

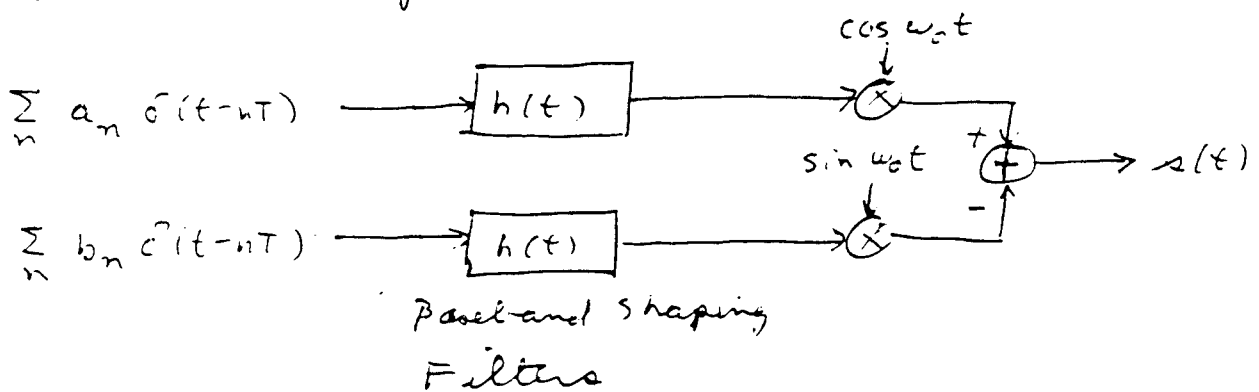
TRELLIS CODES

An Example of Convolutional Coding combined with Modulation and Soft Decoding

References:

1. G. Ungerboeck, "Channel Coding with Multilevel / Phase Signals," IEEE Trans. on Info. Theory, Vol IT-28, No 1, January 1982, pp. 55-67. (First presented at IT Symposium, Sweden, '76)
2. R. E. Blahut, Theory and Practice of Error Control Codes, Addison-Wesley, 1983, pp 473-478.

* Additional references on p 102(4), Ungerboeck survey papers in Comm. Magazines
 Biglieri, Divsalar, McLane, Simon, Intro to Trellis Coded Modulation
 Quadrature Amplitude Modulation (QAM)



no intersymbol interference $h(nT) = \delta_{n,0}$

$$s(t) = \operatorname{Re} \sum_n (a_n + j b_n) h(t-nT) e^{j \omega_c t}$$

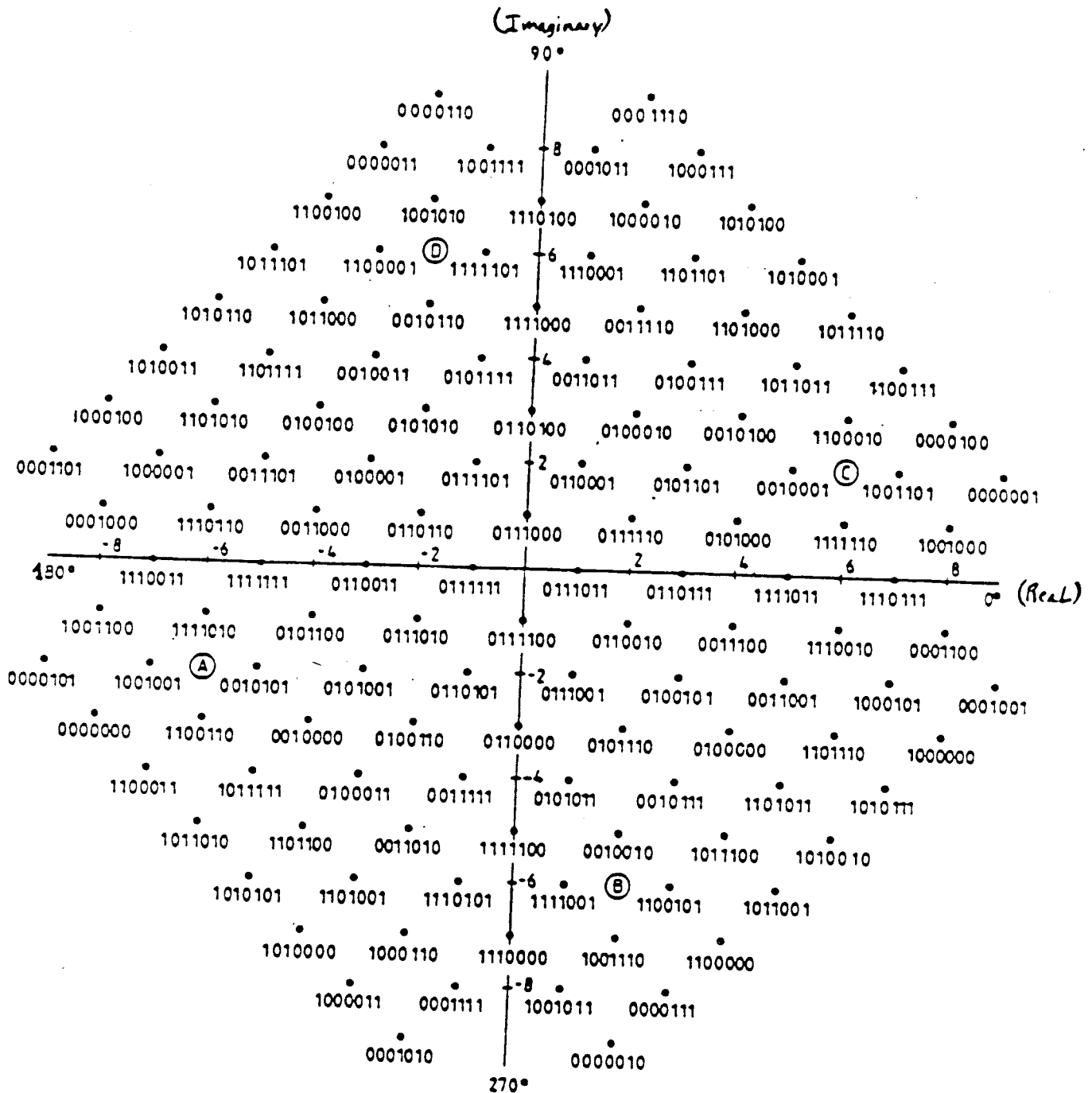
Equivalent baseband signal

$$q(t) = \sum_n (a_n + j b_n) h(t-nT)$$

notice

$$q(nT) = a_n + j b_n = c_n$$

can achieve any 2 dimensional signal "constellation".



Binary numbers refer to Q6_n, Q5_n, Q4_n, Q3_n, Y2_n, Y1_n, Y0_n
A, B, C, D refer to synchronizing signal elements

FIGURE 2/V.33 or V.32bis
SIGNAL SPACE DIAGRAM AND MAPPING FOR TRELLIS-CODED MODULATION
AT 14400 BITS PER SECOND

Received signal

$$r(nT) = c_n + v_n$$

where v_n is white Gaussian complex noise sequence with pdf

$$f(v) = \frac{1}{\pi \sigma_v^2} e^{-\frac{|v|^2}{\sigma_v^2}}$$

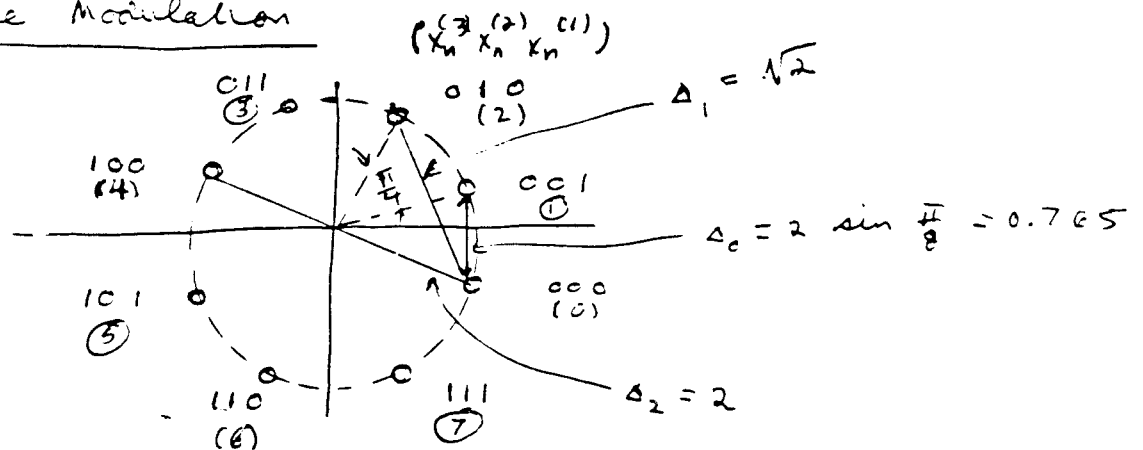
where $\sigma_v^2 = E\{|v_n|^2\}$. Assume c_n 's selected from alphabet $\{d_1, d_2, \dots, d_M\}$. Then channel capacity is

