

Know $\frac{-\alpha|\tau|}{e} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + \omega^2}$

Let $\alpha = \frac{1}{RC}$. Then

$$R_{xx}(\tau) = \frac{kT}{C} e^{-\frac{1}{RC}|\tau|}$$

So noise power at output terminals is

$$E\{x^2(t)\} = R_{xx}(0) = \frac{kT}{C}$$

Energy stored in C is $E = \frac{1}{2} C x^2(t)$, so

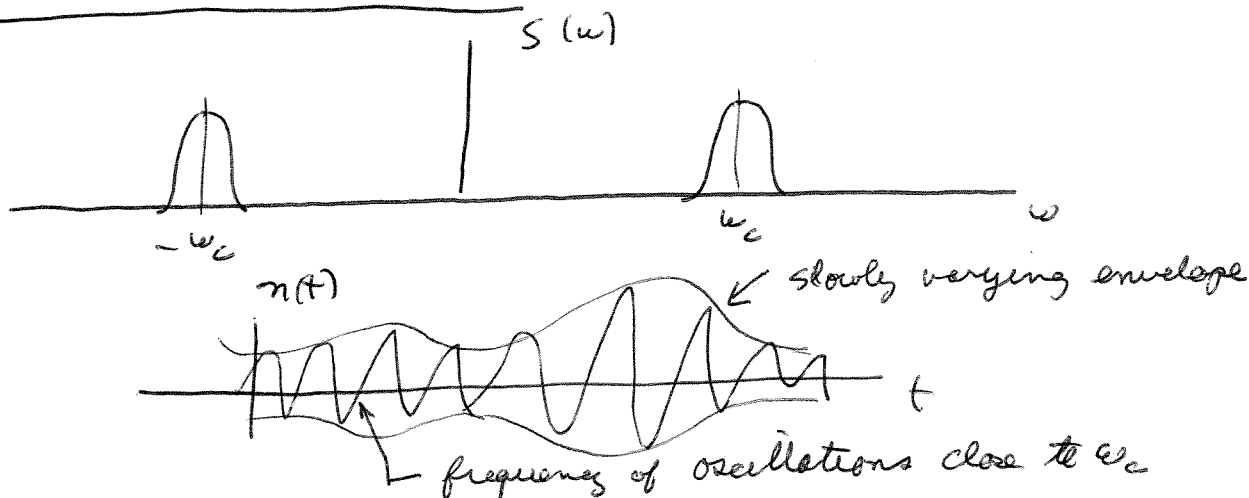
average stored energy is

$$E\{E\} = \frac{1}{2} C E\{x^2\} = \frac{kT}{2}$$

Not function of C or R!

Equipartition Theorem of statistical mechanics says, ^{average} energy stored in each degree of freedom is $kT/2$

Narrowband Noise



Claim: can represent $n(t)$ as

$$n(t) = n_I(t) \cos \omega_c t - n_Q(t) \sin \omega_c t = \text{Re} \{ [n_I(t) + j n_Q(t)] e^{j\omega_c t} \}$$

$n_{\pm}(t)$ = inphase component, low pass spectrum
 $n_Q(t)$ = quadrature component, low pass spectrum

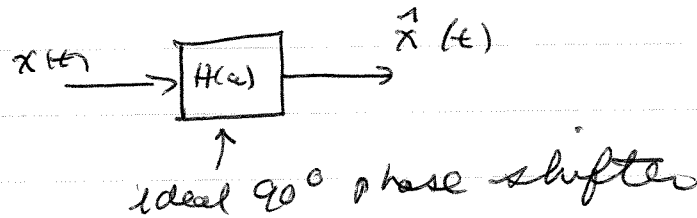
Proof:

Hilbert Transform

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

Let $h(t) = \frac{1}{\pi t}$, Then

$$H(\omega) = \begin{cases} -j & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ j & \text{for } \omega < 0 \end{cases} = -j \operatorname{sign} \omega$$



① Form pre-envelope

$$n_+(t) = n(t) + j \hat{n}(t)$$

② Form complex envelope

$$\tilde{n}(t) = n_+(t) e^{-j\omega_c t} = n_I(t) + j n_Q(t)$$

where $(n_+(t) \text{ demodulated to "baseband"})$ $(\tilde{N}(\omega) = N_+(\omega + \omega_c))$

