information theoretic capacity boils down to finding the transmit covariance matrices of the users. Finding the transmit covariance matrices, in turn, involves two components: finding the optimum transmit directions and finding the optimum power allocation policies. In a single-user MIMO system, when both the receiver and the transmitter have the perfect CSI and the channel is fixed, [1] showed that the optimum transmit directions are the right eigenvectors of the deterministic channel matrix, and the optimum power allocation policy is to waterfill over the eigenvalues of the deterministic channel matrix. In a multi-user MIMO system, when both the receiver and the transmitters have the perfect CSI and the channel is fixed, [2] showed that the optimum transmit directions and the power allocation policies can be found using an iterative algorithm that updates the transmit directions and power policy of one user at a time. When the channel is changing over time due to fading, and perfect and instantaneous CSI is known both at the receiver and at the transmitter side, these solutions extend to water-filling over both the antennas and the channel states in single-user [1], and multi-user [3] MIMO systems. However, in most of the wireless communication scenarios, especially in wireless MIMO communications, it is unrealistic to assume that the transmitter side has the perfect knowledge of the instantaneous CSI. In such scenarios, it might be more realistic to assume that only the receiver side can perfectly estimate the instantaneous CSI, while the transmitter side has only a statistical knowledge of the channel.

When the fading in the channel is assumed to be a Gaussian process, statistics of the channel reduce to the mean and covariance information of the channel. The problem in this setting as well is to find the optimum transmit covariance matrices, i.e., the optimum transmit directions and the optimum power allocation policies. However, in this case the transmit directions and the power allocations are not functions of the channel states, but they are functions of the statistics of the channel states, that are fed by the receiver back to the transmitters. The optimization criteria that we consider are the maximum rate in a single-user system, and the maximum sum rate in a multi-user system. For the covariance feedback case, it was shown in [4] for a multi-input single-output (MISO) system, and in [5], [6] for a MIMO system that the optimal transmit covariance matrix and the channel covariance matrix have the same eigenvectors, i.e., the optimal transmit directions are the eigenvectors of the channel covariance matrix. In [7]–[9], we generalized these results to MIMO-MAC systems. We showed that in a MIMO-MAC with partial CSI at the transmitters, all users should transmit in the direction of the eigenvectors of their *own* channel parameter matrices. Consequently, we showed that, the transmit directions of the users in a MIMO-MAC with partial CSI at the transmitters are independent of the presence of other users, and therefore, that the users maintain their single-user transmit direction strategies even in a multi-user scenario.

On the other hand, in this aforementioned literature, the optimization of the eigenvalues of the transmit covariance matrices, i.e., the power allocation policies, are left as additional optimization problems. Efficient and globally convergent algorithms are needed in order to solve for these optimum eigenvalues. References [10], [11] proposed algorithms that solve this problem for a MISO, and for a MIMO system, respectively. However, in both cases, the convergence proofs for these algorithms were not provided. In a MIMO-MAC with partial CSI available at the transmitters, although the

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eigenvectors of the optimal transmit covariance matrices are known [7]–[9], no algorithm is available to find the optimum eigenvalues in a multi-user setting.

In this paper, first, we give an alternative derivation for the algorithm proposed in [11] for a single-user MIMO system by enforcing the KKT optimality conditions at each iteration. We prove that the convergence point of this algorithm is unique and is equal to the optimum eigenvalue allocation. The proposed algorithm converges to this unique point starting from any point on the space of feasible eigenvalues. Next, we consider the multi-user version of the problem. In this case, the problem is to find the optimum eigenvalues of the transmit covariance matrices of all users that maximize the sum rate of the MIMO-MAC system. We apply the single-user algorithm iteratively to reach the global optimum point. At any given iteration, the multi-user algorithm updates the eigenvalues of one user, using the algorithm proposed for the single-user case, assuming that the eigenvalues of the remaining users are fixed. The algorithm iterates over all users in a round-robin fashion. We prove that, this algorithm converges to the unique global optimum power allocation for all users.