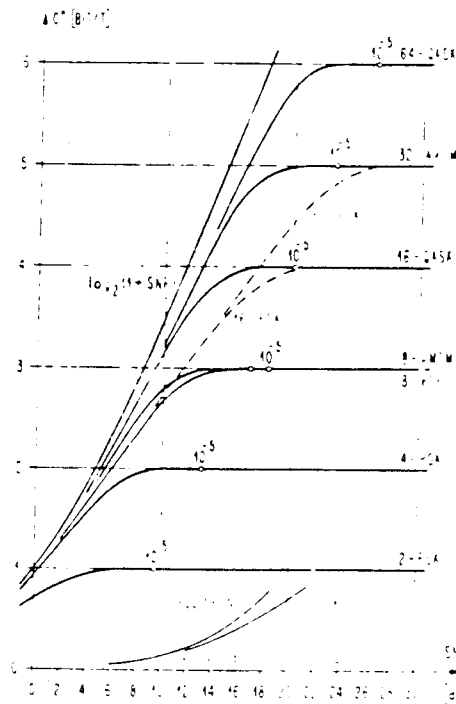
$$C = \max_{\{P_i\}} \sum_{k=1}^M P_k \int f(r|d_k) \log_2 \frac{f(r|d_k)}{\sum_{i=1}^M P_i f(r|d_i)} dr$$

where $P_i = P(c_n = d_i)$. For a symmetric signal set, the maximum is achieved with equally likely inputs, i.e., $P_i = 1/M \forall i$. Thus

$$C = \log_2 M - \frac{1}{M} \sum_{k=1}^M \int \frac{1}{\pi \sigma_v^2} e^{-\frac{|v|^2}{\sigma_v^2}} \log_2 \sum_{i=1}^M e^{-\frac{|v-d_i+d_k|^2}{\sigma_v^2}} dv$$

8-PSK Modulation

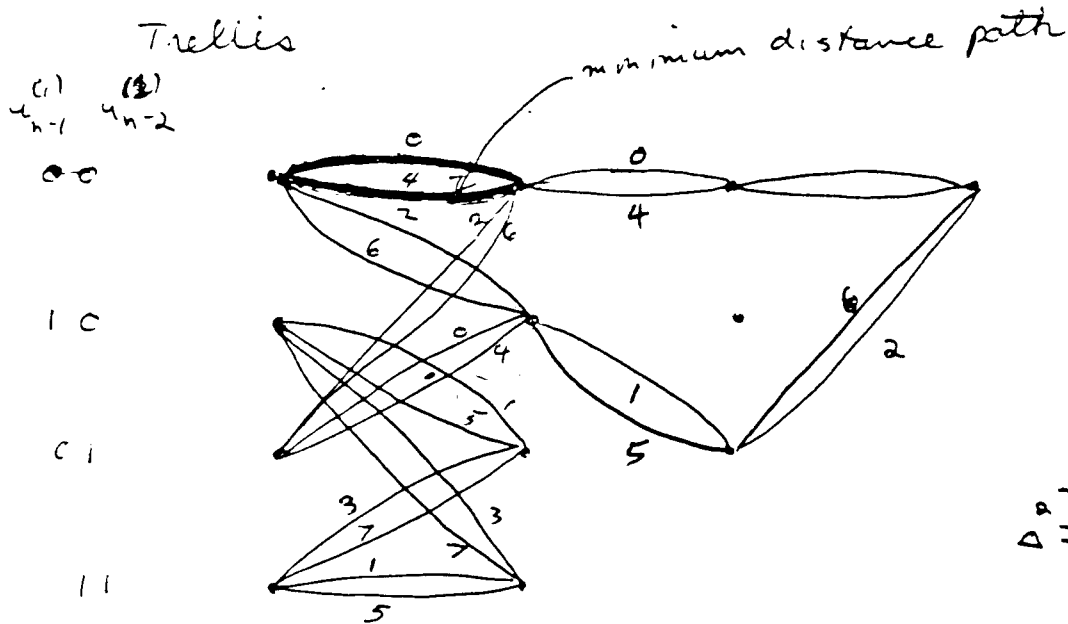




Consider $(x_n^{(3)}, x_n^{(2)}, x_n^{(1)})$ as the binary number

$$N = 4x_n^{(3)} + 2x_n^{(2)} + x_n^{(1)}$$

Then the modulator selects phase N shown on the 8-PSK diagram.



$$d_f = \Delta_2 = 2$$

so gain relative to 4 PSK is

$$20 \log_{10} \frac{4}{\Delta_1} = 20 \log_{10} \frac{2}{\sqrt{2}} = 3 \text{ dB}$$

Notice that at each transmission either the 4-PSK set 0, 2, 4, 6 or the set 1, 3, 5, 7 is chosen so the performance must be at least as good as 4-PSK.

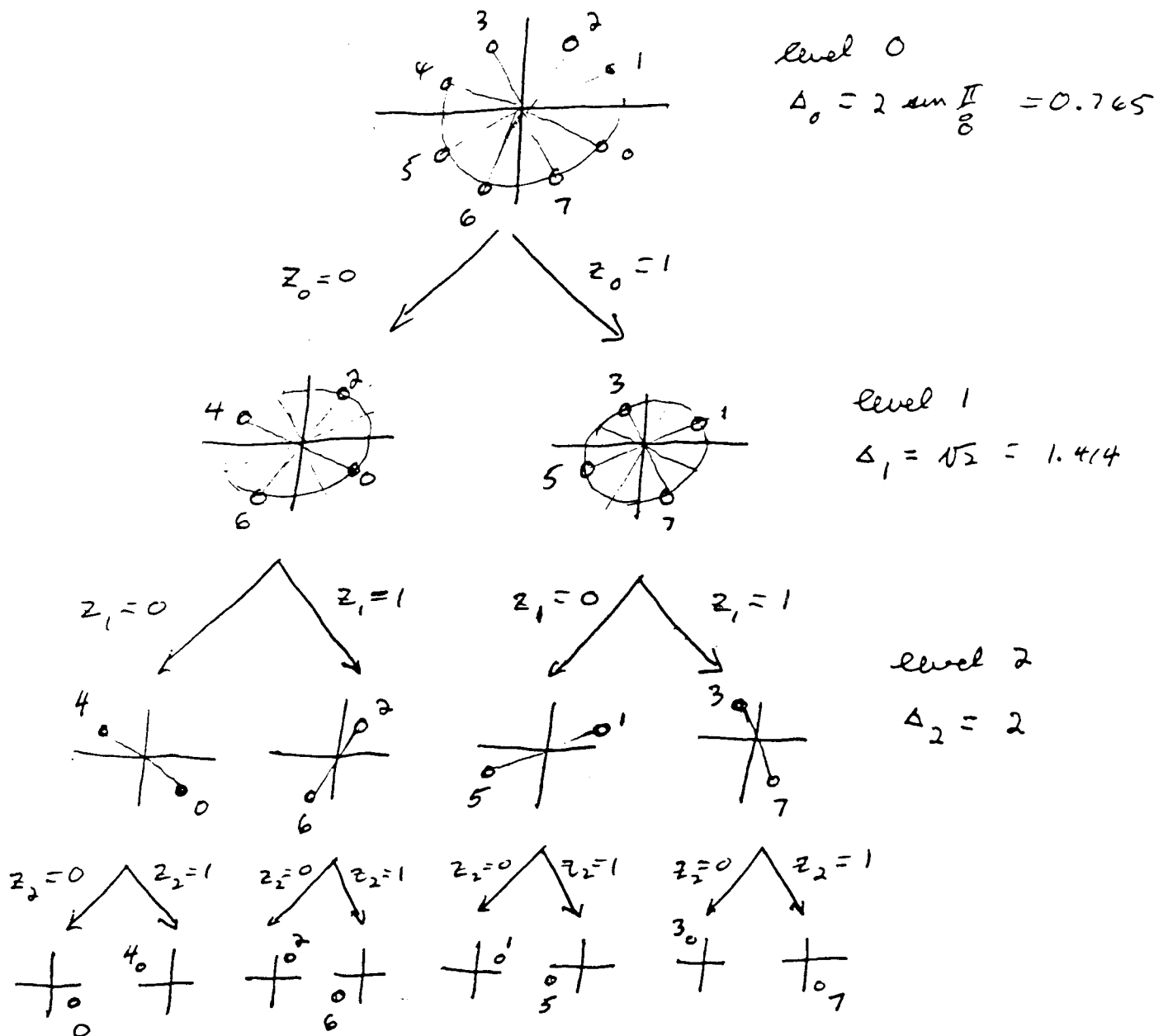
More References:

3. Lee-Fang Wei, "Rotationally Invariant Convolutional Channel Coding with Expanded Signal Space, Part I: 180°," IEEE Transactions on Selected Areas in Communications, Vol SAC-2, No. 5, Sept 1984, pp. 659-671.
4. Lee-Fang Wei, "Rotationally Invariant Channel Coding with Expanded Signal Space - Part II: Nonlinear Codes," op cit, pp 672-686.

More references

5. Lee-Fang Wei, "Trellis-Coded Modulation with Multi-Dimensional Constellations," ~~Submitted to~~ IEEE Trans. Info. Theory, Vol. IT-31, No. 4, ~~Aug.~~ 1985. July 1987, pp. 483-501
6. G. Ungerboeck, "Trellis Coded Modulation with Redundant Signal Sets," Parts I and II, IEEE Communications Magazine, February 1987, Vol. 25, No. 2, pp 5-21.

