Secure Source Coding with a Helper

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***Abstract*— We consider a secure lossless source coding problem with a rate-limited helper. In particular, Alice observes an i.i.d. source X^n and wishes to transmit this source losslessly to Bob at a rate R_x. A helper, say Helen, observes a correlated source Y^n and transmits at a rate R_y to Bob. A passive eavesdropper can observe the coded output of Alice. The equivocation Δ is measured by the conditional entropy $H(X^n|J_x)/n$, where J_x is the coded output of Alice. We first completely characterize the rate-equivocation region for this secure source coding model, where we show that Slepian-Wolf type coding is optimal.**

We next study two generalizations of this model and provide single-letter characterizations for the respective rate-equivocation regions. In particular, we first consider the case of a two-sided helper where Alice also has access to the coded output of Helen. We show that for this case, Slepian-Wolf type coding is suboptimal and one can further decrease the information leakage to the eavesdropper by utilizing the side-information at Alice. We finally generalize this result to the case when there are both secure and insecure rate-limited links from Helen and additional uncoded side informations W^n and Z^n available at Bob and Eve, respectively. For this model, we provide a complete characterization of the rate-equivocation region when $Y^n \to X^n \to (W^n, Z^n)$ forms a Markov chain.

I. Introduction

The study of information theoretic secrecy was initiated by Shannon in [1]. Following Shannon's work, significant contributions were made by Wyner [2] who established the rate-equivocation region of a degraded broadcast channel. Wyner's result was generalized to the case of a general broadcast channel by Csiszar and Korner [3]. Recently, there has been a resurgence of activity in studying multi-terminal and vector extensions of [2], [3].

In this paper, we investigate a secure transmission problem from a source coding perspective. In particular, we first consider a simple setup consisting of four terminals. Terminal 1 (say Alice) observes an i.i.d. source X^n which it intends to transmit losslessly to terminal 2 (say Bob). A malicious but passive user (say Eve) gets to observe the coded output of Alice. In other words, the communication link between Alice and Bob is assumed to be public, i.e., insecure. It is clear that since the malicious user gets the same information as the legitimate user, there cannot be any positive secret rate of transmission. On the other hand, if there is a helper, say Helen, who observes an i.i.d. source Y^n which is correlated with the source X^n and transmits information over a secure rate-limited link to Bob, then one can aim for creating uncertainty at the eavesdropper (see Figure 1[1]). For the model shown in Figure 1, we completely characterize the rate-equivocation region. From our result, we observe that the classical achievablity scheme of Ahlswede and Korner [4] and Wyner [5] for source coding with rate-limited side information is robust in the presence of a passive eavesdropper.

Secondly, we consider the model where Alice also has access to the coded output of Helen and completely characterize the rate-equivocation region. We will call this model the two-sided helper model (see Figure 2). From our result, we observe that the availability of additional coded side information at Alice allows her to increase uncertainty of the source at Eve even though the rate needed by Alice to transmit the source losslessly to Bob remains the same. This observation is in contrast to the case of insecure source coding with side information where providing coded side information to Alice is of no value [4].

We next generalize the setup of Figure 2 to the case when there are both secure and insecure rate-limited links from Helen and there is additional side information W^n at Bob and additional side information Z^n at Eve. In particular, there is a secure link of capacity R_{sec}, whose output is available at Alice and Bob and an insecure link, of capacity R_{ins}, whose output is available at all three terminals, i.e., at Alice, Bob and Eve (see Figure 3). The presence of both secure and insecure links from Helen can be interpreted as a source-coding analogue of a degraded broadcast channel from Helen where Alice and Bob receive both secure and insecure streams J_{sec} and J_{ins}, whereas, Eve only receives the insecure stream J_{ins}. We characterize the set of achievable quadruples $(R_x, R_{sec}, R_{ins}, \Delta)$ for this model when $Y^n \to X^n \to (W^n, Z^n)$ forms a Markov chain.

We explicitly compute the rate-equivocation region for the case of one-sided helper and two-sided helper for a pair of binary symmetric sources. We show that having access to Helen's coded output at Alice yields a strictly larger equivocation than the case of one-sided helper.

II. Related Work

The secure source coding setup shown in Figure 1 was considered in [6] where it was also assumed that Eve has

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[1]In Figures 1, 2 and 3, secure links are shown by bold lines.

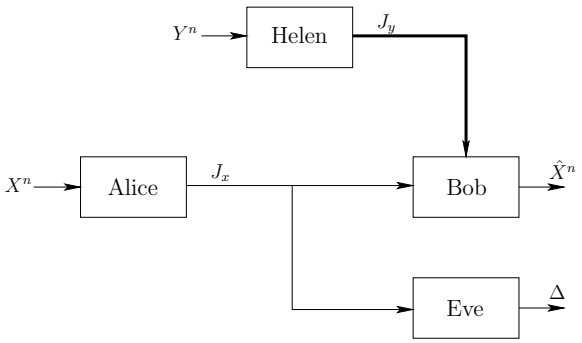

Fig. 1. One-sided helper.

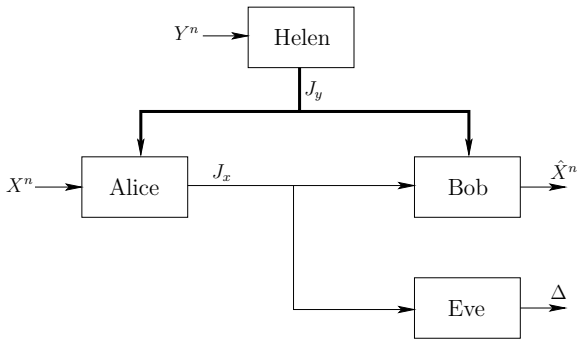

Fig. 2. Two-sided helper.

access to additional correlated side information Z^n. Inner and outer bounds for the rate-equivocation region were provided for this setup, which do not match in general. The rate-equivocation region was completely characterized in [6] for the case when Bob has complete correlated side information Y^n. This result also follows from [7] where a similar three terminal setup was studied and the maximum uncertainty at Eve was characterized under the assumption of no rate constraint in the lossless transmission of the source to Bob. A similar model was also studied in [8] where Bob intends to reconstruct both X^n and Y^n losslessly. It was shown that Slepian-Wolf type coding suffices for characterizing the rate-equivocation region when the eavesdropper does not have additional correlated side information. This setup was generalized in [9] to the case when the eavesdropper has additional side information Z^n, and inner and outer bounds were provided, which do not match in general.