II. System Model

We consider a multiple access channel with multiple transmit antennas at every user and multiple receive antennas at the receiver. The channel between user k and the receiver is represented by a random matrix $\mathbf{H}_k$ with dimensions of $n_R \times n_T$, where n_R and n_T are the number of antennas at the receiver and at the transmitter, respectively. The receiver has the perfect knowledge of the channel, while the transmitters have only the statistical model of the channel. Each transmitter sends a vector $\mathbf{x}_k$, and the received vector is

$$\mathbf{r} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{n} \tag{1}$$

where K is the number of users, $\mathbf{n}$ is a zero-mean, identity-covariance complex Gaussian vector, and the entries of $\mathbf{H}_k$ are complex Gaussian random variables. Let $\mathbf{Q}_k = E[\mathbf{x}_k \mathbf{x}_k^\dagger]$ be the transmit covariance matrix of user k, which has an average power constraint of P_k, $\text{tr}(\mathbf{Q}_k) \leq P_k$.

The statistical model that we consider in this paper is the "partial CSI with covariance feedback" model where each transmitter knows the channel covariance information of all transmitters, in addition to the distribution of the channel. Since the receiver has no physical restrictions, we assume that there is enough spacing between the antenna elements on the receiver such that signals received at different antenna elements are uncorrelated, however there exists correlation between the signals transmitted by different antenna elements at the transmitters. The channel is modeled as [12],

$$\mathbf{H}_k = \mathbf{Z}_k \mathbf{\Sigma}_k^{1/2} \tag{2}$$

where the transmit antenna correlation matrix, $\mathbf{\Sigma}_k$, is the correlation between the signals transmitted from the n_T transmit antennas of user k, and the entries of $\mathbf{Z}_k$ are i.i.d., zero-mean, unit-variance complex Gaussian random variables. From this point on, we will refer to matrix $\mathbf{\Sigma}_k$ as the *channel covariance matrix* of user k. Similar covariance feedback models have been used in [4], [5], [6], [10].

III. Power Allocation for Single-user MIMO

In this section, we will derive our single-user power allocation algorithm. Due to space limitations, we will skip some steps here; a detailed derivation can be found in [13]. In a single-user system the optimization problem becomes,

$$C = \max_{\text{tr}(\mathbf{Q}) \leq P} E\left[\log |\, \mathbf{I}_{n_R} + \mathbf{H}\mathbf{Q}\mathbf{H}^\dagger|\right] \tag{3}$$

where $E[\cdot]$ is the expectation operator with respect to the channel matrix $\mathbf{H}$, and $|\cdot|$ is the determinant operator. The channel covariance matrix $\mathbf{\Sigma}$, which is known at the transmitter, and the transmit covariance matrix $\mathbf{Q}$ have the eigenvalue decompositions $\mathbf{\Sigma} = \mathbf{U}_\Sigma \mathbf{\Lambda}_\Sigma \mathbf{U}_\Sigma^\dagger$, and $\mathbf{Q} = \mathbf{U}_Q \mathbf{\Lambda}_Q \mathbf{U}_Q^\dagger$, respectively. Here, $\mathbf{\Lambda}_\Sigma$ and $\mathbf{\Lambda}_Q$ are the diagonal matrices of ordered eigenvalues of $\mathbf{\Sigma}$, and $\mathbf{Q}$, and $\mathbf{U}_\Sigma$, and $\mathbf{U}_Q$ are unitary matrices.

It has been shown that the eigenvectors of the optimum transmit covariance matrix must be equal to the eigenvectors of the channel covariance matrix, i.e., $\mathbf{U}_Q = \mathbf{U}_\Sigma$ [5]. By inserting this into (3), we get

$$C = \max_{\sum_{i=1}^{n_T} \lambda_i^Q \leq P} E\left[\log \left|\mathbf{I}_{n_R} + \sum_{i=1}^{n_T} \lambda_i^Q \lambda_i^\Sigma \mathbf{z}_i \mathbf{z}_i^\dagger\right|\right] \tag{4}$$

where $\mathbf{z}_i$ is the i^{th} column of $\mathbf{Z}$, i.e., $\{\mathbf{z}_i, i = 1, \ldots, n_T\}$ is a set of $n_R \times 1$ dimensional i.i.d., zero-mean, identity-covariance Gaussian random vectors. Taking the derivatives of the Lagrangian for the above optimization problem, we get the KKT conditions as

$$E\left[\frac{\lambda_i^\Sigma \mathbf{z}_i^\dagger \mathbf{A}_i^{-1} \mathbf{z}_i}{1 + \lambda_i^Q \lambda_i^\Sigma \mathbf{z}_i^\dagger \mathbf{A}_i^{-1} \mathbf{z}_i}\right] \leq \mu, \quad i = 1, \ldots, n_T \tag{5}$$