Set Partitioning Concept



For example code

$$(x_n^{(3)} x_n^{(2)} x_n^{(1)}) \leftrightarrow (z_2 z_1 z_0)$$

General Principles

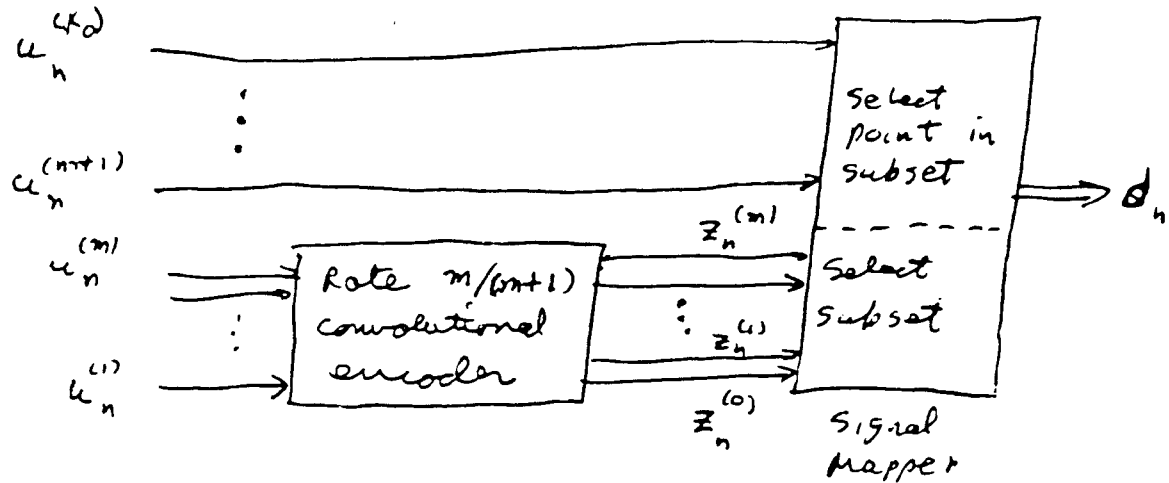
- (1) coded outputs determine subset
($x_n^{(2)}, x_n^{(1)}$ in example), ($z^{(n)}, \dots, z^{(1)}, z^0$)
- (2) uncoded bits determine point within subset
($x_n^{(3)}$ in example)
- (3) branches leaving a given state all belong to same subset at appropriate level
- (4) all parallel transitions belong to same subset at level determined by all coded bits
- (5) branches merging to a given state belong to same subset

Other considerations: might want invariance to 90° rotations, etc.

Design Approach

1. Partition the constellation
2. Assign subsets to transitions according to rules above
3. Search for convolutional code to maximize minimum squared Euclidean distance.

General structure of Encoder/Modulator for Trellis Coded Modulation



There are m coded inputs. We require encoder state to make transitions to 2^m unique states from any given state.

There are $K_0 - m$ uncoded inputs $u_n^{(m)}, \dots, u_n^{(k_0)}$. These specify $2^{K_0 - m}$ parallel branches for each transition. Thus, from a given state there are $2^m 2^{K_0 - m} = 2^{K_0}$ branches leaving the state which allows $K_0 = \log_2 2^{K_0}$ bits to be transmitted on each channel symbol.

Then

$$d_f = \min [\Delta_{m+1}, d_f(m)]$$

where $d_f(m)$ is minimum distance between nonparallel paths.

More Properties of $(m+1, m)$ Codes

Let $[z^{(0)}(D), z^{(1)}(D), \dots, z^{(m)}(D)]$

$$= [u^{(1)}(D), \dots, u^{(m)}(D)] \begin{bmatrix} B(D) & A(D) \\ m \times 1 & m \times m \end{bmatrix}$$

where $\det A(D) = h_0(D) \neq 0$, the equivalent systematic code is

$$\begin{aligned} [z^{(0)}, z^{(1)}, \dots, z^{(m)}] &= [u^{(1)}, \dots, u^{(m)}] [A^{-1}(D)B(D) \mid I_{m \times m}] \\ &= [z^{(1)}, \dots, z^{(m)}] \begin{bmatrix} h_1(D)/h_0(D) & & & \\ h_2(D)/h_0(D) & & & \\ \vdots & & & \\ h_m(D)/h_0(D) & & & \\ & & & I_{m \times m} \end{bmatrix} \end{aligned}$$

where $h_i(D)$ $i = 1, \dots, m$ are also polynomials. Thus, the check symbol $z^{(0)}(D)$ satisfies:

$$z^{(0)}(D) = \sum_{i=1}^m z^{(i)}(D) h_i(D)/h_0(D)$$

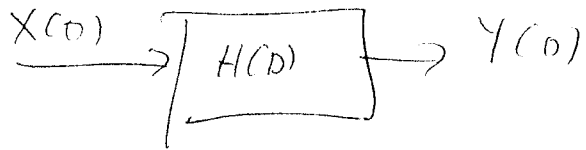
or (for $GF(D)$)

$$\sum_{i=0}^m z^{(i)}(D) h_i(D) = 0$$

Let $\deg h_i(D) \leq v$ for $i = 0, \dots, m$. Then

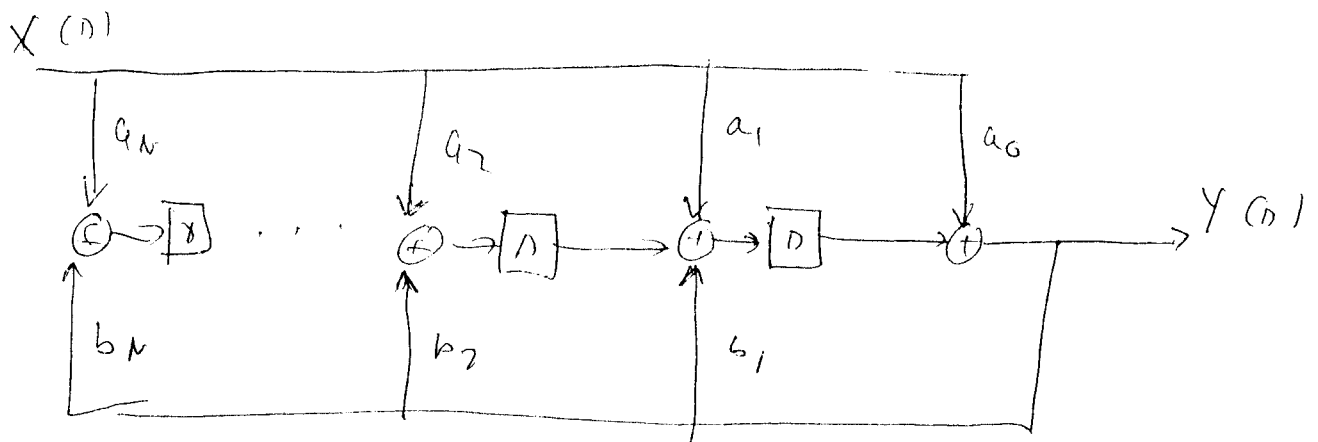
$$\sum_{i=0}^m \sum_{k=0}^v h_{i,k} z_{n-k}^{(i)} = 0 \quad \text{for } n = 0, 1, \dots$$

called parity check equations. Notice that

Realization

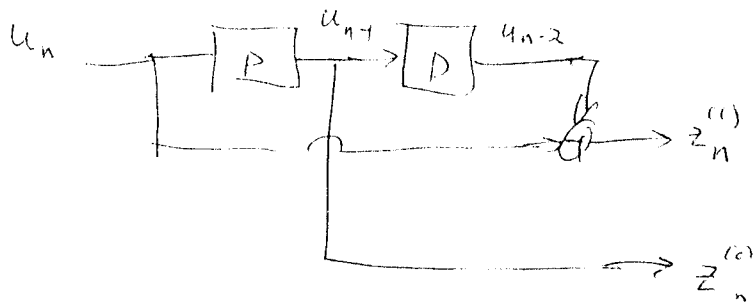
$$\frac{Y(D)}{X(D)} = H(D) = \frac{A(D)}{B(D)} = \frac{a_0 + a_1 D + \dots + a_N D^N}{1 + b_1 D + \dots + b_N D^N}$$

$$\begin{aligned} Y(D) &= X(D) A(D) + Y(D) [b_1 D + \dots + b_N D^N] \\ &= a_0 X(D) + \sum_{i=1}^N [a_i X(D) - b_i Y(D)] D^i \end{aligned}$$



102 (9) b

Ungerboeck 4 state code

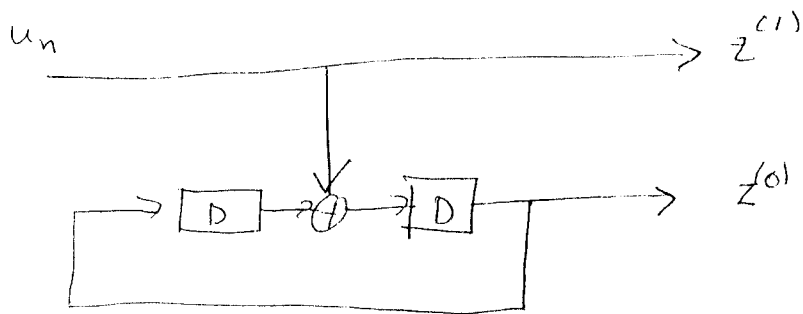


$$\begin{bmatrix} z_n^{(0)}(D) & z_n^{(1)}(D) \end{bmatrix} = u(D) \begin{bmatrix} D & 1+D^2 \end{bmatrix}$$

$$= \underbrace{u(D) (1+D^2)}_{\hat{u}(D)} \begin{bmatrix} \frac{D}{1+D^2} & 1 \end{bmatrix}$$