In another related work [10], a multi-receiver secure broadcasting problem was studied, where Alice intends to transmit a source X^n to K legitimate users. The kth user has access to an *uncoded* correlated source Y_k^n, where $Y_k^n = X^n \oplus B_k^n$, for $k = 1, \dots K$, and eavesdropper has access to Z^n, where $Z^n = X^n \oplus E^n$, and the noise sequences $(B_1^n, \dots, B_K^n, E^n)$ are mutually independent and also independent of the source X^n. Furthermore, it was assumed that Alice also has access to $(Y_1^n, \dots Y_K^n)$. For sources with such modulo-additive structure, it was shown that to maximize the uncertainty at the eavesdropper, Alice cannot do any better than describing the error sequences $(B_1^n, \dots, B_K^n)$ to the legitimate users. Formalizing this idea to the two-sided helper model shown in Figure 2, one can conceive an achievable scheme which makes use of the common coded information available at Alice and Bob and only transmits the resulting error in decoding X^n from the common coded output by using a conditional rate-distortion code. Our results however hold for general discrete memoryless sources (not necessarily modulo-additive) and also in the presence of a rate-limited link from the two-sided helper.

For the model shown in Figure 3, when we set $R_{sec} = R_{ins} = 0$, i.e., in the absence of Helen, we recover the result obtained in [7]. Therefore, our result can also be viewed as a generalization of the result obtained in [7].

By setting $R_{sec} = 0$, i.e., in the absence of the secure rate-limited link, the resulting model is related to the model considered in [11] where the aim is to generate a secret key between two terminals via an insecure rate-limited two-sided helper. In the model studied in this work, the aim is to securely transmit the source X^n to Bob. Note that, when $R_{sec} = 0$ and $W = \phi$, both the secret-key generation capacity [11] and secure transmission rate are zero since Eve has access to both J_{ins} and J_x along with Z^n. On the other hand, in the presence of a secure link, i.e., when $R_{sec} > 0$, even when $W = \phi$, we can still create uncertainty at the eavesdropper. This is possible since Helen can choose not to transmit any information on the insecure link and transmit only a coded description of Y^n by using the secure link at the rate R_{sec}, which plays the role of a secure key. Furthermore, being correlated with the source X^n, the coded description of Y^n also permits Alice to lower the rate of transmission when compared to the case of using an uncorrelated secret key, in which case Alice must transmit at a rate $H(X)$.

III. Summary of Main Results

In Section 4, we present the rate-equivocation region for the case of one-sided helper. We show that Slepian-Wolf type coding alone at Alice is optimal for this case. We present the rate-equivocation region for the case of two-sided helper in Section 5. For the case of two-sided helper, Alice uses a conditional rate-distortion code to create an auxiliary U from the source X and the coded output V received from Helen. This code construction is reminiscent of lossy-source coding with two-sided helper [12], [13]. For the case of lossy source coding, a conditional rate-distortion code is used where U is selected to satisfy the distortion criterion. On the other hand, the purpose of U in secure lossless source coding is to confuse the eavesdropper. From this result, we demonstrate the insufficiency of Slepian-Wolf type coding at Alice by subsequently utilizing the side information at Alice. This observation is further highlighted in Section 6 where we compare the rate-equivocation regions of two-sided helper and one-sided helper for a pair of binary symmetric sources. For this example, we show that for all $R_y > 0$, the information leakage to the eavesdropper for the two-sided helper is strictly less than the case of one-sided helper. We finally generalize the result of two-sided helper to the case when there are both secure and insecure rate-limited links

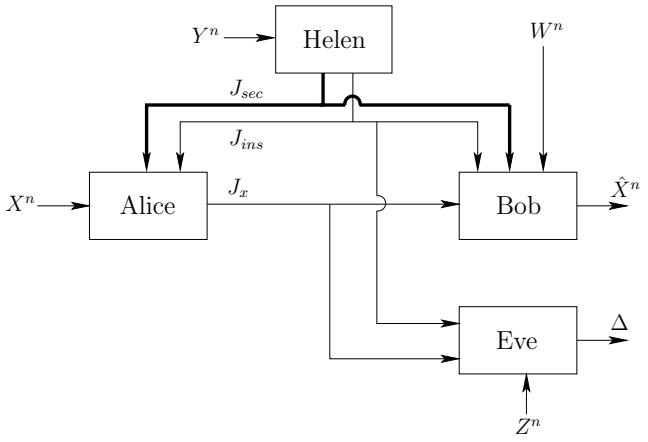

Fig. 3. Secure and insecure links from two-sided helper.

from the two-sided helper and additional side informations W and Z, at Bob and Eve, respectively. The presence of secure and insecure rate-limited links from Helen can be viewed as a source-coding analogue of a degraded broadcast channel from Helen to (Alice, Bob) and Eve. We characterize the rate-equivocation region for this model when the sources satisfy the condition $Y \to X \to (W, Z)$.

