Capacity of a Class of Semi-Deterministic Primitive Relay Channels

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***Abstract*— We characterize the capacity of a class of primitive relay channels. The primitive relay channel considered in this paper consists of a semi-deterministic broadcast channel to the relay and the decoder. The relay can help the message transmission through an orthogonal finite capacity link to the decoder. We show that the capacity of this class of primitive relay channels is given by the cut-set upper bound, and it can be achieved by the partial decode-and-forward scheme. We also show that the rate achievable by the compress-and-forward scheme is strictly smaller than the capacity.**

I. Introduction

In this paper, we consider the primitive relay channel. A primitive relay channel consists of a broadcast channel from the encoder to the decoder and the relay. Furthermore, there is an orthogonal, finite capacity link from the relay to the decoder. Figure 1 shows the primitive relay channel, where the channel input is X, the channel output at the decoder is Y, and the channel output at the relay is Z. The primitive relay channel is relatively simpler to study than the general relay channel [1], since the relay does not have an explicit coded input to the channel.

The capacity of the primitive relay channel is known for the following classes. A class of semi-deterministic primitive relay channels was considered in [2], with the assumption that the channel output at the relay, i.e., Z can be obtained as a deterministic function of the channel input X and the channel output of the decoder, Y. It was shown that the compress-and-forward (CAF) scheme [1, Theorem 6] is capacity achieving and matches the cut-set upper bound [3]. This was the first result to show the optimality of the CAF scheme for any relay channel. Also note that this class of primitive relay channels includes the sub-class, where the broadcast channel is fully deterministic, i.e., when $Z = f(X)$ and $Y = g(X)$, for deterministic functions f and g. For this sub-class, the capacity can be obtained from [2], [4].

An interesting example in the class of semi-deterministic relay channels of [2] is when $Y = X \oplus N$, where $\oplus$ denotes modulo addition and the relay observes the noise N, which is assumed to be independent of the message. Since $N = X \oplus Y$, the capacity can be achieved by CAF scheme and matches the cut-set bound. In an interesting generalization of this example, it was shown in [5] that if the relay observes a noisy version of the forward noise, i.e., the relay observes $\tilde{N}$ where $\tilde{N} = N \oplus V$, and V is independent of N, then the CAF scheme is capacity achieving, although, the capacity can be strictly smaller than the cut-set bound. This was the first result to show the sub-optimality of the cut-set bound for any relay channel.

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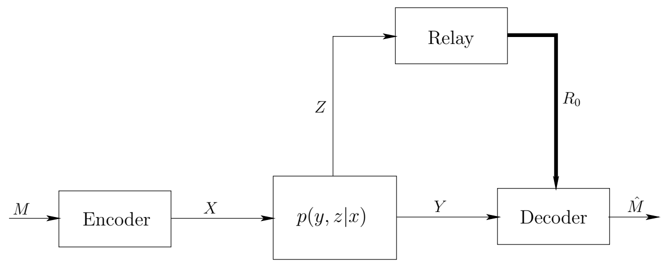

Fig. 1. The primitive relay channel.

In this paper, we focus on a special class of primitive relay channels whose capacity is not known. Figure 2 shows the broadcast component, $p(y, z|x)$, for the primitive relay channel in consideration. The channel input X is ternary, taking values in $\{0, 1, 2\}$ and the channel outputs Y and Z are binary, taking values in $\{0, 1\}$.

Note that Y is a deterministic function of X, whereas Z is not. Furthermore, for any $p \in (0, 1)$, $Z \neq f(X, Y)$, i.e., Z cannot be expressed as a deterministic function of (X, Y). To observe this, note that if $X = 2$, then $Y = 0$, but Z could be either 0 or 1. In other words, this primitive relay channel does not fall in the class of primitive relay channels studied in [2], [4]. This particular semi-deterministic broadcast channel is also referred to as the noisy Blackwell channel and was studied in [6].

Observe that when $p = 1$, then $Z = Y = f(X)$ and the capacity is given by $\mathcal{C}(R_0) = \max_{p(x)} H(Y) = 1$ for all values of R_0. This implies that the relay is not useful at all. On the other hand, when $p = 0$, then $Z = f(X)$ and $Y = g(X)$, and the resulting primitive relay channel falls in the class studied in [2], [4]. We therefore consider the case when $p \in (0, 1)$. The main contribution of this paper is to characterize the capacity of this primitive relay channel. We show that the capacity is given by the cut-set upper bound and it can be achieved by using the partial decode-and-forward scheme (PDAF). We also show that for this class of primitive relay channels, the CAF scheme is sub-optimal.