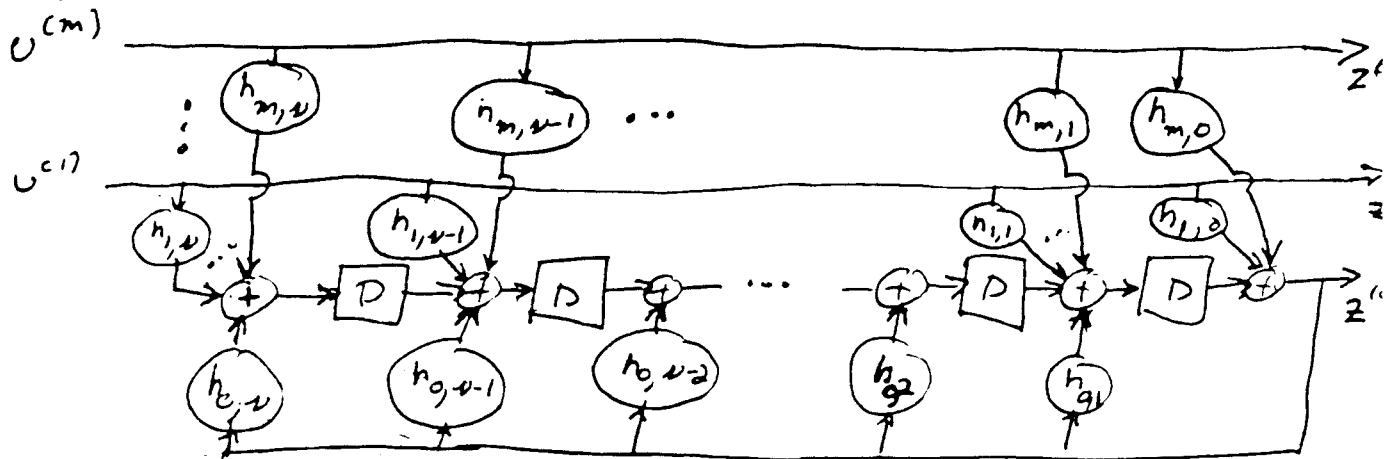
$$\hat{u}(D) = (1+D^2) u(D)$$

Systematic Form



the sequences $\{z_n^{(1)}, \dots, z_n^{(m)}\}$ can be selected arbitrarily with $\{z_n^{(0)}\}$ chosen to satisfy parity check equations.

Encoder Realization



$h_{0,v} = 1$, and

Notice that if $h_{i,0} = 0$ and $h_{i,v} = 0$ for $i = 1, \dots, m$, then $z_n^{(0)}$ same for all paths diverging from a state and also $z_n^{(0)}$ same for all paths converging to a state. This guarantees at least distance Δ_1 between such branches.

a lower bound on $d_f(m)$

Note: $\underline{z}_n = [z_n^{(m)} z_n^{(m-1)} \dots z_n^{(1)} z_n^{(0)}]$

Consider code sequences $\{\underline{z}_n\}$ and $\{\underline{z}'_n\} = \{\underline{z}_n + \underline{e}_n\}$.

Then $\{\underline{e}_n\}$ is also a code sequence. Then

$$d_f^2(m) \geq \min_{\{\underline{e}_n\} \neq \{0\}} \sum_n \Delta^2 q(\underline{e}_n)$$

where

$\{\underline{e}_n\}$ is code word that diverges from $\{0\}$ & later re merges forever

$q(\underline{e}_n)$ = number of 0's on right hand side of $\underline{e}_n$
 = number of trailing positions in which subset labels $\underline{z}_n$ and $\underline{z}'_n$ agree.

and $\Delta q(0) = 0$ (since symbols from same subset)

Minimization is taken over all code sequences $\{z_n\}$ that diverge from $\{z\}$ and later remerge. The fact that $z_n^{(n)}, \dots, z_n^{(1)}$ are arbitrary guarantees that there exists two coded sequences $\{z_n\}$ and $\{z'_n\}$ that at each time have channel signals that are closest together in the subsets whose labels z_n and z'_n differ by z_n . Usually, this closest distance is $\Delta q(z_n)$ for all z_n and the above bound becomes an equality. Only when the subsets contain very few signals may the bound not be satisfied with equality.

This bound is the basis for search for optimum codes. We can use known techniques for binary codes using the Hamming distance metric but with Hamming distance replaced by $\Delta^2 q(z_n)$.

Coding Gain for General Constellations

$$\begin{array}{lll} \text{minimum squared} & & \text{Power} \\ \text{distance} & & \\ \text{Coded:} & d_f & , \text{ power } P_c \\ \text{Uncoded:} & d_u^2 & , P_u \end{array}$$

Want to compare with same average signal powers. So scale uncoded constellation by $\left(\frac{P_c}{P_u}\right)^{1/2}$ so

$$d_u^2 \rightarrow d_u^2 \frac{P_c}{P_u} \text{ and } P_u \rightarrow P_u \left(\frac{P_c}{P_u}\right) = P_c$$

Then

$$\begin{aligned} \text{coding gain} = \gamma_c &= 10 \log_{10} \frac{d_f^2}{d_u^2 \frac{P_c}{P_u}} \\ &= 10 \log_{10} \frac{[d_f^2 / P_c]}{(d_u^2 / P_u)} \end{aligned}$$