IV. One-Sided Helper

A. System model

We consider the following source coding problem. Alice observes an n-length source sequence X^n, which is intended to be transmitted losslessly to Bob. The coded output of Alice can be observed by the malicious user Eve. Moreover, Helen observes a correlated source Y^n and there exists a noiseless rate-limited channel from Helen to Bob. We assume that the link from Helen to Bob is a secure link and the coded output of Helen is not observed by Eve (see Figure 1). The sources (X^n, Y^n) are generated i.i.d. according to $p(x, y)$ where $p(x, y)$ is defined over the finite product alphabet $\mathcal{X} \times \mathcal{Y}$. The aim of Alice is to create maximum uncertainty at Eve regarding the source X^n while losslessly transmitting the source to Bob.

An $(n, 2^{nR_x}, 2^{nR_y})$ code for this model consists of an encoding function at Alice, $f_x : X^n \to \{1, \ldots, 2^{nR_x}\}$, an encoding function at Helen, $f_y : Y^n \to \{1, \ldots, 2^{nR_y}\}$, and a decoding function at Bob, $g : \{1, \ldots, 2^{nR_x}\} \times \{1, \ldots, 2^{nR_y}\} \to X^n$. The uncertainty about the source X^n at Eve is measured by $H(X^n|f_x(X^n))/n$. The probability of error in the reconstruction of X^n at Bob is defined as $P_e^n = \Pr(g(f_x(X^n), f_y(Y^n)) \neq X^n)$. A triple (R_x, R_y, Δ) is achievable if for any $\epsilon > 0$, there exists a $(n, 2^{nR_x}, 2^{nR_y})$ code such that $P_e^n \leq \epsilon$ and $H(X^n|f_x(X^n))/n \geq \Delta$. We denote the set of all achievable (R_x, R_y, Δ) rate triples as $\mathcal{R}_{1-sided}$.

B. Result

The main result is given in the following theorem.

Theorem 1: The set of achievable rate triples $\mathcal{R}_{1-sided}$ for secure source coding with one-sided helper is given as

$$\mathcal{R}_{1-sided} = \Big\{(R_x, R_y, \Delta) : R_x \geq H(X|V) \quad (1)$$
$$R_y \geq I(Y; V) \quad (2)$$
$$\Delta \leq I(X; V)\Big\} \quad (3)$$

where the joint distribution of the involved random variables is as follows,

$$p(x, y, v) = p(x, y)p(v|y) \quad (4)$$

and it suffices to consider such distributions for which $|\mathcal{V}| \leq |\mathcal{Y}| + 2$.

The proof of Theorem 1 is omitted here due to space limitations. It can be found in [14].

We note that inner and outer bounds for source coding model considered in this section can be obtained from the results presented in [6] although these bounds do not match in general. These bounds match when Bob has complete uncoded side information Y^n, i.e., when $R_y \geq H(Y)$.

The achievability scheme which yields the rate region described in Theorem 1 is summarized as follows:

1) Helen uses a rate-distortion code to describe the correlated source Y to Bob.
2) Alice performs Slepian-Wolf binning of the source X with respect to the coded side information at Bob.

Therefore, our result shows that the achievable scheme of Ahlswede, Korner [4] and Wyner [5] is optimal in the presence of an eavesdropper. Moreover, on dropping the security constraint, Theorem 1 yields the result of [4], [5].

V. Two-Sided Helper

A. System model

We next consider the following generalization of the model considered in Section 4. In this model, Alice also has access to the coded output of Helen besides the source sequence X^n (see Figure 2). An $(n, 2^{nR_x}, 2^{nR_y})$ code for this model consists of an encoding function at Alice, $f_x : X^n \times \{1, \ldots, 2^{nR_y}\} \to \{1, \ldots, 2^{nR_x}\}$, an encoding function at Helen, $f_y : Y^n \to \{1, \ldots, 2^{nR_y}\}$, and a decoding function at Bob, $g : \{1, \ldots, 2^{nR_x}\} \times \{1, \ldots, 2^{nR_y}\} \to X^n$. The uncertainty about the source X^n at Eve is measured by $H(X^n|f_x(X^n))/n$. The probability of error in the reconstruction of X^n at Bob is defined as $P_e^n = \Pr(g(f_x(X^n, f_y(Y^n)), f_y(Y^n)) \neq X^n)$. A triple (R_x, R_y, Δ) is achievable if for any $\epsilon > 0$, there exists a $(n, 2^{nR_x}, 2^{nR_y})$ code such that $P_e^n \leq \epsilon$ and $H(X^n|f_x(X^n))/n \geq \Delta$. We denote the set of all achievable (R_x, R_y, Δ) rate triples as $\mathcal{R}_{2-sided}$.

B. Result

The main result is given in the following theorem.

Theorem 2: The set of achievable rate triples $\mathcal{R}_{2-sided}$

for secure source coding with two-sided helper is given as

$$\mathcal{R}_{2-sided} = \Big\{(R_x, R_y, \Delta) : R_x \geq H(X|V) \quad (5)$$
$$R_y \geq I(Y;V) \quad (6)$$
$$\Delta \leq \min(I(X;V|U), R_y)\Big\} \quad (7)$$

where the joint distribution of the involved random variables is as follows,

$$p(x,y,v,u) = p(x,y)p(v|y)p(u|x,v) \quad (8)$$

and it suffices to consider distributions such that $|\mathcal{V}| \leq |\mathcal{Y}|+2$ and $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{Y}| + 2|\mathcal{X}|$.

The proof of Theorem 2 is given in the Appendix. We remark here that the proof of the converse for Theorem 2 is closely related to the proof of the converse of the rate-distortion function with a two-sided helper [12], [13], [15].

The achievability scheme which yields the rate region described in Theorem 2 is summarized as follows:

1) Helen uses a rate-distortion code to describe the correlated source Y to both Bob and Alice through a coded output V.
2) Using the coded output V and the source X, Alice jointly quantizes (X,V) to an auxiliary random variable U. She subsequently performs Wyner-Ziv coding, i.e., bins the U sequences at the rate $I(X;U|V)$ such that Bob can decode U by using the side-information V from Helen.
3) Alice also bins the source X at a rate $H(X|U,V)$ so that having access to (U,V), Bob can correctly decode the source X. The total rate used by Alice is $I(X;U|V) + H(X|U,V) = H(X|V)$[2].