II. Main Result

We now state the main result of this paper in the following theorem.

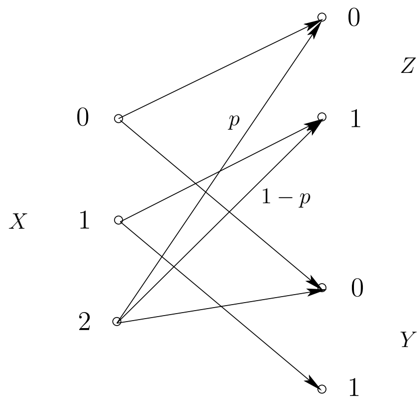

Fig. 2. The broadcast component of the primitive relay channel in consideration.

Theorem 1: The capacity of the primitive relay channel with a broadcast channel as shown in Figure 2 is given as,

$$\mathcal{C}(R_0) = \max_{\alpha \in [0,1/2]} h(\alpha) + \min(R_0, (1-\alpha)K(p)) \quad (1)$$

where, $K(p) = \log(1 + (1-p)p^{p/(1-p)})$. The capacity, $\mathcal{C}(R_0)$, is equal to the cut-set upper bound and can be achieved by the partial decode-and-forward scheme.

We will prove Theorem 1 in two steps. We first explicitly evaluate the cut-set upper bound [3] and then show that it can be achieved by the PDAF scheme [7].

A. Evaluation of the Cut-set Bound

We begin by first evaluating the cut-set bound for this relay channel [3]:

$$\mathcal{CS}(R_0) = \max_{p(x)} \min(I(X;Y,Z), I(X;Y) + R_0) \quad (2)$$

which can be simplified to

$$\mathcal{CS}(R_0) = \max_{p(x)} (H(Y) + \min(I(X;Z|Y), R_0)) \quad (3)$$

by using the fact that $H(Y|X) = 0$.

Let us fix an input distribution, $p(x)$, as

$$p_X(1) = \alpha, \quad p_X(0) = (1-\pi)(1-\alpha), \quad p_X(2) = \pi(1-\alpha) \quad (4)$$

where $\alpha, \pi \in [0,1]$. For this input distribution, we have,

$$H(Y) = h(\alpha) \quad (5)$$

where $h(\alpha)$ is the binary entropy function, and

$$\begin{aligned} I(X;Z|Y) &= P_Y(1)I(X;Z|Y=1) \\ &\quad + P_Y(0)I(X;Z|Y=0) \quad (6) \\ &= P_Y(0)I(X;Z|Y=0) \quad (7) \\ &= (1-\alpha)I(X;Z|Y=0) \quad (8) \end{aligned}$$

Now, note that

$$\begin{aligned} I(X;Z|Y=0) &= h(\pi(1-p)) - \pi h(p) \quad (9) \\ &\le \max_{\pi \in [0,1]} h(\pi(1-p)) - \pi h(p) \quad (10) \\ &= K(p) \quad (11) \end{aligned}$$

where $K(p)$ is the capacity of a Z-channel with crossover probability p [8], and is given as,

$$K(p) = \log\left(1 + (1-p)p^{p/(1-p)}\right) \quad (12)$$

By using (5), (8), and (11), we have an expression for the cut-set bound as,

$$\begin{aligned} \mathcal{CS}(R_0) &= \max_{p(x)} H(Y) + \min(R_0, I(X;Z|Y)) \quad (13) \\ &= \max_{\alpha \in [0,1]} h(\alpha) + \min(R_0, (1-\alpha)K(p)) \quad (14) \\ &= \max_{\alpha \in [0,1/2]} h(\alpha) + \min(R_0, (1-\alpha)K(p)) \quad (15) \end{aligned}$$

where (15) follows from the fact that $h(\alpha)$ is symmetric around $1/2$.