Decoding

Squared Euclidean distance metric

Reset metrics at each stage
by subtracting smallest from each

~~Slide parallel~~

Slide to best point in each
subset first for parallel
transitions

V32 code

unzipped page p13 fig 2

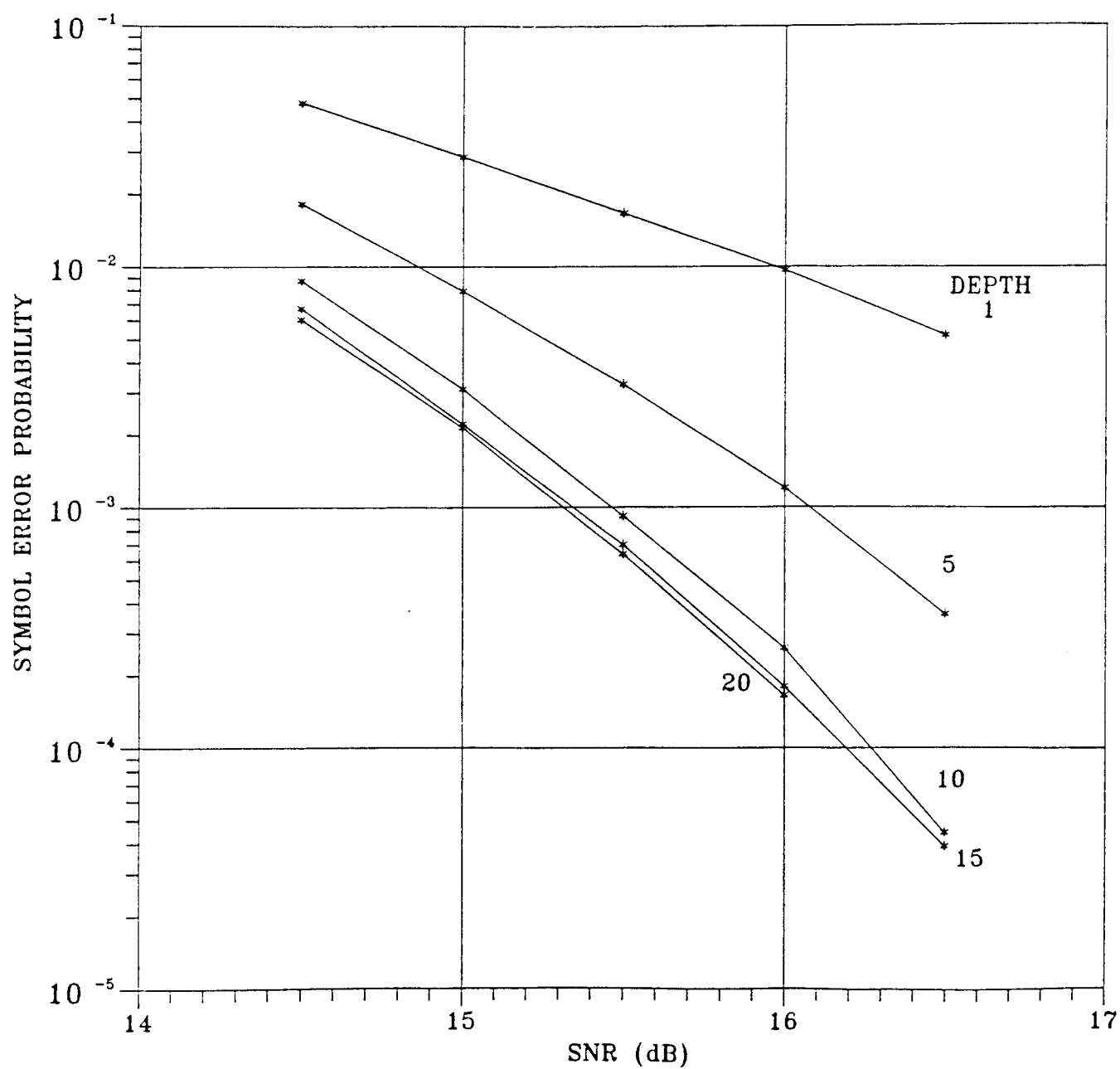
T's and $\square$'s

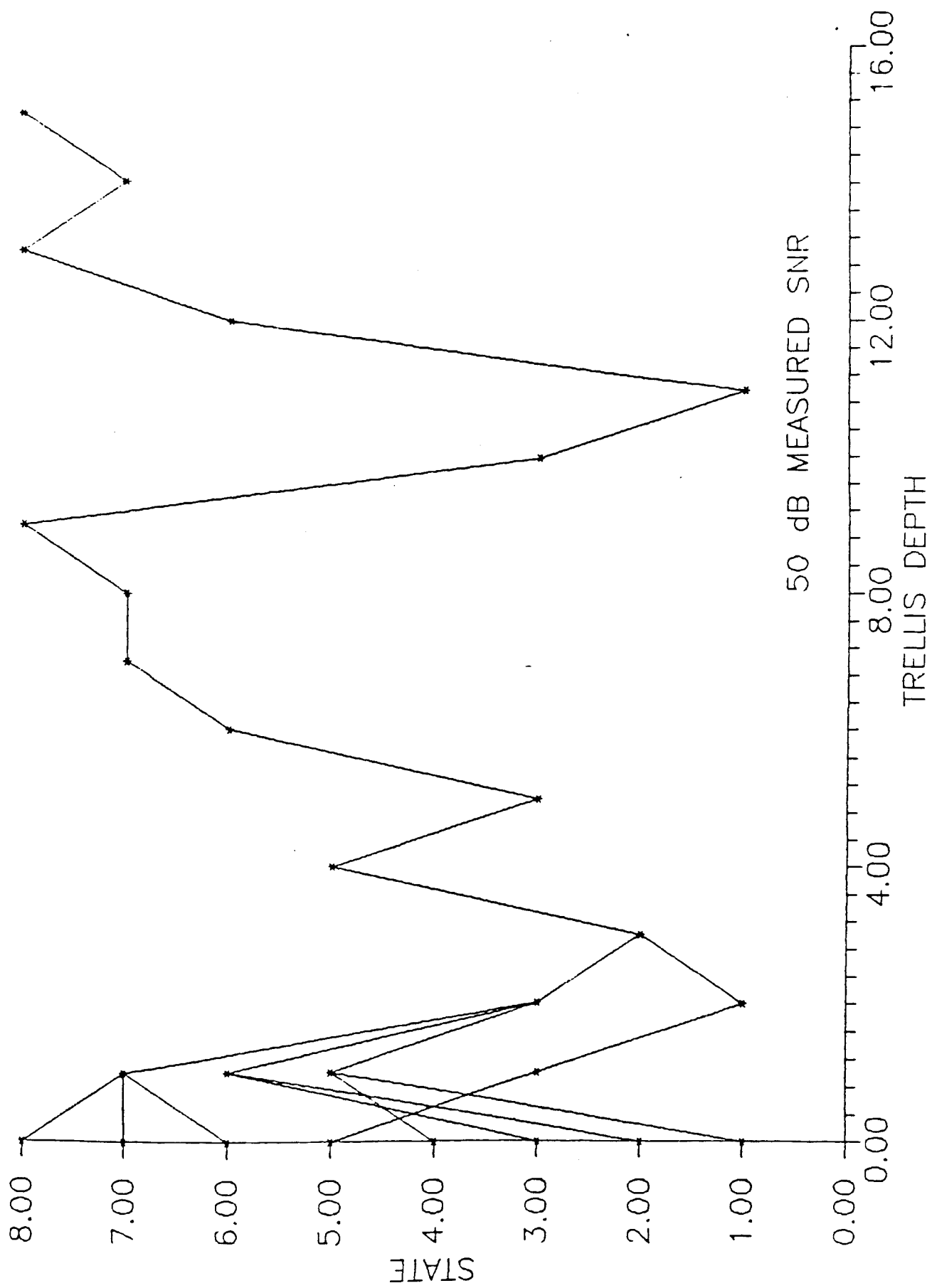
32 points
8 subsets

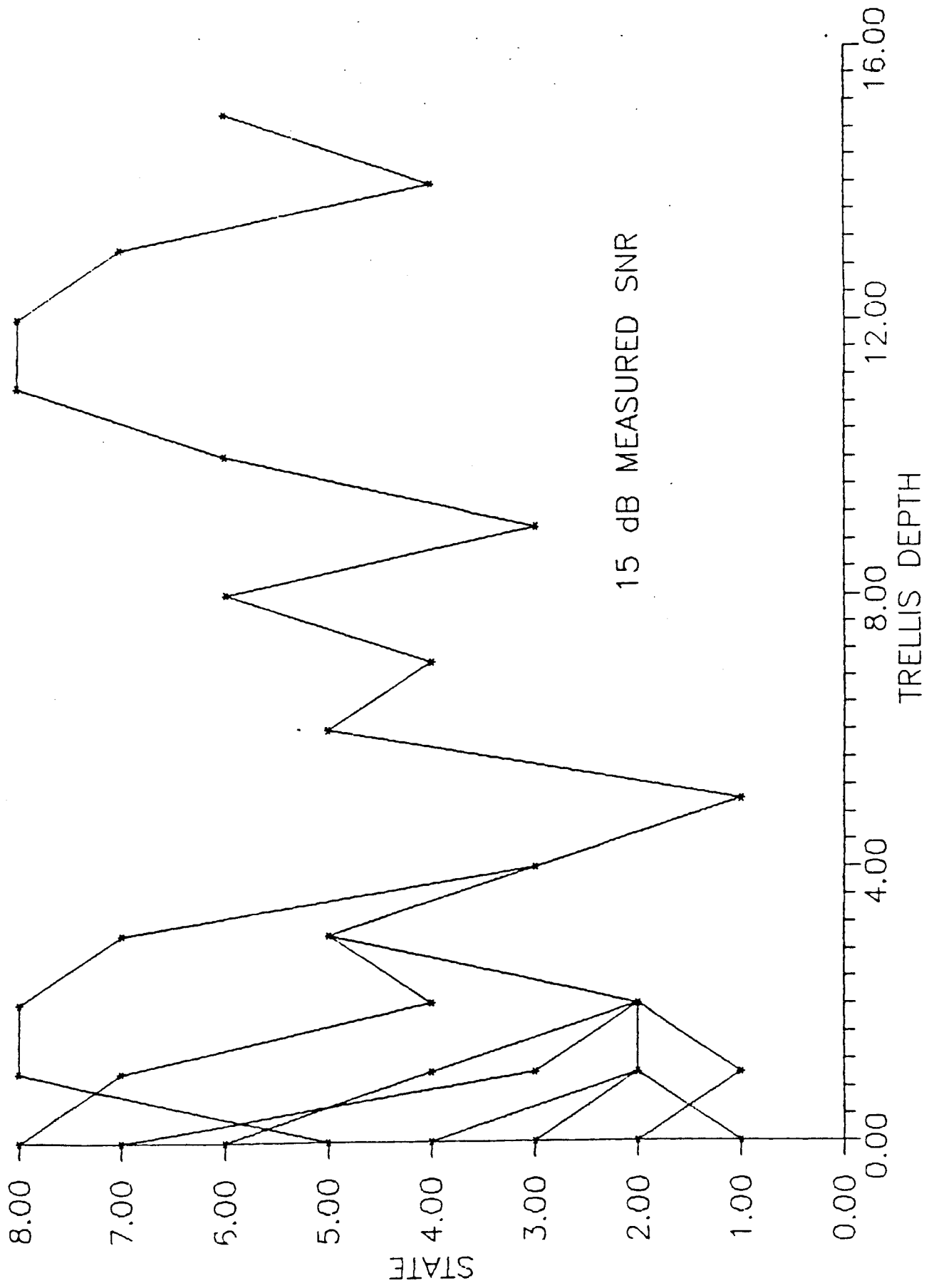
$$\frac{32}{8} = 4 \text{ pts / subset}$$