Therefore, the main difference between the achievability schemes for Theorems 1 and 2 is at the encoding at Alice and decoding at Bob. Also note that selecting a constant U in Theorem 2 corresponds to Slepian-Wolf type coding at Alice which resulted in an equivocation of $I(X;V)$ in Theorem 1. We will show in the next section through an example that the uncertainty about the source at Eve for the case of two-sided helper can be strictly larger than the case of one-sided helper and selecting U as a constant is clearly suboptimal.

Besides reflecting the fact that the uncertainty at Eve can be strictly larger than the case of a one-sided helper, Theorem 2 has another interesting interpretation. If Alice and Helen can use sufficiently large rates to securely transmit the source X^n to Bob, then the helper can simply transmit a secret key of entropy $H(X)$ to both Alice and Bob. Alice can then use this secret key to losslessly transmit the source to Bob in perfect secrecy [1]. In other words, when R_x and R_y are larger than $H(X)$, one can immediately obtain this result from Theorem 2 by selecting V to be independent of (X,Y) and uniformly distributed on $\{1,\ldots,|\mathcal{X}|\}$. Finally, selecting $U = X \oplus V$, we observe that $\min(R_y, I(X;V|U)) = H(X)$, yielding perfect secrecy.

[2]This achievability scheme can be alternately interpreted as follows: on receiving V from Helen, Alice uses a conditional rate-distortion code to transmit U at a rate $I(X;U|V)$, where U is suitably chosen so that Bob can recover X using U and V, and V acts as the common side information at Alice and Bob.

VI. An Example: Binary Symmetric Sources

Before proceeding to further generalizations of Theorems 1 and 2, we explicitly evaluate Theorems 1 and 2 for a pair of binary sources.

Let X and Y be binary sources with $X \sim \text{Ber}(1/2), Y \sim \text{Ber}(1/2)$ and $X = Y \oplus E$, where $E \sim \text{Ber}(\delta)$. For this pair of sources, the region described in Theorem 1 can be completely characterized as,

$$\mathcal{R}_{1-sided}(R_y) = \big\{(R_x,\Delta) : R_x \geq h(\delta * h^{-1}(1-R_y))$$
$$\Delta \leq 1 - h(\delta * h^{-1}(1-R_y))\big\} \quad (9)$$

and the region in Theorem 2 can be completely characterized as,

$$\mathcal{R}_{2-sided}(R_y) = \big\{(R_x,\Delta) : R_x \geq h(\delta * h^{-1}(1-R_y))$$
$$\Delta \leq \min(R_y, 1)\big\} \quad (10)$$

We start with the derivation of (9). Without loss of generality, we assume that $R_y \leq H(Y)$. Achievability follows by selecting $V = Y \oplus N$, where $N \sim \text{Ber}(\alpha)$, where

$$\alpha = h^{-1}(1-R_y) \quad (11)$$

where $h(.)$ is the binary entropy function. Substituting, we obtain

$$H(X|V) = h(\delta * h^{-1}(1-R_y)) \quad (12)$$
$$I(X;V) = 1 - h(\delta * h^{-1}(1-R_y)) \quad (13)$$

which completes the achievability. Converse follows by simple application of Mrs. Gerber's lemma [16] as follows. Let us be given $R_y \in (0,1)$. We have

$$R_y \geq I(Y;V) \quad (14)$$
$$= H(Y) - H(Y|V) \quad (15)$$
$$= 1 - H(Y|V) \quad (16)$$

which implies $H(Y|V) \geq 1 - R_y$. Mrs. Gerber's lemma states that for $X = Y \oplus E$, with $E \sim \text{Ber}(\delta)$, if $H(Y|V) \geq \beta$, then $H(X|V) \geq h(\delta * h^{-1}(\beta))$. We therefore have,

$$R_x \geq H(X|V) \quad (17)$$
$$\geq h(\delta * h^{-1}(1-R_y)) \quad (18)$$

and

$$\Delta \leq I(X;V) \quad (19)$$
$$= H(X) - H(X|V) \quad (20)$$
$$= 1 - H(X|V) \quad (21)$$
$$\leq 1 - h(\delta * h^{-1}(1-R_y)) \quad (22)$$

This completes the converse.

For the case of two-sided helper, we compute the equivocation Δ as follows. We choose V as $V = Y \oplus N$ where $N \sim \text{Ber}(\alpha)$ as in the case of one-sided helper. We choose

U as,

$$U = X \oplus V \tag{23}$$

We next compute the term $I(X;V|U)$ as follows,

$$\begin{aligned} I(X;V|U) &= I(X;V|X\oplus V) && (24)\\ &= H(X, X\oplus V) - H(X\oplus V) && (25)\\ &= H(X) + H(V|X) - H(X\oplus V) && (26)\\ &= H(X) + H(Y\oplus N|X) \\ &\quad - H(X\oplus Y\oplus N) && (27)\\ &= H(X) + H(X\oplus E\oplus N|X) \\ &\quad - H(X\oplus X\oplus E\oplus N) && (28)\\ &= H(X) && (29)\\ &= 1 && (30) \end{aligned}$$

and therefore

$$\min(R_y, I(X;V|U)) = \min(R_y, 1) \tag{31}$$

For the converse part, we also have that

$$\begin{aligned} \Delta &\le \min(R_y, I(X;V|U)) && (32)\\ &\le \min(R_y, H(X)) && (33)\\ &= \min(R_y, 1) && (34) \end{aligned}$$

The rate from Alice, R_x and the equivocation Δ for the cases of one-sided and two-sided helper are shown in Figure 4 for the case when $\delta = 0.05$. For the one-sided helper, we can observe a trade-off in the amount of information Alice needs to send versus the uncertainty at Eve. For small values of R_y, Alice needs to send more information thereby leaking out more information to Eve. The amount of information leaked is exactly the information sent by Alice.