where $\mathbf{A} = \mathbf{I}_{n_R} + \sum_{i=1}^{n_T} \lambda_i^Q \lambda_i^\Sigma \mathbf{z}_i \mathbf{z}_i^\dagger$, and $\mathbf{A}_i = \mathbf{A} - \lambda_i^Q \lambda_i^\Sigma \mathbf{z}_i \mathbf{z}_i^\dagger$. The inequalities in (5) are satisfied with equality whenever λ_i^Q is non-zero. Unlike the classical water-filling solution, it is not possible to solve for λ_i^Q in (5) explicitly. Instead, we multiply both sides of (5) by λ_i^Q. Then, by pulling λ_i^Q from that equation and summing over all antennas, we can find the conditions which have to be satisfied by the optimum power values,

$$\lambda_i^Q = \frac{E\left[\frac{\lambda_i^Q \lambda_i^\Sigma \mathbf{z}_i^\dagger \mathbf{A}_i^{-1} \mathbf{z}_i}{1 + \lambda_i^Q \lambda_i^\Sigma \mathbf{z}_i^\dagger \mathbf{A}_i^{-1} \mathbf{z}_i}\right]}{\sum_{j=1}^{n_T} E\left[\frac{\lambda_j^Q \lambda_j^\Sigma \mathbf{z}_j^\dagger \mathbf{A}_j^{-1} \mathbf{z}_j}{1 + \lambda_j^Q \lambda_j^\Sigma \mathbf{z}_j^\dagger \mathbf{A}_j^{-1} \mathbf{z}_i}\right]} P \triangleq f_i(\boldsymbol{\lambda}^Q) \tag{6}$$

where $\boldsymbol{\lambda}^Q = [\lambda_1^Q, \ldots, \lambda_{n_T}^Q]$, and we defined the right hand side of (6) which depends on all of the eigenvalues as $f_i(\boldsymbol{\lambda}^Q)$. In order to write (6) in a more compact form, let us define $E_i(\boldsymbol{\lambda}^Q)$ as the numerator in (6). Now, we can write the function $f_i(\boldsymbol{\lambda}^Q)$

as

$$f_i(\boldsymbol{\lambda}^Q) = \frac{E_i(\boldsymbol{\lambda}^Q)}{\sum_j E_j(\boldsymbol{\lambda}^Q)} P = \frac{P}{\sum_j \frac{E_j(\boldsymbol{\lambda}^Q)}{E_i(\boldsymbol{\lambda}^Q)}} \quad (7)$$

We propose to use the following algorithm

$$\boldsymbol{\lambda}^Q(n+1) = \mathbf{f}(\boldsymbol{\lambda}^Q(n)) \quad (8)$$

where $\mathbf{f} = [f_1, \ldots, f_K]$. In order to solve for the optimum eigenvalues, (8) updates the eigenvalues at step $n+1$ as a function of the eigenvalues at step n. We claim that this algorithm converges and that the unique fixed point of the algorithm is equal to the optimum eigenvalues. Although this independently obtained algorithm is the same as the one in [11], here, we also provide a convergence proof.

IV. Convergence Proof

A. Convergence from the Corner Points and the Uniqueness

As stated in (4), the constraint set of the optimization problem is $\sum_{i=1}^{n} \lambda_i^Q \leq P$. Since this inequality should be satisfied with equality in order to maximize the capacity, the resulting equality, i.e., $\sum_{i=1}^{n} \lambda_i^Q = P$, defines a simplex in the n_T-dimensional space (see Fig. 1), and all feasible eigenvalue vectors are located on this simplex. Note that if the algorithm is initiated at the exact corner points of the simplex, then the updates stay at the same point indefinitely. The reason for this is that by multiplying both sides of (5) with λ_i^Q, we created some artificial fixed points. In order to overcome this difficulty, we will re-define the corner points of the simplex as follows. Let $\mathbf{c}_j$ be the j^{th} corner point of the simplex, then the j^{th} component of $\mathbf{c}_j$ is $P - (n_T - 1)\epsilon$, and the rest of the components are ϵ, which is a very small positive number. This point, which we refer to as the j^{th} corner point, is, in fact, slightly inside of the actual j^{th} corner point of the n_T-dimensional simplex. We will prove that the algorithm converges, if we start the algorithm from $\mathbf{c}_j$, for all $j = 1, \ldots, n_T$. The main ingredient of our convergence proof is the following lemma.