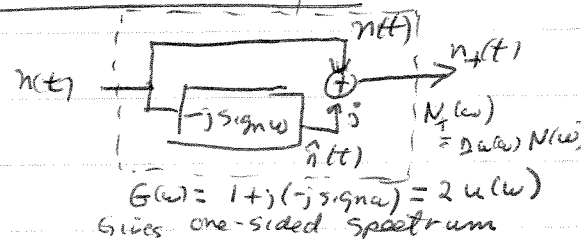
$$n_I(t) = \operatorname{Re} \{ n_+(t) e^{-j\omega_c t} \} = \operatorname{Re} \{ [n(t) + j \hat{n}(t)] [\cos \omega_c t - j \sin \omega_c t] \}$$

$$= n(t) \cos \omega_c t + \hat{n}(t) \sin \omega_c t$$

$$n_Q(t) = \operatorname{Im} \{ n_+(t) e^{-j\omega_c t} \} = \hat{n}(t) \cos \omega_c t - n(t) \sin \omega_c t$$

Then

$$n(t) = \operatorname{Re} n_+(t) = \operatorname{Re} \{ \tilde{n}(t) e^{j\omega_c t} \} = \operatorname{Re} \{ [n_I(t) + j n_Q(t)] \times [\cos \omega_c t + j \sin \omega_c t] \} = n_I(t) \cos \omega_c t - n_Q(t) \sin \omega_c t$$



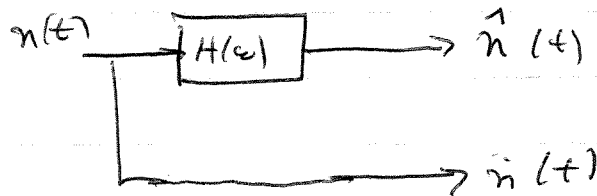
Autocorrelation Functions and Spectral Densities

For pre-envelope

$$R_+(\tau) = E \{ n_+(t+\tau) \overline{n_+(t)} \}$$

$$= E \{ [n(t+\tau) + j \hat{n}(t+\tau)] [n(t) - j \hat{n}(t)] \}$$

$$= R_{nn}(\tau) + R_{\hat{n}\hat{n}}(\tau) + j R_{\hat{n}n}(\tau) - j R_{n\hat{n}}(\tau)$$



$$S_{\hat{n}\hat{n}}(\omega) = \underbrace{|H(\omega)|^2}_1 S_{nn}(\omega) = S_{nn}(\omega)$$

$$\Rightarrow R_{\hat{n}\hat{n}}(\tau) = R_{nn}(\tau)$$

$$S_{n\hat{n}}(\omega) = H(\omega) \times 1 \times S_{nn}(\omega)$$

$$\Rightarrow R_{n\hat{n}}(\tau) = \hat{R}_{nn}(\tau)$$

$$S_{\hat{n}n}(\omega) = 1 \times H(-\omega) \times S_{nn}(\omega) = (j \operatorname{sign} \omega) S_{nn}(\omega)$$

$$= - \underbrace{(-j \operatorname{sign} \omega)}_{|H(\omega)|} S_{nn}(\omega)$$

$$\Rightarrow R_{\hat{n}n}(\tau) = - \hat{R}_{nn}(\tau)$$

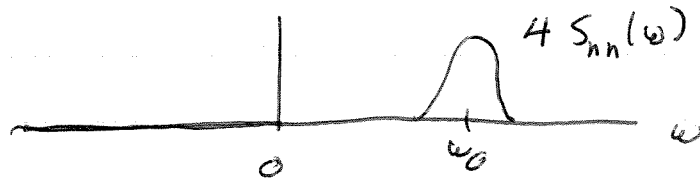
So

$$R_+(\tau) = 2 [R_{nn}(\tau) + j \hat{R}_{nn}(\tau)] \Rightarrow R_{nn}(\tau) = \frac{1}{2} \operatorname{Re} R_+(\tau)$$

$$S_+(\omega) = 2 [S_{nn}(\omega) + j (-j \operatorname{sign} \omega) S_{nn}(\omega)]$$

$$= 2 S_{nn}(\omega) \underbrace{[1 + \operatorname{sign} \omega]}_{2u(\omega)} = 4 S_{nn}(\omega) u(\omega)$$

Has one-sided spectrum!