On the other hand, for the case of two-sided helper, the uncertainty at the eavesdropper is always strictly larger than the uncertainty in the one-sided case. Also note that for this pair of sources, perfect secrecy is possible for the case of two-sided helper when $R_y \ge H(Y)$ which is not possible for the case of one-sided helper.

VII. Secure and Insecure Links from Two-Sided Helper

A. System model

In this section, we consider a generalization of the model considered in Section 5. We consider the case when there are two links from Helen (see Figure 3). One link of rate R_{sec} is secure, i.e., the output of this link is available to only Alice and Bob. The second link of rate R_{ins} is public and the output of this link is available to Alice, Bob and Eve. The sources (X^n, Y^n, W^n, Z^n) are generated i.i.d. according to $p(x,y,w,z) = p(x,y)p(w,z|x)$ where $p(x,y,w,z)$ is defined over the finite product alphabet $\mathcal{X}\times\mathcal{Y}\times\mathcal{W}\times\mathcal{Z}$.

A $(n, 2^{nR_x}, 2^{nR_{sec}}, 2^{nR_{ins}})$ code for this model consists of an encoding function at Helen, $f_y : Y^n \to J_{sec}\times J_{ins}$, where $|J_{sec}| \le 2^{nR_{sec}}$, $|J_{ins}| \le 2^{nR_{ins}}$, an encoding function at Alice, $f_x : X^n \times J_{sec}\times J_{ins} \to \{1,\dots,2^{nR_x}\}$, and a decoding function at Bob, $g : \{1,\dots,2^{nR_x}\}\times J_{sec}\times J_{ins}\times W^n \to X^n$.

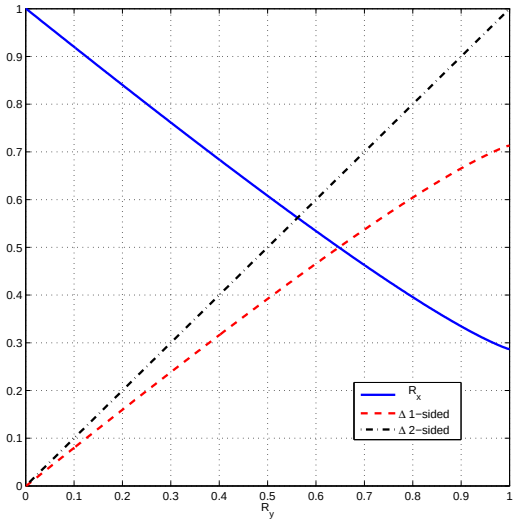

Fig. 4. The rate-equivocation region for a pair of binary symmetric sources.

The uncertainty about the source X^n at Eve is measured by $H(X^n|f_x(X^n, J_{sec}, J_{ins}), J_{ins}, Z^n)/n$. The probability of error in the reconstruction of X^n at Bob is defined as $P_e^n = \Pr(g(f_x(X^n, J_{sec}, J_{ins}), J_{sec}, J_{ins}, W^n) \ne X^n)$. A quadruple $(R_x, R_{sec}, R_{ins}, \Delta)$ is achievable if for any $\epsilon > 0$, there exists a $(n, 2^{nR_x}, 2^{nR_{sec}}, 2^{nR_{ins}})$ code such that $P_e^n \le \epsilon$ and $H(X^n|f_x(X^n, J_{sec}, J_{ins}), J_{ins}, Z^n)/n \ge \Delta$. We denote the set of all achievable $(R_x, R_{sec}, R_{ins}, \Delta)$ rate quadruples as $\mathcal{R}_{2-sided}^{W,Z}$.

B. Result

The main result is given in the following theorem.

Theorem 3: The set of achievable rate quadruples $\mathcal{R}_{2-sided}^{W,Z}$ for secure source coding with secure and insecure links from a two-sided helper, additional side information W at Bob and Z at Eve is given as

$$\begin{aligned} \mathcal{R}_{2-sided}^{W,Z} = \Big\{ &(R_x, R_{sec}, R_{ins}, \Delta) : \\ &R_x \ge H(X|V_1, V_2, W) && (35)\\ &R_{sec} \ge I(Y;V_1|W) && (36)\\ &R_{ins} \ge I(Y;V_2|W, V_1) && (37)\\ &\Delta \le \min(R_{sec}, I(X;V_1|U, V_2, W)) \\ &\qquad + I(X;W|U,V_2) - I(X;Z|U,V_2)\Big\} && (38) \end{aligned}$$

where the joint distribution of the involved random variables is as follows,

$$\begin{aligned} &p(x,y,w,z,v_1,v_2,u) \\ &\quad = p(x,y)p(w,z|x)p(v_1,v_2|y)p(u|x,v_1,v_2) \end{aligned} \tag{39}$$

and it suffices to consider such distributions for which $|\mathcal{V}_1| \le |\mathcal{Y}|+3$, $|\mathcal{V}_2| \le |\mathcal{Y}|+4$ and $|\mathcal{U}| \le |\mathcal{X}||\mathcal{Y}|^2+7|\mathcal{X}||\mathcal{Y}|+12|\mathcal{X}|+2$.

The proof of Theorem 3 is omitted here due to space limitations. It can be found in [14].