Lemma 1: Let us have two feasible vectors on the simplex, $\boldsymbol{\lambda}^Q$ and $\bar{\boldsymbol{\lambda}}^Q$, such that $\lambda_i^Q > \bar{\lambda}_i^Q$, then $f_i(\boldsymbol{\lambda}^Q) > f_i(\bar{\boldsymbol{\lambda}}^Q)$.

This lemma shows us the monotonicity property of the algorithm. In the following lemmas, we will use this monotonicity property to prove that the algorithm always converges if some conditions are satisfied. Finally, we will show that these conditions define the unique optimum point.

Lemma 2: For a given i, if $f_i(\mathbf{c}_j) > \epsilon$ for $j \neq i$, then starting from $\mathbf{c}_j$, the algorithm results in a monotonically increasing sequence for all $j \neq i$ that converges to a positive number.

Lemma 3: For a given i, if $f_i(\mathbf{c}_j) < \epsilon$ for $j \neq i$, then starting from $\mathbf{c}_j$, the algorithm results in a monotonically decreasing sequence for all $j \neq i$ that converges to zero.

Lemma 2 and Lemma 3 state that the algorithm converges for a given i, if some conditions are satisfied. These conditions are related to the fact that whether the optimum point of the problem has zero components or not. In the next lemma, we

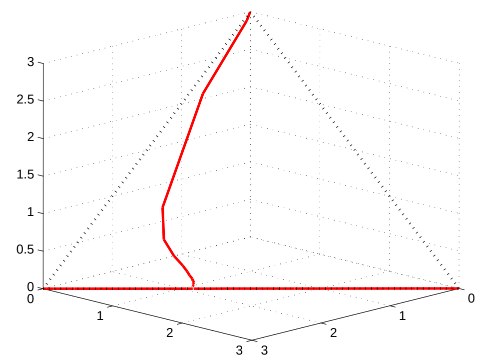

Fig. 1. The trajectories of the single-user algorithm when it is started from the corner points for the case where one of the optimal eigenvalues is zero.

will show the following. If the optimum λ_i^Q of the maximization problem is strictly positive, then after one iteration that is applied to the j^{th} corner of the simplex, $\mathbf{c}_j$, the algorithm gets further away from that corner point. That is, all components that are equal to ϵ at the initial point will get larger than ϵ after the first iteration. Since all components sum to P, the largest component of the corner point, c_{jj} gets smaller. Conversely, if starting from $\mathbf{c}_j$, $j \neq i$, one iteration of the algorithm moves us away from the corner point of the simplex, then the optimum λ_i^Q must be non-zero. Similarly, if the optimum point of the maximization problem has zero components, then starting from $\mathbf{c}_j$, we move towards the exact corner point of the simplex.

Lemma 4: For a given i, $f_i(\mathbf{c}_j) > \epsilon$ for $j \neq i$, if and only if the optimum λ_i^Q is non-zero.

This lemma tells us that the artificial fixed points are unstable. If we start ϵ away from the artificial fixed points, we move further away from them. Now, we can combine these results and show that if we start the algorithm from $\mathbf{c}_j$, for any j, it converges to the unique optimum point. Let us consider λ_i^Q for $i \neq j$. If the optimum value of λ_i^Q is non-zero, then due to Lemma 4, $f_i(\mathbf{c}_j) > \epsilon$, and due to Lemma 2, λ_i^Q converges. On the other hand, if the optimum value of λ_i^Q is zero, then due to Lemma 4, $f_i(\mathbf{c}_j) < \epsilon$, and due to Lemma 3, λ_i^Q converges to zero. Since λ_i^Q converges for all $i \neq j$, and the summation of λ_i^Q's is equal to P, λ_j^Q also converges. This proves the claim that for all i, the algorithm converges starting from any corner point of the simplex, $\mathbf{c}_j$.