B. Evaluation of the Partial Decode-and-Forward Rate

The PDAF scheme yields the following rate for the primitive relay channel [7],

$$R_{PDF}(R_0) = \max_{p(u,x)} \min(H(Y|U) + I(U;Z), H(Y) + R_0) \quad (16)$$

We consider the following input distribution $p(u,x)$. Select $|\mathcal{U}| = 2$, and the distribution $p(u,x)$ as follows. Fix an $\alpha \in [0,1/2]$, and define,

$$\beta = \begin{cases} \frac{1+(1-p)p^{p/(1-p)}}{1+(1/\alpha-p)p^{p/(1-p)}} & \text{if } \alpha > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

and

$$\gamma = \begin{cases} \frac{\alpha}{\beta} & \text{if } \alpha > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

We now describe the input distribution $p(u,x)$ as,

$$P_U(0) = 1-\gamma, \quad P_U(1) = \gamma \quad (19)$$

and

$$P_{X|U}(0|0) = 1, \quad P_{X|U}(1|1) = \beta, \quad P_{X|U}(2|1) = 1-\beta. \quad (20)$$

To observe the input distribution, see Figure 3.

For this input distribution, we have $P_Y(1) = \gamma\beta = \alpha$, so that

$$H(Y) = h(\alpha) \quad (21)$$

We will now show that,

$$\begin{aligned} &H(Y|U) + I(U;Z) \\ &\quad = h(\alpha) + (1-\alpha)\log\left(1 + (1-p)p^{p/(1-p)}\right) \quad (22) \end{aligned}$$

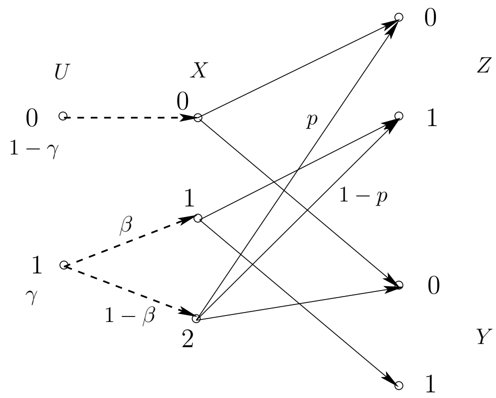

Fig. 3. Construction of the optimal input distribution $p(u, x)$.

$$= h(\alpha) + (1-\alpha)K(p) \tag{23}$$

which is equivalent to showing that,

$$I(U;Z) - I(U;Y) = (1-\alpha)K(p) \tag{24}$$

The proof of (24) is given in the Appendix. Therefore, the rates for PDAF can be lower bounded as,

$$R_{PDF}(R_0) \geq \max_{\alpha\in[0,1/2]} \min(h(\alpha)+R_0, h(\alpha)+(1-\alpha)K(p)) \tag{25}$$

$$= \max_{\alpha\in[0,1/2]} h(\alpha) + \min(R_0, (1-\alpha)K(p)) \tag{26}$$

Using (15) and (26), the capacity of the primitive relay channel in consideration is given as,

$$\mathcal{C}(R_0) = \max_{\alpha\in[0,1/2]} h(\alpha) + \min(R_0, (1-\alpha)K(p)) \tag{27}$$

III. On the Sub-Optimality of the CAF Scheme

In the previous section, we showed that the PDAF scheme achieves the capacity of a particular class of primitive relay channels. Besides the PDAF scheme, another natural achievable scheme for the primitive relay channel is the CAF scheme. In this scheme, the relay does not make use of the codebook structure and compresses its output using decoder's channel output as the side information. In this section, we will show that for the primitive relay channel in consideration, the CAF scheme is strictly sub-optimal. We will show this by obtaining an upper bound on the rate of the CAF scheme. We will then show that this upper bound is strictly smaller than the capacity for certain values of R_0.

The achievable rate of the CAF scheme is given as,

$$R_{CAF}(R_0) = \max_{p(x),p(\hat{z}|z)} I(X;Y,\hat{Z}) \tag{28}$$

$$\text{such that } I(Z;\hat{Z}|Y) \leq R_0 \tag{29}$$

We remark here that evaluation of the rates of CAF scheme is a difficult problem, primarily due to the presence of the auxiliary random variable $\hat{Z}$, for which there are no cardinality bounds.