The achievability scheme which yields the rate region described in Theorem 3 is summarized as follows:

1) Helen first uses the secure link to describe the source Y to both Alice and Bob at a rate $I(Y;V_1|W)$, where W plays the role of side information. Subsequently, the insecure link is used to provide another description of the correlated source Y at a rate $I(Y;V_2|W,V_1)$, where (W,V_1) plays the role of side information. Therefore, the key idea is to first use the secure link to build common randomness at Alice and Bob and subsequently use this common randomness to send information over the insecure link at a lower rate.
2) Having access to the coded outputs (V_1,V_2) from Helen and the source X, Alice jointly quantizes (X,V_1,V_2) to an auxiliary random variable U. She subsequently performs Wyner-Ziv coding, i.e., bins the U sequences at the rate $I(X;U|V_1,V_2,W)$ such that Bob can decode U by using W and the coded outputs (V_1,V_2) from the helper.
3) She also bins the source X at a rate $H(X|U,V_1,V_2,W)$ so that having access to (U,V_1,V_2,W), Bob can correctly decode the source X. The total rate used by Alice is $I(X;U|V_1,V_2,W) + H(X|U,V_1,V_2,W) = H(X|V_1,V_2,W)$.

We note here that using Theorem 3, we can recover Theorem 2 by setting $R_{ins}=0$, and selecting $W=Z=V_2=\phi$.

The resulting model when $R_{ins}=0$ and $W=\phi$, is related to the model considered in [10], where K legitimate users are interested in lossless reconstruction of the source X. The kth legitimate user has access to uncoded correlated side information Y_k, where $Y_k^n = X^n \oplus B_k^n$, for $k=1,\dots K$, and eavesdropper has access to Z^n, where $Z^n = X^n \oplus E^n$, and the noise sequences $(B_1^n,\dots,B_K^n,E^n)$ are mutually independent and also independent of the source X^n. Furthermore, it was assumed that Alice also has access to $(Y_1^n,\dots Y_K^n)$. For the case when eavesdropper does not have additional side information, it was shown in [10] that to maximize the uncertainty at the eavesdropper, Alice cannot do any better than describing the error sequences $(B_1^n,\dots,B_K^n)$ to the legitimate users. Formalizing this idea to general sources for our model, this is tantamount to using a conditional rate-distortion code for transmitting the resulting error in decoding X^n from the common coded output available at Alice and Bob.

On setting $R_{sec}=0$, the resulting model is similar to the one considered in [11] although the aim in [11] is to generate a secret key to be shared by Alice and Bob. On the other hand, we are interested in the secure transmission of the source X.

If $R_{sec}=R_{ins}=0$, then we recover the result of [7] as a special case by setting $V_1=V_2=\phi$. Therefore, Theorem 3 can also be viewed as a generalization of the result of [7] where no rate constraint is imposed on the transmission of Alice and the goal is to maximize the uncertainty at Eve while losslessly transmitting the source to Bob.

VIII. Conclusions

In this paper, we considered several secure source coding problems. We first provided the characterization of the rate-equivocation region for a secure source coding problem with coded side information at the legitimate user. We next generalized this result for two different models with increasing complexity. We characterized the rate-equivocation region for the case of two-sided helper. The value of two-sided coded side information is emphasized by comparing the respective equivocations for a pair of binary sources. It is shown for this example that Slepian-Wolf type coding alone is insufficient and using our achievable scheme, one attains strictly larger uncertainty at the eavesdropper than the case of one-sided helper. We next considered the case when there are both secure and insecure rate-limited links from the helper and characterized the rate-equivocation region.

IX. Appendix

A. Proof of Theorem 2

Achievability: Fix the distribution $p(x,y,v) = p(x,y)p(v|y)p(u|x,v)$.

1) Codebook generation at Helen: From the conditional probability distribution $p(v|y)$ compute $p(v) = \sum_y p(y)p(v|y)$. Generate $2^{nI(V;Y)}$ codewords $v(l)$ independently according to $\prod_{i=1}^n p(v_i)$, where $l = 1,\dots,2^{nI(V;Y)}$.
2) Codebook generation at Alice: From the distribution $p(u|x,v)$, compute $p(u)$. Generate $2^{nI(X,V;U)}$ sequences $u(s)$ independently according to $\prod_{i=1}^n p(u_i)$, where $s = 1,\dots,2^{nI(X,V;U)}$. Next, bin these sequences uniformly into $2^{nI(X;U|V)}$ bins.
   Also, randomly bin the x^n sequences into $2^{nH(X|U,V)}$ bins and index these bins as $m=1,\dots,2^{nH(X|U,V)}$.
3) Encoding at Helen: On observing the sequence y^n, Helen tries to find a sequence $v(l)$ such that $(v(l),y^n)$ are jointly typical. From rate-distortion theory, we know that there exists one such sequence. Helen sends the index l of the sequence $v(l)$.
4) Encoding at Alice: The key difference from the one-sided helper case is in the encoding at Alice. On observing the sequence x^n, Alice first finds the bin index m_X in which the sequence x^n falls. Alice also has the sequence $v(l)$ received from Helen. Alice next finds a sequence u such that $(u,v(l),x^n)$ are jointly typical. Let the bin index of this resulting u sequence be s_U.
   Alice transmits the pair (s_U,m_X) which is received by Bob and Eve. The total rate used by Alice is $I(X;U|V)+H(X|U,V)=H(X|V)$.
5) Decoding at Bob: On receiving the pair (s_U,m_X) from Alice and the index l from Helen, Bob first searches the bin s_U for a sequence $\hat{u}$ such that $(\hat{u},v(l))$ are

jointly typical. This is possible since the number of u sequences in each auxiliary bin is approximately $2^{nI(X,V;U)}/2^{nI(X;U|V)}$ which is $2^{nI(U;V)}$ and therefore with high probability, Bob will be able to obtain the correct u sequence.

Using the estimate $\hat{u}$ and $v(l)$, Bob searches for a unique x^n sequence in the bin m_X such that $(\hat{u}, v(l), x^n)$ are jointly typical. This is possible since the number of x^n sequences in each bin is approximately $2^{nH(X)}/2^{nH(X|U,V)}$ which is $2^{nI(U,V;X)}$.