Therefore, we proved that the algorithm in (8) converges to the fixed points of (6) when it is started from the corner points of the simplex. We know as a result of Lemma 4 that the algorithm cannot converge to the corner points unless that corner point is optimum. That is, the algorithm never converges to an artificial fixed point. As a result, the point that the algorithm converges to, always satisfies the KKT conditions. We know that the KKT conditions are the necessary and sufficient conditions for optimality, and for a strictly concave objective function with convex constraint set, there is a unique global optimum point. This unique global optimum point has to be the unique solution of the KKT conditions. Hence, when the algorithm converges, it does so to the unique optimum power allocation.

B. Proof of Convergence from an Arbitrary Point

In this section, we will show that starting from any arbitrary but feasible $\boldsymbol{\lambda}^Q$ which is not on the boundary of the simplex, the algorithm will converge to the unique convergence point. Let us define, three different starting points, $\boldsymbol{\lambda}^Q(0)$, which is arbitrary but feasible, $\bar{\boldsymbol{\lambda}}^Q(0) = \mathbf{c}_j$, and $\tilde{\boldsymbol{\lambda}}^Q(0) = \mathbf{c}_{n_T}$. For $i \neq n_T$, we have

$$\epsilon < \lambda_i^Q(0) < P - (n_T - 1)\epsilon \tag{9}$$

since, for any component of an arbitrary but feasible eigenvalue vector which is not on the boundary of the simplex, we can find sufficiently small ϵ so that this component lies between ϵ and $P-(n_T-1)\epsilon$. Also note that $c_{n_T i} = \epsilon$, and $c_{ii} = P-(n_T-1)\epsilon$. By inserting these into (9), we have

$$\bar{\lambda}_i^Q(0) = c_{n_T i} < \lambda_i^Q(0) < c_{ii} = \tilde{\lambda}_i^Q(0) \tag{10}$$

Applying Lemma 1 to (10), we have

$$\bar{\lambda}_i^Q(1) < \lambda_i^Q(1) < \tilde{\lambda}_i^Q(1) \tag{11}$$

Now, by applying Lemma 1 to (11), and continuing this way, we have

$$\bar{\lambda}_i^Q(n) < \lambda_i^Q(n) < \tilde{\lambda}_i^Q(n) \tag{12}$$

We showed in Section IV-A that the sequences, $\{\bar{\lambda}_i^Q(n)\}_{n=0}^{\infty}$, and $\{\tilde{\lambda}_i^Q(n)\}_{n=0}^{\infty}$ are guaranteed to converge, since they have started from the corner points of the simplex. Moreover, due to the uniqueness of the convergence point, they converge to the same point. Now, using a sandwiching argument, the sequence in the middle, which started from an arbitrary point has to converge to the unique convergence point as well. Note that this is true for all $i \neq n_T$. However, $\lambda_{n_T}^Q$ is uniquely determined by λ_i^Q, for all $i \neq n_T$, since $\sum_{j=1}^{n_T} \lambda_j^Q = P$. Therefore, $\lambda_{n_T}^Q$ has to converge to the unique convergence point as well.

V. Power Allocation for Multi-user MIMO

In this section, we will leave much of the derivation of the multi-user algorithm to [13] due to space limitations. The sum capacity of a multi-user MIMO-MAC is given as,

$$C_{sum} = \max_{\substack{\mathrm{tr}(\mathbf{Q}_k) \leq P_k \\ k=1,\ldots,K}} E\left[\log\left|\mathbf{I}_{n_R} + \sum_{k=1}^{K} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^{\dagger}\right|\right] \tag{13}$$

Let $\boldsymbol{\Sigma}_k = \mathbf{U}_{\Sigma_k} \boldsymbol{\Lambda}_{\Sigma_k} \mathbf{U}_{\Sigma_k}^{\dagger}$ be the spectral decomposition of the channel covariance matrix of user k. Then, the optimum transmit covariance matrix $\mathbf{Q}_k$ of user k has the form $\mathbf{Q}_k = \mathbf{U}_{\Sigma_k} \boldsymbol{\Lambda}_{Q_k} \mathbf{U}_{\Sigma_k}^{\dagger}$, for all users [9]. Inserting this into (13), the sum capacity of a MIMO-MAC is given as [9],

$$C_{sum} = \max_{\substack{\sum_{i=1}^{n_T} \lambda_{ki}^Q \leq P_k \\ k=1,\ldots,K}} E\left[\log\left|\mathbf{I}_{n_R} + \sum_{k=1}^{K}\sum_{i=1}^{n_T} \lambda_{ki}^Q \lambda_{ki}^{\Sigma} \mathbf{z}_{ki} \mathbf{z}_{ki}^{\dagger}\right|\right] \tag{14}$$

where $\mathbf{z}_{ki}$ is the i^{th} column of $\mathbf{Z}_k$, i.e., $\{\mathbf{z}_{ki}, k = 1, \ldots, K, i = 1, \ldots, n_T\}$ is a set of $n_R \times 1$ dimensional i.i.d., zero-mean, identity-covariance Gaussian random vectors.