To obtain an upper bound on $R_{CAF}(R_0)$, we fix an arbitrary input distribution $p(x)$ as follows,

$$p_X(1) = \alpha, \quad p_X(0) = (1-\pi)(1-\alpha), \quad p_X(2) = \pi(1-\alpha) \tag{30}$$

where $\alpha, \pi \in [0,1]$. For this input distribution, we have,

$$H(X) = h(\alpha) + (1-\alpha)h(\pi) \tag{31}$$

$$H(Z|Y) = (1-\alpha)h(\pi(1-p)) \tag{32}$$

We also have,

$$I(X;Y,\hat{Z}) = H(X) - H(X|Y,\hat{Z}) \tag{33}$$

$$= h(\alpha) + (1-\alpha)h(\pi) - H(X|Y,\hat{Z}) \tag{34}$$

$$= h(\alpha) + (1-\alpha)h(\pi) - (1-\alpha)H(X|Y=0,\hat{Z}) \tag{35}$$

Note that only such conditional distributions $p(\hat{z}|z)$ are permitted such that,

$$R_0 \geq I(Z;\hat{Z}|Y) \tag{36}$$

$$= H(Z|Y) - H(Z|Y,\hat{Z}) \tag{37}$$

$$= (1-\alpha)h(\pi(1-p)) - H(Z|Y,\hat{Z}) \tag{38}$$

$$= (1-\alpha)h(\pi(1-p)) - (1-\alpha)H(Z|Y=0,\hat{Z}) \tag{39}$$

This implies that,

$$H(Z|Y=0,\hat{Z}) \geq \left(h(\pi(1-p)) - \frac{R_0}{(1-\alpha)}\right)^+ \tag{40}$$

$$= \lambda(\alpha,\pi,R_0) \tag{41}$$

where $x^+ = \max(0,x)$. To obtain an upper bound on the rate achievable by the CAF scheme, we will lower bound the term $H(X|Y=0,\hat{Z})$ appearing in (35). For this purpose, we will use (40) in the following manner. For simplicity, we define the following random variables $X^{'}, Z^{'}, \hat{Z}^{'}$, where $|\mathcal{X}^{'}| = \{0,2\}, |\mathcal{Z}^{'}| = |\mathcal{Z}|$ and $|\hat{\mathcal{Z}}^{'}| = |\hat{\mathcal{Z}}|$. The random variables have the following joint distribution,

$$p_{X^{'},Z^{'},\hat{Z}^{'}}(x^{'},z^{'},\hat{z}^{'}) = p_{X,Z|Y}(x^{'},z^{'}|y=0)p_{\hat{Z}|Z}(\hat{z}^{'}|z^{'}) \tag{42}$$

for all $(x^{'},z^{'},\hat{z}^{'})$. By the construction of $(X^{'},Z^{'},\hat{Z}^{'})$, the following is a valid Markov chain.

$$\hat{Z}^{'} \rightarrow Z^{'} \rightarrow X^{'} \tag{43}$$

Moreover, we also have,

$$H(X^{'}|\hat{Z}^{'}) = H(X|Y=0,\hat{Z}) \tag{44}$$

$$H(Z^{'}|\hat{Z}^{'}) = H(Z|Y=0,\hat{Z}) \tag{45}$$

We will now make use of the following result on pairs of dependent random variables [9, Theorem 4], [10]. If $\hat{Z}^{'} \rightarrow Z^{'} \rightarrow X^{'}$ forms a Markov chain, and,

$$H(Z^{'}|\hat{Z}^{'}) \geq (1-\pi(1-p))\frac{h(\gamma)}{\gamma} \tag{46}$$

for some $\gamma \in [(1-\pi(1-p)), 1]$, then,

$$H(X^{'}|\hat{Z}^{'}) \geq (1-\pi(1-p))\frac{h(\epsilon * \gamma)}{\gamma} \quad (47)$$

where,

$$\epsilon = \frac{(1-\pi)}{(1-\pi(1-p))} \quad (48)$$

Continuing from (35), we have,

$$\begin{aligned} & I(X;Y,\hat{Z}) \\ &= h(\alpha) + (1-\alpha)h(\pi) - (1-\alpha)H(X|Y=0,\hat{Z}) && (49) \\ &= h(\alpha) + (1-\alpha)h(\pi) - (1-\alpha)H(X^{'}|\hat{Z}^{'}) && (50) \\ &\leq h(\alpha) + (1-\alpha)h(\pi) - (1-\alpha)(1-\pi(1-p))\frac{h(\epsilon * \gamma)}{\gamma} && (51) \\ &= h(\alpha) + (1-\alpha)h(\pi) - (1-\alpha)\mu(\alpha,\pi,R_0) && (52) \end{aligned}$$