6) Equivocation:

$$\begin{aligned}
H(X^n|s_U, m_X) &= H(X^n, m_X, s_U) - H(m_X, s_U) && (40)\\
&= H(X^n) + H(m_X, s_U|X^n) \\
&\quad - H(m_X, s_U) && (41)\\
&= H(X^n) + H(s_U|X^n) \\
&\quad - H(m_X, s_U) && (42)\\
&\geq H(X^n) + H(s_U|X^n) - H(s_U) \\
&\quad - H(m_X) && (43)\\
&= H(X^n) - H(m_X) \\
&\quad - I(s_U; X^n) && (44)\\
&\geq H(X^n) - nH(X|U,V) \\
&\quad - I(s_U; X^n) && (45)\\
&= nI(X;U,V) - I(s_U; X^n) && (46)\\
&\geq nI(X;U,V) - I(U^n; X^n) && (47)\\
&\geq nI(X;U,V) - nI(U;X) && (48)\\
&= nI(X;V|U) && (49)\\
&\geq n\min(I(X;V|U), R_y) && (50)
\end{aligned}$$

where (42) follows from the fact that m_X is the bin index of the sequence X^n, (43) follows from the fact that conditioning reduces entropy, (45) follows from the fact that $H(m_X) \leq \log(2^{nH(X|U,V)})$, (47) follows from the fact that s_U is the bin index of the sequence U^n, i.e., $s_U \to U^n \to X^n$ forms a Markov chain and subsequently using the data-processing inequality. Finally, (50) follows from the fact that $\min(I(X;V|U), R_y) \leq I(X;V|U)$. We therefore have

$$\Delta \leq \min(I(X;V|U), R_y) \tag{51}$$

This completes the achievability part for the case of two-sided helper.

Converse: Let the output of Helen be J_y, and the output of Alice be denoted J_x, i.e.,

$$J_y = f_y(Y^n), \quad J_x = f_x(X^n, J_y) \tag{52}$$

First note that, for noiseless reconstruction of the sequence X^n at the legitimate decoder, we have by Fano's inequality [17]

$$H(X^n|J_x, J_y) \leq n\epsilon_n \tag{53}$$

We start by obtaining a lower bound on R_x as follows

$$\begin{aligned}
nR_x &\geq H(J_x) && (54)\\
&\geq H(J_x|J_y) && (55)\\
&= H(X^n, J_x|J_y) - H(X^n|J_x, J_y) && (56)\\
&\geq H(X^n, J_x|J_y) - n\epsilon_n && (57)\\
&\geq H(X^n|J_y) - n\epsilon_n && (58)\\
&= \sum_{i=1}^{n} H(X_i|X^{i-1}, J_y) - n\epsilon_n && (59)\\
&= \sum_{i=1}^{n} H(X_i|V_i) - n\epsilon_n && (60)\\
&= nH(X_Q|V_Q, Q) - n\epsilon_n && (61)\\
&= nH(X|V) - n\epsilon_n && (62)
\end{aligned}$$

where (57) follows by (53). In (60), we have defined

$$V_i = (J_y, X^{i-1}) \tag{63}$$

Next, we obtain a lower bound on R_y, the rate of Helen,

$$\begin{aligned}
nR_y &\geq H(J_y) && (64)\\
&\geq I(J_y; Y^n) && (65)\\
&= \sum_{i=1}^{n} I(J_y, Y^{i-1}; Y_i) && (66)\\
&= \sum_{i=1}^{n} I(J_y, Y^{i-1}, X^{i-1}; Y_i) && (67)\\
&\geq \sum_{i=1}^{n} I(J_y, X^{i-1}; Y_i) && (68)\\
&= \sum_{i=1}^{n} I(V_i; Y_i) && (69)\\
&= nI(V_Q; Y_Q|Q) && (70)\\
&= nI(V_Q, Q; Y_Q) && (71)\\
&= nI(V; Y) && (72)
\end{aligned}$$

where (67) follows from the Markov chain

$$X^{i-1} \to (J_y, Y^{i-1}) \to Y_i \tag{73}$$

We will now derive an upper bound on the equivocation rate of the eavesdropper:

$$\begin{aligned}
H(X^n|J_x) &= H(X^n, J_y|J_x) - H(J_y|X^n, J_x) && (74)\\
&= H(J_y|J_x) - H(J_y|X^n, J_x) \\
&\quad + H(X^n|J_x, J_y) && (75)\\
&\leq I(X^n; J_y|J_x) + n\epsilon_n && (76)\\
&= \sum_{i=1}^{n} I(X_i; J_y|J_x, X^{i-1}) + n\epsilon_n && (77)\\
&= \sum_{i=1}^{n} I(X_i; J_y, X^{i-1}|J_x, X^{i-1}) + n\epsilon_n && (78)
\end{aligned}$$

$$= \sum_{i=1}^{n} I(X_i; V_i|U_i) + n\epsilon_n \tag{79}$$
$$= nI(X_Q; V_Q|U_Q, Q) + n\epsilon_n \tag{80}$$
$$= nI(X_Q; V_Q, Q|U_Q, Q) + n\epsilon_n \tag{81}$$
$$= nI(X; V|U) + n\epsilon_n \tag{82}$$

where (76) follows from (53) and (79) follows by defining

$$U_i = (J_x, X^{i-1}) \tag{83}$$

Finally, in going from (60) to (62), from (69) to (72), and from (79) to (82), we have defined

$$X = X_Q, \quad Y = Y_Q, \tag{84}$$
$$U = (U_Q, Q), \quad V = (V_Q, Q) \tag{85}$$

where Q is uniformly distributed on $\{1, \ldots, n\}$ and is independent of all other random variables.