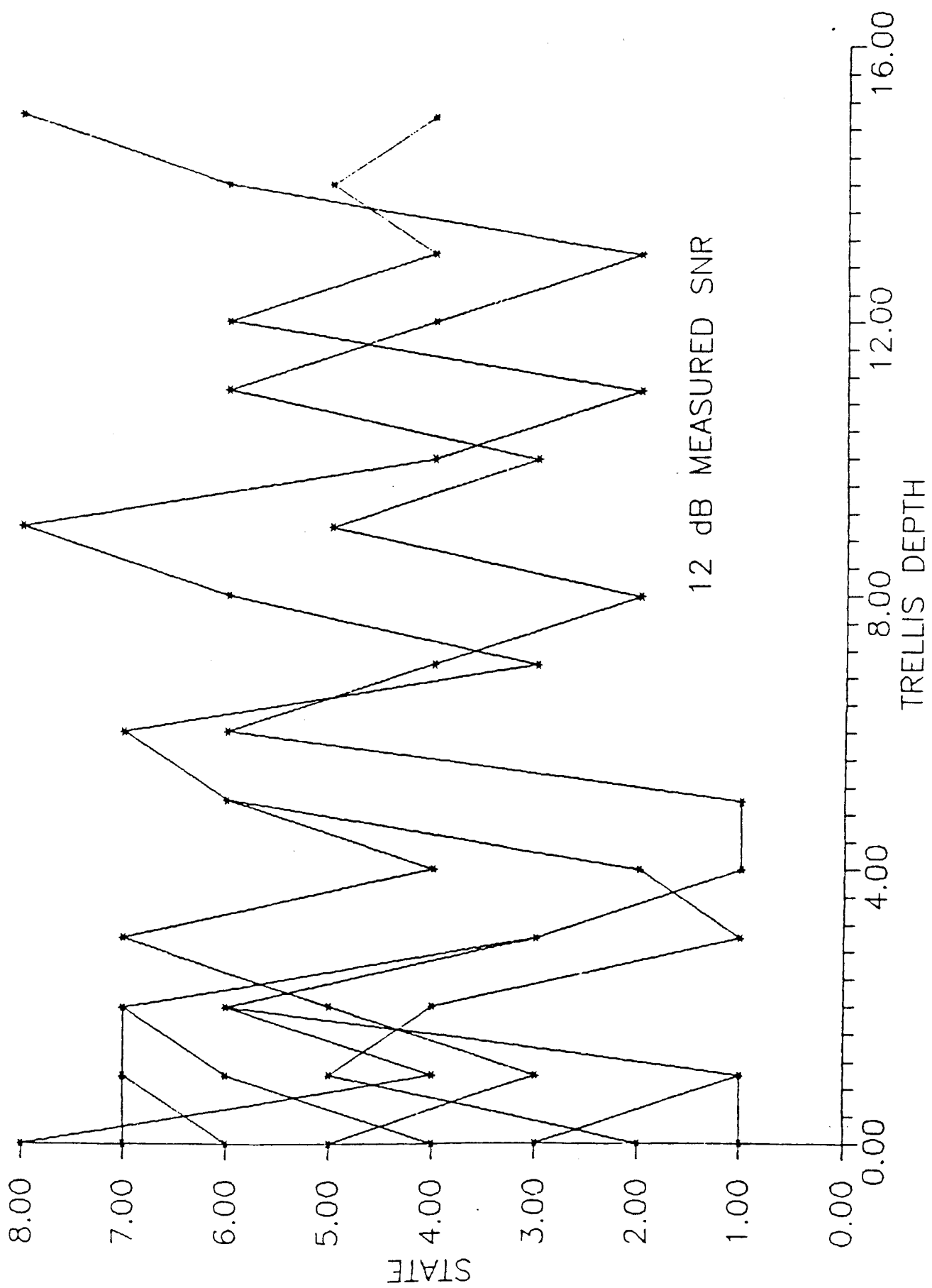
Sliding by table look-up to save time
Speed - memory tradeoff

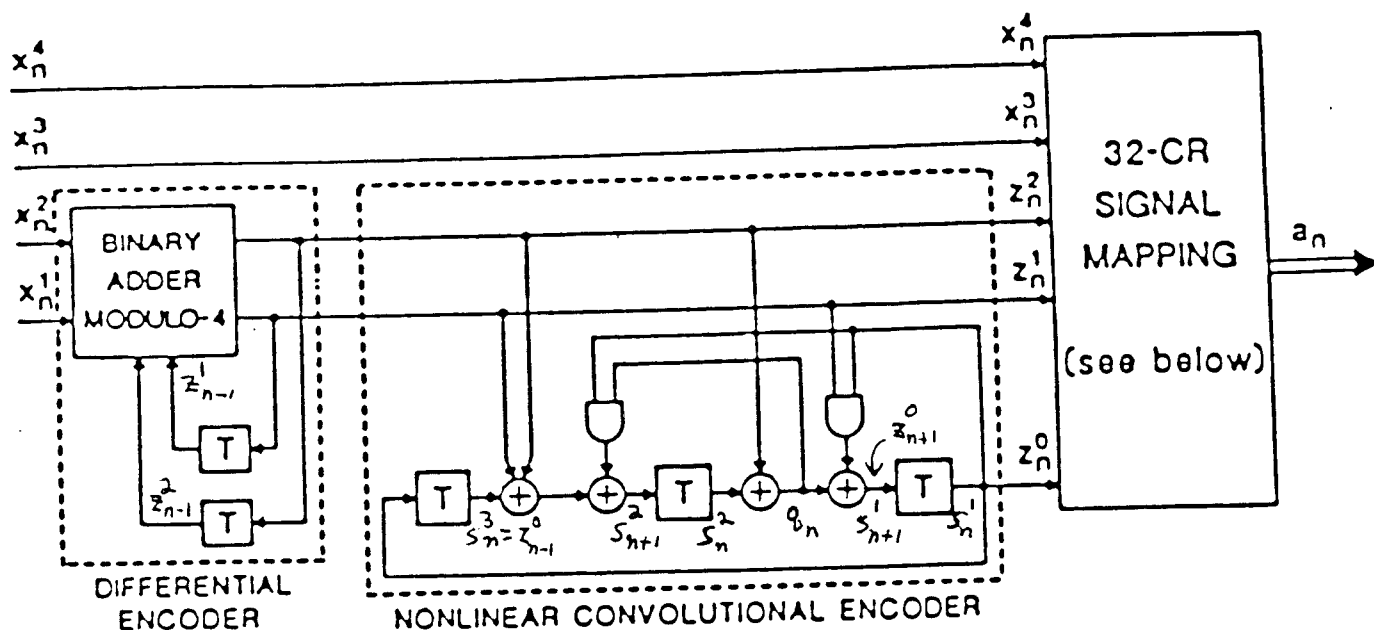
SIMULATION RESULTS FOR V32 TRELLIS CODE ERROR PROBABILITY VS TRELLIS DEPTH











$$\begin{aligned}
 (z_n^2 z_n^1) &= (z_{n-1}^2 z_{n-1}^1) + (x_n^2 x_n^1) \bmod 4 \\
 &= (z_{n-1}^2 \oplus x_n^2 \oplus (z_{n-1}^1 \cdot x_n^1), z_{n-1}^1 \oplus x_n^1)
 \end{aligned}$$

NUMBER BENEATH SIGNAL:

$$x_n^4 x_n^3 z_n^2 z_n^1 z_n^0$$

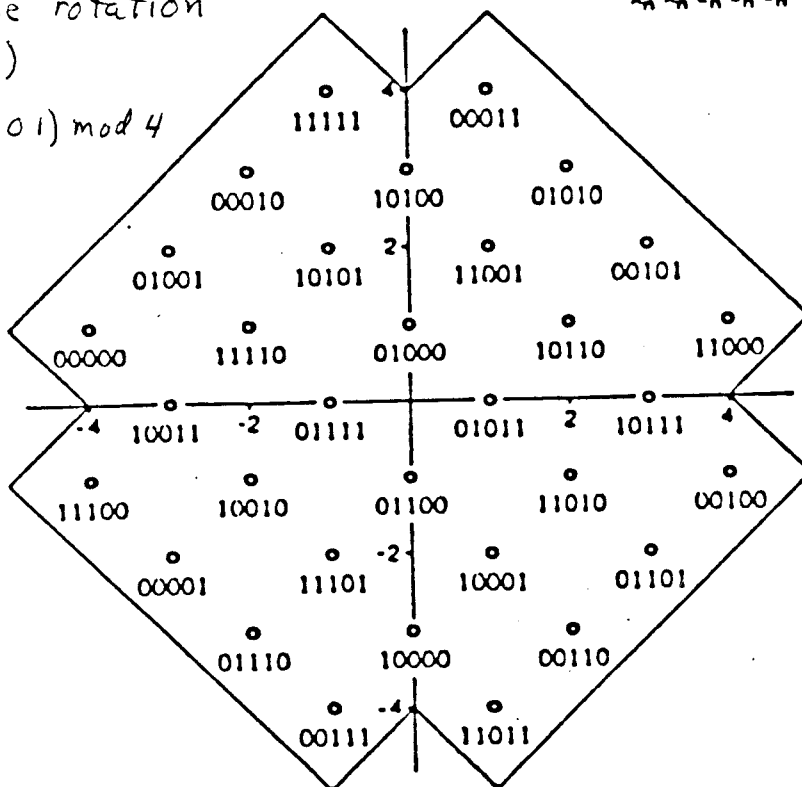
Under 90° clockwise rotation

$$(x_n^4 x_n^3) \rightarrow (x_n^4 x_n^3)$$

$$(z_n^2 z_n^1) \rightarrow (z_n^2 z_n^1) + (01) \bmod 4$$

$$= (z_n^2 \oplus z_n^1, z_n^1)$$

$$z_n^0 \rightarrow \overline{z_n^0}$$



Nonlinear 8-state encoder modulator with 32-CROSS signal set and differential encoding, as in CCITT Recommendation V.32.