where (50) follows from (44), (51) follows from (47), and in (52), we have defined

$$\mu(\alpha,\pi,R_0) = (1-\pi(1-p))\frac{h(\epsilon * \gamma)}{\gamma} \quad (53)$$

where $\gamma \in [(1-\pi(1-p)), 1]$ is the solution of the equation,

$$\lambda(\alpha,\pi,R_0) = (1-\pi(1-p))\frac{h(\gamma)}{\gamma} \quad (54)$$

Using (52), we have an upper bound on the rates of CAF scheme as follows,

$$R_{CAF}(R_0) \leq \max_{\alpha,\pi}\left[h(\alpha) + (1-\alpha)(h(\pi) - \mu(\alpha,\pi,R_0))\right] \quad (55)$$

The capacity as a function of R_0 is shown in Figure 4 for various values of parameter p. The figure also shows the upper bound on the rate of CAF scheme, obtained in (55) for $p = 0.2$. We observe that the rate of CAF scheme is strictly below the capacity when $p = 0.2$.

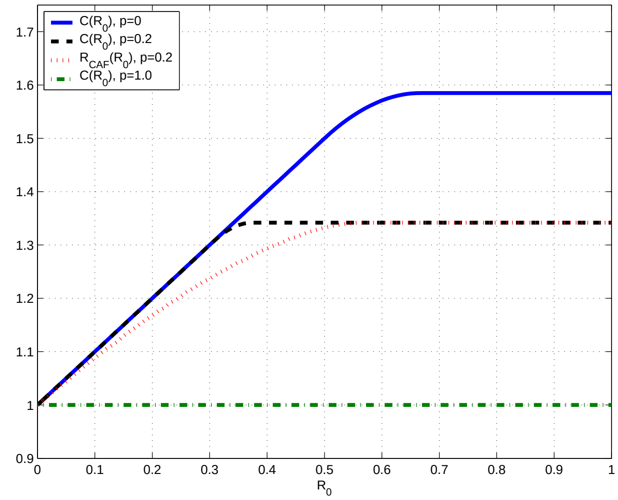

Fig. 4. Capacity curve $\mathcal{C}(R_0)$ for various values of parameter p.

IV. Conclusions

We considered a class of primitive relay channel whose capacity was not previously known. We characterized the capacity of this class of channels. It is shown that the cut-set upper bound is tight and it can be achieved by partial decode-and-forward scheme. We also show that the compress-and-forward scheme is strictly sub-optimal for this relay channel.

V. Appendix

A. Proof of (24)

To prove (24), we define the following variables,

$$\eta = (1-p)p^{p/(1-p)} \quad (56)$$

$$\Delta = \frac{(1-\alpha)}{\alpha(1-p)} \quad (57)$$

so that we have,

$$\beta = \frac{1+\eta}{(1+\eta+\eta\Delta)} \quad (58)$$

Using these variables, we have the following,

$$\begin{aligned} H(Z) &= h\left(\frac{\alpha+\eta}{1+\eta}\right) && (59) \\ H(Y) &= h(\alpha) && (60) \\ H(Y|U) &= \frac{\alpha(1+\eta+\eta\Delta)}{(1+\eta)}h\left(\frac{1+\eta}{1+\eta+\eta\Delta}\right) && (61) \\ H(Z|U) &= \frac{\alpha(1+\eta+\eta\Delta)}{(1+\eta)}h\left(\frac{\eta\Delta p}{1+\eta+\eta\Delta}\right) && (62) \end{aligned}$$

We therefore have,

$$\begin{aligned} & I(U;Z) - I(U;Y) \\ &= h\left(\frac{\alpha+\eta}{1+\eta}\right) - h(\alpha) \\ &\quad - \frac{\alpha(1+\eta+\eta\Delta)}{(1+\eta)}\left[h\left(\frac{\eta\Delta p}{1+\eta+\eta\Delta}\right)\right. \\ &\quad \left. - h\left(\frac{1+\eta}{1+\eta+\eta\Delta}\right)\right] && (63) \\ &= (1-\alpha)\log(1+\eta) && (64) \\ &= (1-\alpha)\log(1+(1-p)p^{p/(1-p)}) && (65) \\ &= (1-\alpha)K(p) && (66) \end{aligned}$$

where (64) follows from direct simplification of the expression.

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