Unlike the decoupling in transmit directions, the amount of power each user allocates in each direction depends on both the transmit directions and the power allocations of other users. By using the same ideas as in the single-user case, we find the fixed point equation that is satisfied by all eigenvalues of all users

$$\lambda_{ki}^Q = g_{ki}(\boldsymbol{\lambda}^Q) = \frac{E_{ki}(\boldsymbol{\lambda}^Q)}{\sum_j E_{kj}(\boldsymbol{\lambda}^Q)} P_k \tag{15}$$

where $\boldsymbol{\lambda}^Q = [\boldsymbol{\lambda}_1^Q, \ldots, \boldsymbol{\lambda}_K^Q]$, $\boldsymbol{\lambda}_k^Q = [\lambda_{k1}^Q, \ldots, \lambda_{kn_T}^Q]$ is the eigenvalue vector of user k, and $E_{ki}(\boldsymbol{\lambda}^Q)$ is defined as

$$E_{ki}(\boldsymbol{\lambda}^Q) = E\left[\frac{\lambda_{ki}^Q \lambda_{ki}^{\Sigma} \mathbf{z}_{ki}^{\dagger} \mathbf{A}_{ki}^{-1} \mathbf{z}_{ki}}{1 + \lambda_{ki}^Q \lambda_{ki}^{\Sigma} \mathbf{z}_{ki}^{\dagger} \mathbf{A}_{ki}^{-1} \mathbf{z}_{ki}}\right] \tag{16}$$

where $\mathbf{A} = \mathbf{I}_{n_R} + \sum_{k=1}^{K}\sum_{i=1}^{n_T} \lambda_{ki}^Q \lambda_{ki}^{\Sigma} \mathbf{z}_{ki} \mathbf{z}_{ki}^{\dagger}$, and $\mathbf{A}_{ki} = \mathbf{A} - \lambda_{ki}^Q \lambda_{ki}^{\Sigma} \mathbf{z}_{ki} \mathbf{z}_{ki}^{\dagger}$.

The following algorithm enforces (15),

$$\boldsymbol{\lambda}_k^Q(n+1) = \mathbf{g}_k\left(\boldsymbol{\lambda}_1^Q, \ldots, \boldsymbol{\lambda}_{k-1}^Q, \boldsymbol{\lambda}_k^Q(n), \boldsymbol{\lambda}_{k+1}^Q, \ldots, \boldsymbol{\lambda}_K^Q\right) \tag{17}$$

where $\mathbf{g}_k = [g_{k1}, \ldots, g_{kn_T}]$ is the vector valued update function of user k. This algorithm finds the optimum eigenvalues of a given user by assuming that the eigenvalues of the rest of the users are fixed. In that sense, complete update of one user is equivalent to the single-user algorithm proposed in (8). The algorithm moves to another user, after (17) converges. We know from the previous section that the algorithm in (17) converges to the unique optimum point, at each complete update corresponding to user k. Such an algorithm is guaranteed to converge to the global optimum [14, Page 219], since (14) is a concave function of λ_{ki} for all k and i, C_k is a strictly concave function of λ_{ki} for all i, and the constraint set is convex and has a Cartesian product structure among the users.

In order to improve the convergence rate, we also propose the following multi-user algorithm,

$$\begin{aligned}\boldsymbol{\lambda}_{k'}^Q(n+1) = \mathbf{g}_{k'}\Big(&\boldsymbol{\lambda}_1^Q(n+1), \ldots, \boldsymbol{\lambda}_{k'-1}^Q(n+1), \\ &\boldsymbol{\lambda}_{k'}^Q(n), \boldsymbol{\lambda}_{k'+1}^Q(n), \ldots, \boldsymbol{\lambda}_K^Q(n)\Big)\end{aligned} \tag{18}$$

where $k' = (n+1) \mod K$. At a given time $n+1$, this algorithm updates the eigenvalues of user k'. In the next iteration, it moves to another user. We will see in Section VI that this algorithm converges much faster than the algorithm in (17).