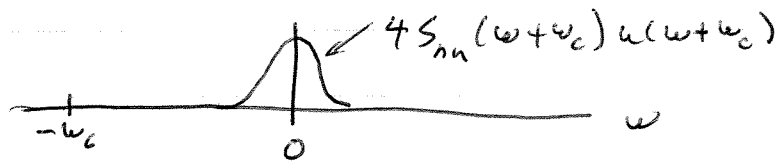


For complex envelope

$$\begin{aligned}\tilde{R}(\tau) &= E\{\tilde{n}(t+\tau) \overline{\tilde{n}(t)}\} \\ &= E\{n_+(t+\tau) e^{-j\omega_c(t+\tau)} \overline{n_+(t) e^{-j\omega_c t}}\} \\ &= E\{n_+(t+\tau) n_+(t)\} e^{-j\omega_c \tau} \\ &= R_+(\tau) e^{-j\omega_c \tau}\end{aligned}$$

So

$$\tilde{S}(\omega) = S_+(\omega + \omega_c) = 4 S_{nn}(\omega + \omega_c) u(\omega + \omega_c)$$



For inphase component

$$\begin{aligned}R_{\pm}(\tau) &= E\{n_{\pm}(t+\tau) n_{\pm}(t)\} \\ &= E\{[n(t+\tau) \cos \omega_c(t+\tau) + \hat{n}(t+\tau) \sin \omega_c(t+\tau)] \times \\ &\quad [n(t) \cos \omega_c t + \hat{n}(t) \sin \omega_c t]\} - \hat{R}_{nn}(\tau) \\ &= R_{nn}(\tau) \cos \omega_c(t+\tau) \cos \omega_c t + \underbrace{R_{n\hat{n}}(\tau)}_{\hat{R}_{nn}(\tau)} \cos \omega_c(t+\tau) \sin \omega_c t \\ &\quad + \underbrace{R_{\hat{n}n}(\tau)}_{\hat{R}_{nn}(\tau)} \sin \omega_c(t+\tau) \cos \omega_c t + \underbrace{R_{\hat{n}\hat{n}}(\tau)}_{\hat{R}_{nn}(\tau)} \sin \omega_c(t+\tau) \sin \omega_c t\end{aligned}$$

$$R_{\pm}(\tau) = R_{nn}(\tau) \cos \omega_c \tau + \hat{R}_{nn}(\tau) \sin \omega_c \tau$$

$$= \frac{1}{2} \text{Re} \{ R_+(\tau) e^{-j\omega_c \tau} \} = \frac{1}{2} \text{Re} \{ \tilde{R}(\tau) \}$$

use result of p. 15

For quadrature component: similarly, can show

$$R_Q(\tau) = R_{\pm}(\tau) = \frac{1}{2} \text{Re} \{ R_+(\tau) e^{-j\omega_c \tau} \}$$

Power Properties

$$E\{n_I^2(t)\} = E\{n_Q^2(t)\} = R_I(0) = R_{nn}(0) = E\{n^2(t)\}$$

Cross correlation of I and Q components

$$\begin{aligned} R_{IQ}(\tau) &= E\{n_I(t+\tau) n_Q(t)\} \\ &= E\{[n(t+\tau) \cos \omega_c(t+\tau) + \tilde{n}(t+\tau) \sin \omega_c(t+\tau)] \\ &\quad \times [\tilde{n}(t) \cos \omega_c t - n(t) \sin \omega_c t]\} \\ &= R_{nn}(\tau) \cos \omega_c(t+\tau) \cos \omega_c t - R_{nn}(\tau) \cos \omega_c(t+\tau) \sin \omega_c t \\ &\quad + \underbrace{R_{\tilde{n}\tilde{n}}(\tau)}_{R_{nn}(\tau)} \sin \omega_c(t+\tau) \cos \omega_c t - \underbrace{R_{\tilde{n}n}(\tau)}_{R_{nn}(\tau)} \sin \omega_c(t+\tau) \sin \omega_c t \\ &= R_{nn}(\tau) \sin \omega_c \tau - R_{nn}(\tau) \cos \omega_c \tau \\ &= -\frac{1}{2} \operatorname{Im} [R_+(\tau) e^{-j\omega_c \tau}] \end{aligned}$$

Can show $R_{QI}(\tau) = -R_{IQ}(\tau)$

Then

$$R_I(\tau) - j R_{IQ}(\tau) = \frac{1}{2} R_+(\tau) e^{-j\omega_c \tau} = \frac{1}{2} \tilde{R}(\tau)$$

or

$$\tilde{R}(\tau) = R_+(\tau) e^{-j\omega_c \tau} = 2 [R_I(\tau) - j R_{IQ}(\tau)]$$

also $R_{nn}(\tau) = \frac{1}{2} \operatorname{Re} R_+(\tau) = \operatorname{Re} \{ [R_I(\tau) - j R_{IQ}(\tau)] e^{j\omega_c \tau} \}$
 $= R_I(\tau) \cos \omega_c \tau + R_{IQ}(\tau) \sin \omega_c \tau$