Le-Fang Wei code

90° Rotational Invariance of V.32 Trellis Code

Notice that $(x_n^4 x_n^3)$ remains constant under 90° rotations. Under 90° clockwise rotations $(z_n^2 z_n^1) \rightarrow (z_n^2 z_n^1) + (01) \text{ mod } 4$ so $z_n^1 \rightarrow \overline{z_n^1}$, $z_n^2 \rightarrow z_n^2 \oplus z_n^1$

The DIFFERENTIAL ENCODER solves this problem.

$$(z_n^2 z_n^1) = (z_{n-1}^2 z_{n-1}^1) + (x_n^2 x_n^1) \text{ mod } 4$$

The decoder computes

$$(x_n^2 x_n^1) = (z_n^2 z_n^1) - (z_{n-1}^2 z_{n-1}^1) \text{ mod } 4$$

so any constant change in $(z_n^2 z_n^1)$ is cancelled.

Also, under 90° rotation $z_n^0 \rightarrow \overline{z_n^0}$.

The parity check equation for the nonlinear encoder will now be derived.

$$s_{n+1}^1 = z_{n+1}^0 = z_n^0 z_n^1 \oplus q_n \text{ so } q_n = z_{n+1}^0 \oplus z_n^0 z_n^1$$

$$s_{n+1}^2 = z_{n+1}^0 \oplus z_n^1 \oplus z_n^2 \oplus z_n^0 q_n = z_{n+1}^0 \oplus z_n^1 \oplus z_n^2 \oplus z_n^0 [z_{n+1}^0 \oplus z_n^0 z_n^1]$$

$$\text{so } s_n^2 = z_{n-2}^0 \oplus z_{n-1}^1 \oplus z_{n-1}^2 \oplus z_{n-1}^0 z_n^0 \oplus z_{n-1}^0 z_{n-1}^1$$

$$z_{n+1}^0 = s_n^2 \oplus z_n^2 \oplus z_n^0 z_n^1 = z_{n-2}^0 \oplus z_{n-1}^1 \oplus z_{n-1}^2 \oplus z_{n-1}^0 z_n^0 \oplus z_{n-1}^0 z_{n-1}^1 \oplus z_n^2 \oplus z_n^0 z_n^1$$

Parity Check Equation

If the parity check equation is satisfied after a 90° clockwise rotation, the trellis code will be transparent to a 90° rotation. Making the substitution

$$z_n^1 \rightarrow \overline{z_n^1} = 1 \oplus z_n^1, \quad z_n^2 \rightarrow z_n^2 \oplus z_n^1, \quad z_n^0 \rightarrow \overline{z_n^0} = z_n^0 \oplus 1 \text{ gives}$$

$$1 \oplus z_{n+1}^0 = (1 \oplus z_{n-2}^0) \oplus (1 \oplus z_{n-1}^1) \oplus (z_{n-1}^2 \oplus z_{n-1}^1) \oplus (1 \oplus z_{n-1}^0)(1 \oplus z_n^0) \\ \oplus (1 \oplus z_{n-1}^0)(1 \oplus z_{n-1}^1) \oplus (z_n^2 \oplus z_n^1) \oplus (1 \oplus z_n^0)(1 \oplus z_n^1)$$

Expanding products and combining terms modulo 2 gives the original parity check equation, so the z sequence for the rotated signal also is a codeword in the nonlinear code.

CGnet Codes

R. Calderbank (ATT Bell Labs)
G.D. Forney (Motorola / Codex)

Def: N-Dimensional Lattice

Let $\underline{g}_1, \underline{g}_2, \dots, \underline{g}_N$ be linearly independent vectors in an L -dimensional vector space. (In our examples we will let $L=N$ so $\underline{g}_i$ are N -tuples over space of real numbers $\mathbb{R}$). The lattice Λ is the set of points

$$\Lambda = \{ \underline{x} = \lambda_1 \underline{g}_1 + \lambda_2 \underline{g}_2 + \dots + \lambda_N \underline{g}_N \mid \lambda_k \in \mathbb{Z}, k=1, \dots, N \}$$

where $\mathbb{Z}$ is the set of integers. So

$$\underline{x} = [\lambda_1, \lambda_2, \dots, \lambda_N] \begin{bmatrix} \underline{g}_1 \\ \vdots \\ \underline{g}_N \end{bmatrix} = \underline{I} \underline{G}$$

Lattice points form a group under vector addition.

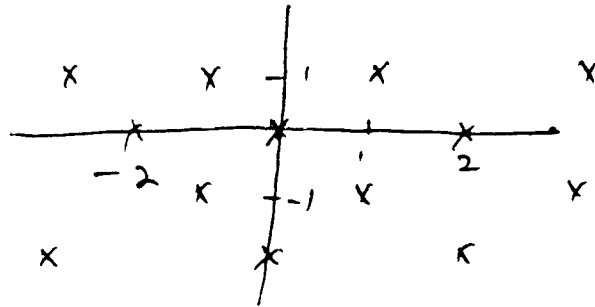
EX $\Lambda = \mathbb{Z} = \{ \text{integers} \}$
 $N=1, \quad \underline{g} = [1]$

EX $\Lambda = \mathbb{Z}^2 = \{ (\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 \in \mathbb{Z} \}$

EX $\mathbb{R}\mathbb{Z}^2$

$$\mathbb{R}\mathbb{Z}^2 = \{ (i_1, i_2) \mid (i_1, i_2) \in \mathbb{Z}^2, i_1 + i_2 \text{ even} \}$$

$$= \{ \mathbb{Z}^2 \text{ rotated by } 45^\circ, \text{ scaled by } \sqrt{2} \}$$



$$G = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Def: Sublattice

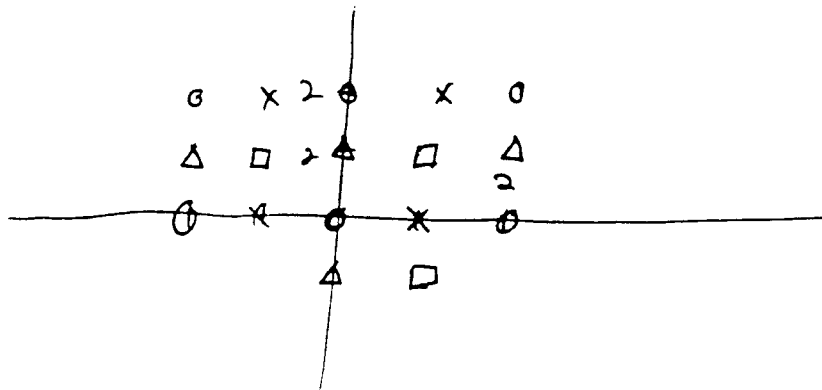
A lattice Λ' which is a subset of Λ is called a sublattice of Λ

EX: $\mathbb{R}\mathbb{Z}^2$ and $2\mathbb{Z}^2$ are sublattices of $\mathbb{Z}^2$

Lattice Partitions

Can form cosets of Λ' with respect to Λ . Set of cosets called lattice partition and denoted by Λ/Λ' . The number of cosets is denoted by $|\Lambda/\Lambda'|$. A set of coset leaders is denoted by $\{\Lambda/\Lambda'\}$.