VI. Numerical Results and Remarks

In this section, we will provide numerical examples for the performances of the proposed algorithms. In Fig. 1, we plot the trajectories of the iterations of the proposed single-user algorithm for a MIMO system with $m = n = P = 3$. We run the algorithm three times for each figure with different initial points, which are the three corner points of the 3-dimensional simplex. In the system for this figure, one of the optimum

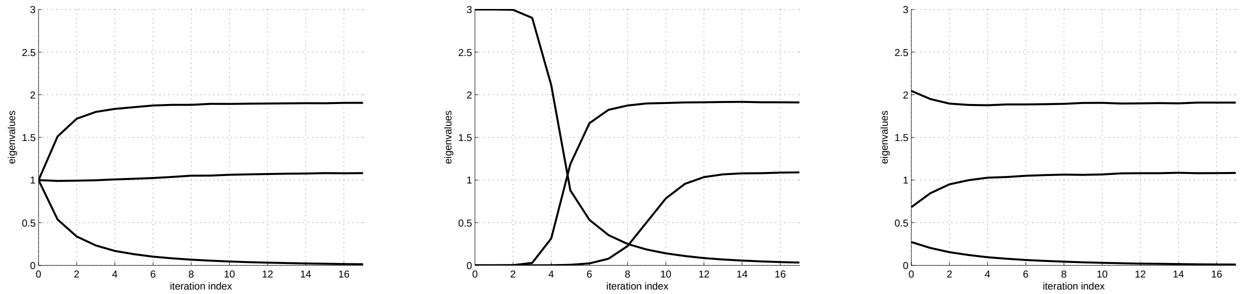

Fig. 2. The convergence of the algorithm starting from various points, when one of the optimal eigenvalues is zero: (a) convergence of all three eigenvalues from $(1, 1, 1)$; (b) convergence of all three eigenvalues from $\mathbf{c}_3$; (c) convergence of all three eigenvalues from the normalized channel eigenvalue vector.

eigenvalues is zero. We observe that the algorithm converges to the unique optimum point.

In Fig. 2, we plot the eigenvalues as a function of the iteration index. We observe that the eigenvalues converge to the same unique convergence point starting from various initial points. In addition to a corner point, the other initial points are: the all-one vector, and the point corresponding to the channel covariance matrix eigenvalues, which is normalized to satisfy the power constraint. Although the algorithm converges to the optimum point from any arbitrary initial point, it needs much less time to converge to the optimum point when it is started from the normalized channel covariance eigenvalue point compared to the cases when it is started from the corner points of the simplex. This is true mainly because of an argument similar to the water-filling argument, where we allocate more power to the strongest channel.

Finally, we consider a multi-user MIMO-MAC scenario. Note that, for a given user, the multi-user algorithm given in (17) demonstrates the same convergence behavior as in Fig. 2, when the eigenvalues of the other users are kept constant. Therefore, we plot Fig. 3 by running the multi-user algorithm proposed in (18). In this figure, we consider 3 users with different channel covariance matrices. At the end of the first iteration, all users have run the algorithm in (18) once. We can see in Fig. 3 that the multi-user algorithm converges quite quickly, and at the end of the fourth iteration, all users are almost at their optimum eigenvalue points.

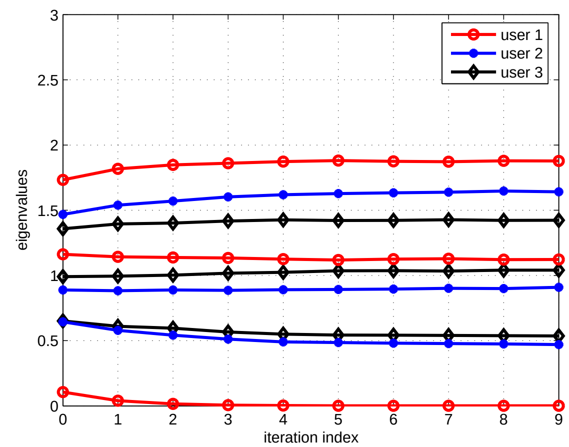

Fig. 3. The convergence of the multi-user algorithm where each iteration corresponds to a single update of all users.

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