We also have,

$$H(X^n|J_x) = H(X^n, J_y|J_x) - H(J_y|X^n, J_x) \tag{86}$$
$$= H(J_y|J_x) - H(J_y|X^n, J_x) + H(X^n|J_x, J_y) \tag{87}$$
$$\leq H(J_y|J_x) + n\epsilon_n \tag{88}$$
$$\leq H(J_y) + n\epsilon_n \tag{89}$$
$$\leq nR_y + n\epsilon_n \tag{90}$$

This together with (82) implies

$$\Delta \leq \min(I(X; V|U), R_y) \tag{91}$$

The joint distribution of the involved random variables is as follows,

$$p^{out}(x, y, v, u) = p(x, y)p(v|y)p^{out}(u|x, v, y) \tag{92}$$

Note that in the achievability proof of Theorem 2, joint distributions of the following form are permitted,

$$p^{ach}(x, y, v, u) = p(x, y)p(v|y)p^{ach}(u|x, v) \tag{93}$$

i.e, we have the Markov chain, $Y \to (X, V) \to U$. With the definition of V and U as in (63) and (83), these random variables do not satisfy this Markov chain. This implies that what we have shown so far is the following,

$$\mathcal{R}_{2-sided} \subseteq \mathcal{R}_{out} \tag{94}$$

where

$$\mathcal{R}_{out} = \Big\{(R_x, R_y, \Delta) : R_x \geq H(X|V) \tag{95}$$
$$R_y \geq I(Y; V) \tag{96}$$
$$\Delta \leq \min(I(X; V|U), R_y)\Big\} \tag{97}$$

where the joint distribution of the involved random variables is as given in (92).

However, we observe that the term $I(X; V|U)$ depends only on the marginal $p^{out}(u|x, v)$. Similarly, the terms $H(X|V)$ and $I(X; V)$ depend only on the marginal $p(x, v)$. We use these observations to show that the region $\mathcal{R}_{out}$ is the same when it is evaluated using the joint distributions of the form given in (93). This is clear by noting that once we are given a distribution of the form given in (92), we can construct a new distribution of the form given in (93) with the same rate expressions. Consider any distribution $p^{out}(x, y, v, u)$ of the form given in (92). Using $p^{out}(x, y, v, u)$, compute the marginal $p^{out}(u|x, v)$ as,

$$p^{out}(u|x, v) = \frac{\sum_y p^{out}(x, y, u, v)}{p(x, v)} \tag{98}$$

We now construct a distribution $p^{ach}(x, y, v, u) \in \mathcal{P}_{ach}$ as,

$$p^{ach}(x, y, v, u) = p(x, y)p(v|y)p^{out}(u|x, v) \tag{99}$$

such that the terms $I(X; V|U)$, $H(X|V)$ and $I(X; V)$ are the same whether they are evaluated according to $p^{out}(x, y, v, u)$ or according to $p^{ach}(x, y, v, u)$. Therefore, we conclude that it suffices to consider input distributions satisfying the Markov chain $Y \to (X, V) \to U$ when evaluating $\mathcal{R}_{out}$ and hence $\mathcal{R}_{out} = \mathcal{R}_{ach}$. This completes the converse part. We remark here that this observation was useful in obtaining the converse for the rate-distortion function with common coded side information [12], [13], [15].

REFERENCES

[1] C. E. Shannon. Communication theory of secrecy systems. *Bell System Technical Journal*, 28(4):656–715, October 1949.

[2] A. D. Wyner. The wire-tap channel. *Bell System Technical Journal*, 54(8):1335–1387, January 1975.

[3] I. Csiszar and J. Korner. Broadcast channels with confidential messages. *IEEE Trans. on Information Theory*, 24(3):339–348, May 1978.

[4] R. Ahlswede and J. Korner. Source coding with side information and a converse for degraded broadcast channels. *IEEE Trans. on Information Theory*, 21(6):629–637, Nov 1975.

[5] A. D. Wyner. On source coding with side information at the decoder. *IEEE Trans. on Information Theory*, 21(3):294–300, May 1975.

[6] D. Gunduz, E. Erkip, and H. V. Poor. Secure lossless compression with side information. In *IEEE Information Theory Workshop*, 2008.

[7] V. Prabhakaran and K. Ramchandran. On secure distributed source coding. In *IEEE Information Theory Workshop*, 2007.

[8] W. Luh and D. Kundur. Distributed keyless secret sharing over noiseless channels. In *IEEE Globecom*, 2007.

[9] D. Gunduz, E. Erkip, and H. V. Poor. Lossless compression with security constraints. In *IEEE International Symposium on Information Theory*, 2008.

[10] L. Grokop, A. Sahai, and M. Gastpar. Discriminatory source coding for a noiseless broadcast channel. In *IEEE International Symposium on Information Theory*, 2005.

[11] I. Csiszar and P. Narayan. Common randomness and secret key generation with a helper. *IEEE Trans. on Information Theory*, 46(2):344–366, March 2000.

[12] A. Kaspi and T. Berger. Rate-distortion for correlated sources with partially separated encoders. *IEEE Trans. on Information Theory*, 28(6):828–840, Nov 1982.

[13] D. Vasudevan and E. Perron. Cooperative source coding with encoder breakdown. In *IEEE International Symposium on Information Theory*, 2007.

[14] R. Tandon, S. Ulukus, and K. Ramchandran. Secure source coding with a helper. To be submitted.

[15] H. Permuter, Y. Steinberg, and T. Weissman. Problems we can solve with a helper. In *IEEE Information Theory Workshop*, 2009.

[16] A. D. Wyner and J. Ziv. A theorem on the entropy of certain binary sequences and applications-I. *IEEE Trans. on Information Theory*, 19(6):769–772, Nov 1973.

[17] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. New York:Wiley, 1991.