EX: $z^2 / 2z^2$

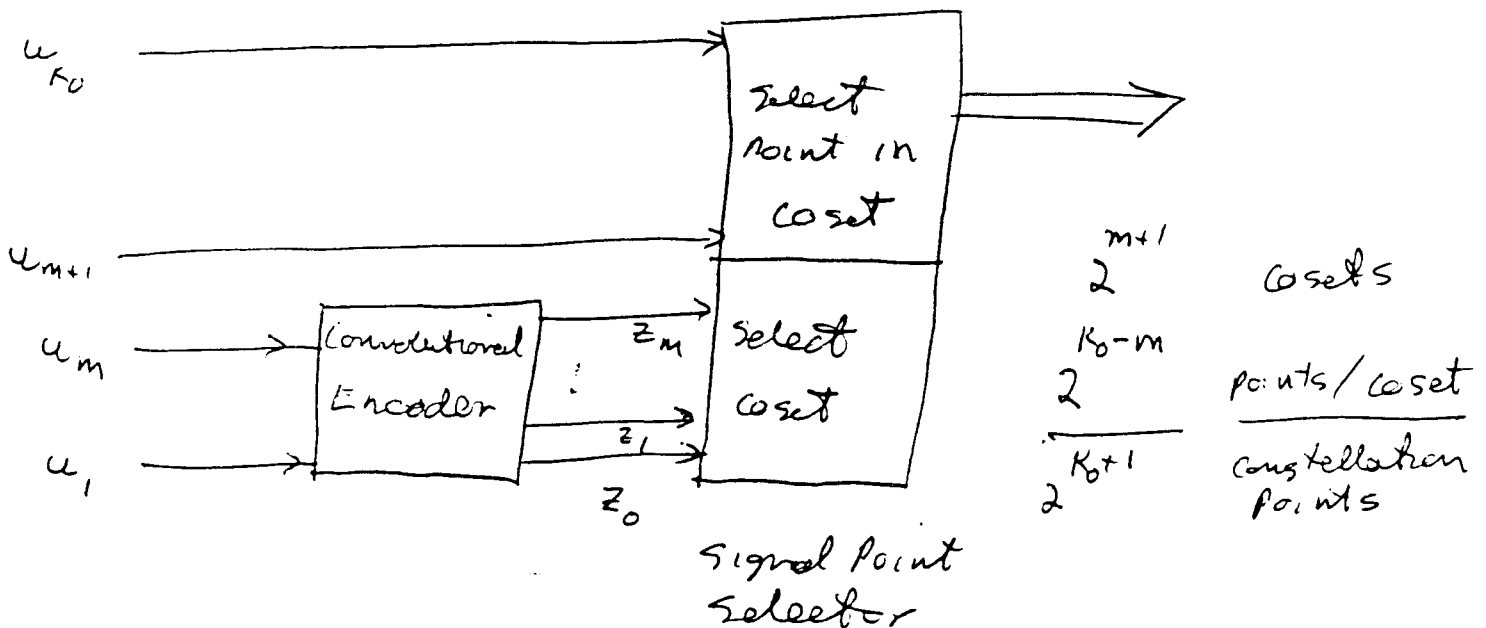


$$|z^2 / 2z^2| = 4$$

$$z^2 = 2z^2 \cup \{2z^2 + (0,1)\} \cup \{2z^2 + (1,0)\} \cup \{2z^2 + (1,1)\}$$

Can use $\{z^2 / 2z^2\} = \{(0,0), (0,1), (1,0), (1,1)\}$

Encoder

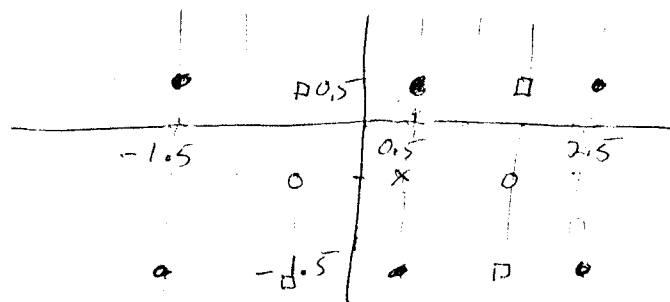


The 4-Dimensional V.34 Constellation

Basic approach: Partition the 2D constellation into 8 subsets and then combine pairs of 2D subsets into 16 4D subsets.

The 2D Constellation

Consider "translated" lattice $2\mathbb{Z}^2 + (0.5, 0.5) = \Lambda_0$



Can generate $\mathbb{Z}^2$ by union of 90° rotations of $2\mathbb{Z}^2 + (0.5, 0.5)$.

Let 90° clockwise rotation of 2D point (x, y) be $R(x, y)$. Then

$$R(x, y) = (y, -x)$$

Let $\tau_0 = (2n_1 + 0.5, 2n_2 + 0.5) \in \Lambda_0$. Then

$$(1) \quad R(\tau_0) = (2n_2 + 0.5, -2n_1 - 0.5) = (2n_2, -2n_1) + (0.5, -0.5) \in 2\mathbb{Z}^2 + (0.5, -0.5) \equiv \mathcal{R}\Lambda_0$$

but

$$\begin{aligned} \Lambda_1 &= 2\mathbb{Z}^2 + (0.5, -0.5) = 2\mathbb{Z}^2 + (0.5, 0.5) + (0, -1) \\ &= 2\mathbb{Z}^2 + (0.5, 0.5) + (0, 1) = \Lambda_0 + (0, 1) = \mathcal{R}\Lambda_0 \end{aligned}$$

(2)

$$\begin{aligned} \Lambda_2 &= \mathcal{R}\Lambda_1 = \mathcal{R}\Lambda_0 + \mathcal{R}(0, 1) = [\Lambda_0 + (0, 1)] + (1, 0) \\ &= \Lambda_0 + (1, 1) \end{aligned}$$

$$(3) \quad \begin{aligned} \Lambda_3 &= \mathcal{R}\Lambda_2 = \mathcal{R}\Lambda_0 + \mathcal{R}(1, 1) = \Lambda_0 + (0, 1) + (1, 1) = \Lambda_0 + (1, 2) \\ &= \Lambda_0 + (1, 2) - (0, 2) = \Lambda_0 + (1, 0) \end{aligned}$$

Partitioning the 2D Constellation into 8 subsets

V.34 partitioning based on lattice partition chain $Z^2 / 2Z^2 / 4Z^2 / 8Z^2$

Each subpartition has order 2 so

$$|Z^2 / 2Z^2| = 2^2 = 4$$

(can represent subsets by binary 3-tuple, (J_2, J_1, J_0) with

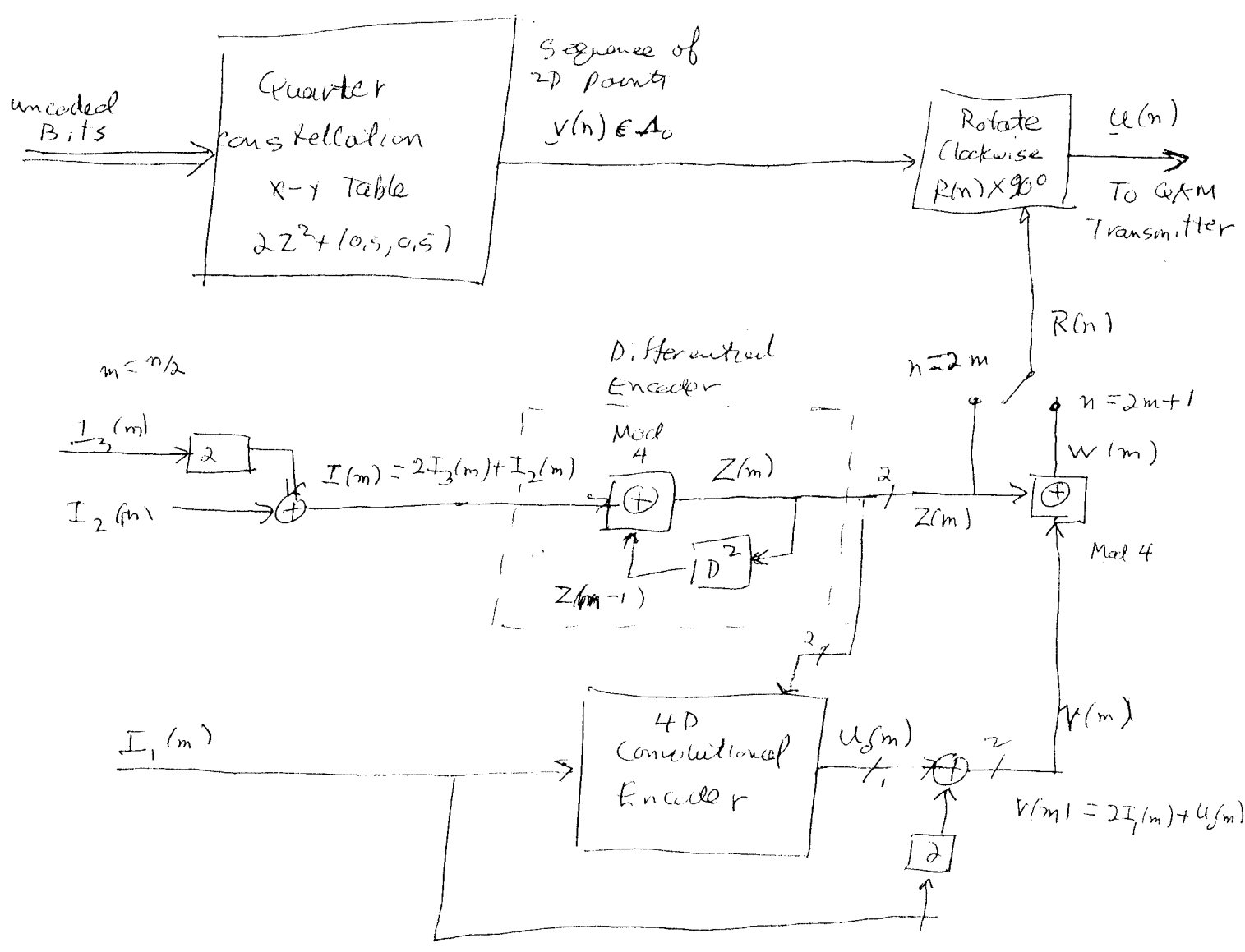
$$\Lambda(J_2, J_1, J_0) = 2RZ^2 + [J_2 \oplus (J_0 \cdot \bar{J}_1)](2, 0) + J_1(1, 1) + J_0(0, 1) + (0.5, 0.5)$$

J_2, J_1, J_0

000	a	$2RZ^2 + (0, 0) + (0.5, 0.5)$
001	b	$2RZ^2 + (2, 1) + (0.5, 0.5)$
010	c	$2RZ^2 + (1, 1) + (0.5, 0.5)$
011	d	$2RZ^2 + (1, 2) + (0.5, 0.5)$
100	A	$2RZ^2 + (2, 0) + (0.5, 0.5)$
101	B	$2RZ^2 + (0, 1) + (0.5, 0.5)$
110	C	$2RZ^2 + (3, 1) + (0.5, 0.5)$
111	D	$2RZ^2 + (3, 2) + (0.5, 0.5)$

The 4D Constellation

A 4D lattice point consists of a pair of 2D points. The V.34 4D constellation is transparent to 90° rotations.



4D points consist of pairs of 2D points
 $\{u(2m), u(2m+1)\}$
 $\uparrow \quad \quad \uparrow$
 early late
 2D symbol 2D symbol

The uncoded bits select an initial 4D point
 $[v(2m) \quad v(2m+1)]$

The early final 4D point $u(2m)$
 $u(2m) = R^{Z(m)} v(2m) \in A_{Z(m)} \quad Z(m) \in \{0, 1, 2, 3\}$