Formulas for Power Spectral Densities of I and Q components

Remember $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$
 $\overline{x(t)} \xleftrightarrow{\mathcal{F}} \overline{X(\omega)}$

$$\tilde{R}(\tau) \xleftrightarrow{\mathcal{F}} \tilde{S}(\omega) = S_+(\omega + \omega_c) = 2 [S_I(\omega) - j S_{IQ}(\omega)] \quad (1)$$

$$\overline{\tilde{R}(\tau)} \xleftrightarrow{\mathcal{F}} \overline{\tilde{S}(\omega)} = S_+(\omega_c - \omega) = 2 [S_I(\omega) + j S_{IQ}(\omega)] \quad (2)$$

can ignore, since $S_x(\omega)$ is real

$$\frac{①+②}{4}$$

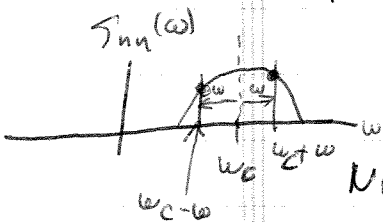
$$S_{\pm}(\omega) = S_{\phi}(\omega) = \frac{S_{+}(\omega + \omega_c) + S_{+}(\omega_c - \omega)}{4}$$

$$= \frac{\tilde{S}(\omega) + \tilde{S}(-\omega)}{4} = \frac{S_{nn}(\omega + \omega_c) u(\omega + \omega_c) + S_{nn}(\omega_c - \omega) u(\omega_c - \omega)}{4}$$

and

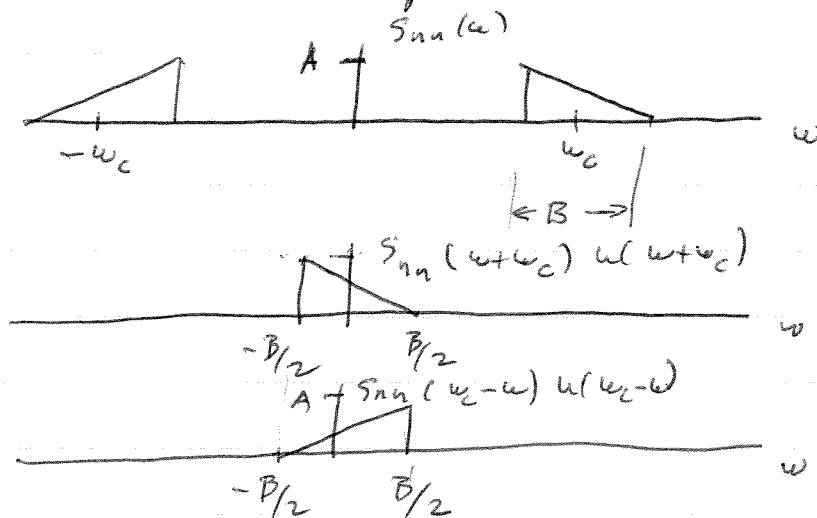
$$\frac{③-①}{4} \quad S_{IQ}(\omega) = \frac{S_{+}(\omega_c - \omega) - S_{+}(\omega_c + \omega)}{4} = \frac{\tilde{S}(-\omega) - \tilde{S}(\omega)}{4}$$

$$= S_{nn}(\omega_c - \omega) u(\omega_c - \omega) - S_{nn}(\omega_c + \omega) u(\omega_c + \omega)$$

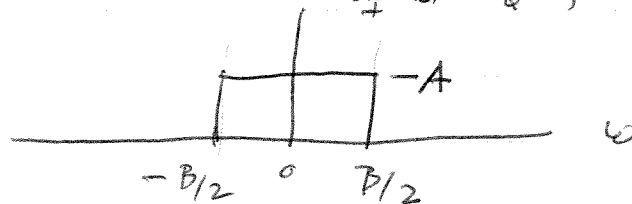


Note: If $S_{nn}(\omega)$ symmetric about ω_c , then $S_{IQ}(\omega) = 0 \Rightarrow n_c(t), n_s(t)$ uncorrelated

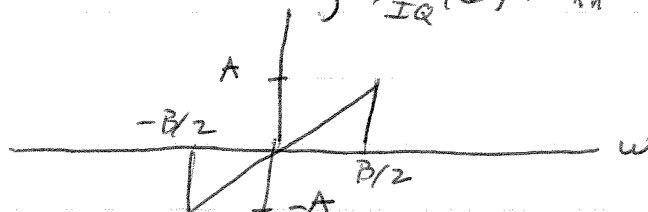
EX: ~~Flat~~ Triangular Bandpass Noise



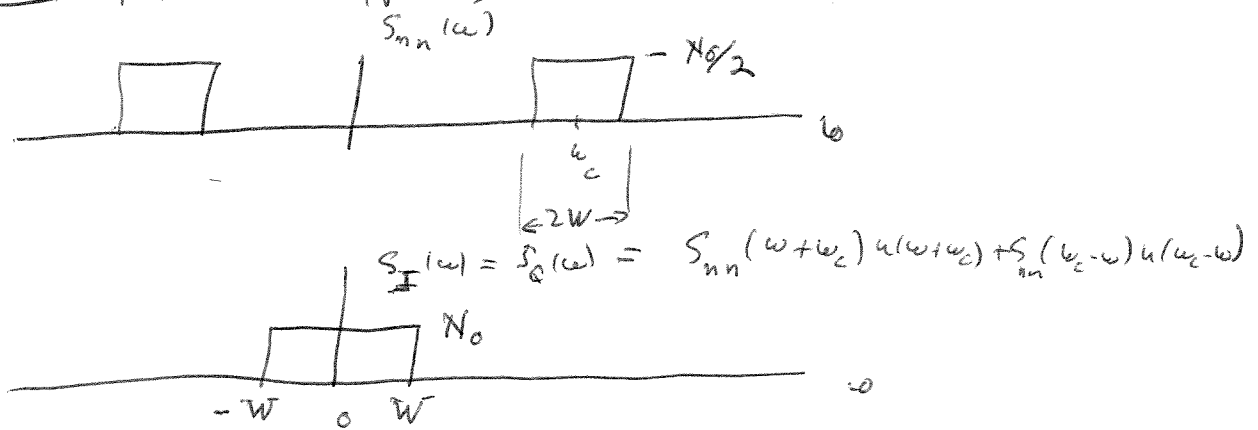
$$S_{\pm}(\omega) = S_{\phi}(\omega) = S_{nn}(\omega + \omega_c) u(\omega + \omega_c) + S_{nn}(\omega_c - \omega) u(\omega_c - \omega)$$



$$S_{IQ}(\omega) = S_{nn}(\omega_c - \omega) u(\omega_c - \omega) - S_{nn}(\omega_c + \omega) u(\omega_c + \omega)$$



EX: Flat Bandpass Noise



$S_{IQ}(\omega) = 0$ since $S_{nn}(\omega)$ symmetric about $\omega_c \Rightarrow R_{IQ}(\tau) = 0$

$$R_I(\tau) = \frac{1}{2\pi} \int_{-W}^W N_0 e^{j\omega\tau} d\omega = \frac{N_0}{2\pi} \frac{e^{jW\tau} - e^{-jW\tau}}{j\tau} = \frac{N_0}{\pi} \frac{\sin W\tau}{\tau}$$

From p. 15

$$R_{nn}(\tau) = \frac{1}{2} \text{Re}\{R_+(\tau)\} = \frac{1}{2} \text{Re}\{\tilde{R}(\tau) e^{j\omega_c\tau}\}$$

$$= \frac{1}{2} \text{Re}\{2R_+(\tau) e^{j\omega_c\tau}\}$$

$$= R_I(\tau) \cos \omega_c\tau = \frac{N_0}{\pi} \frac{\sin W\tau}{\tau} \cos \omega_c\tau$$

$$E\{n^2(t)\} = R_{nn}(0) = \frac{N_0 W}{\pi}$$

$$\text{or } E\{n^2(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{nn}(\omega) d\omega = (2W) \frac{N_0}{2} \times \frac{1}{2\pi} = \frac{N_0 W}{\pi}$$

$$E\{n_I^2(t)\} = \frac{1}{2\pi} \int_{-W}^W N_0 d\omega = \frac{2W N_0}{2\pi} = \frac{N_0 W}{\pi}$$

$$= E\{n^2(t)\}$$