Notes on Fourier Series

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October 30, 2013

Contents

Let $x(t)$ be a periodic signal with period T_0 and *fundamental frequency* $\omega_0 = 2\pi/T_0$. Fourier showed that these signals can be represented by a sum of scaled sines and cosines at multiples of the fundamental frequency. The series can also be expressed as sums of scaled complex exponentials at multiples of the fundamental frequency. A sinusoid at frequency $n\omega_0$ is called an nth harmonic. This document presents the approach I have taken to Fourier series in my lectures for ENEE 322 Signal and System Theory. Unless stated otherwise, it will be assumed that $x(t)$ is a real, not complex, signal. However, periodic complex signals can also be represented by Fourier series.

1 The Real Form Fourier Series

as follows:

$$\boxed{x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t} \tag{1}$$

This is called a trigonometric series. We will call it the real form of the Fourier series.

To derive formulas for the Fourier coefficients, that is, the $a's$ and $b's$, we need trigonometric identities for the products of cosines and sines. You should already know the following formulas for the cosine of the sum and difference of two angles.

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{2}$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \tag{3}$$

Adding these two equations and dividing by two gives

$$\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b) \tag{4}$$

Subtracting the second equation from the first and dividing by 2 gives

$$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b) \tag{5}$$

You should already know the following two trigonometric identities:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{6}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \tag{7}$$

Adding these last two equations give

$$\sin a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b) \tag{8}$$

To find $a_0/2$ consider the integral of $x(t)$ over one complete period T_0. For some conveniently chosen starting time t_0 this is

$$\int_{t_0}^{t_0+T_0} x(t)\,dt = T_0\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T_0} \cos n\omega_0 t\,dt + b_n \int_{t_0}^{t_0+T_0} \sin n\omega_0 t\,dt \tag{9}$$

Each integral in the sum is over n complete periods of a sine or cosine and is zero. Therefore

$$\boxed{\frac{a_0}{2} = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,dt} \tag{10}$$

which is just the DC value of $x(t)$.

To find a_n for $n \geq 1$ consider the following integral for $k \geq 1$:

$$\begin{aligned}\int_{t_0}^{t_0+T_0} x(t)\cos k\omega_0 t\,dt = \int_{t_0}^{t_0+T_0} \frac{a_0}{2}\cos k\omega_0 t\,dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T_0} \cos n\omega_0 t \cos k\omega_0 t\,dt \\ + b_n \int_{t_0}^{t_0+T_0} \sin n\omega_0 t \cos k\omega_0 t\,dt\end{aligned} \tag{11}$$

Using identity (4) gives

$$\int_{t_0}^{t_0+T_0} \cos n\omega_0 t \cos k\omega_0 t\,dt = \frac{1}{2}\int_{t_0}^{t_0+T_0} \cos[(n-k)\omega_0 t]\,dt + \frac{1}{2}\int_{t_0}^{t_0+T_0} \cos[(n+k)\omega_0 t]\,dt \tag{12}$$

The first integral on the right-hand-side is over $|n-k|$ complete periods of the cosine and is zero when $n \neq k$ and $T_0/2$ when $n = k$. The second integral is over $n+k$ periods and is always zero. Using identity (8) gives

$$\int_{t_0}^{t_0+T_0} \sin n\omega_0 t \cos k\omega_0 t \, dt = \frac{1}{2} \int_{t_0}^{t_0+T_0} \sin[(n+k)\omega_0 t] \, dt + \frac{1}{2} \int_{t_0}^{t_0+T_0} \sin[(n-k)\omega_0 t] \, dt \tag{13}$$

The two integrals on the right are zero even when $n = k$. Therefore

$$\boxed{a_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos n\omega_0 t \, dt} \quad \text{for} \quad n \geq 1 \tag{14}$$

To find b_n for $n \geq 1$ consider the following integral for $k \geq 1$:

$$\int_{t_0}^{t_0+T_0} x(t) \sin k\omega_0 t \, dt = \int_{t_0}^{t_0+T_0} \frac{a_0}{2} \sin k\omega_0 t \, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T_0} \cos n\omega_0 t \sin k\omega_0 t \, dt$$
$$+ b_n \int_{t_0}^{t_0+T_0} \sin n\omega_0 t \sin k\omega_0 t \, dt \tag{15}$$

Using identity (8) gives

$$\int_{t_0}^{t_0+T_0} \cos n\omega_0 t \sin k\omega_0 t \, dt = \frac{1}{2} \int_{t_0}^{t_0+T_0} \sin[(k+n)\omega_0 t] \, dt + \frac{1}{2} \int_{t_0}^{t_0+T_0} \sin[(k-n)\omega_0 t] \, dt \tag{16}$$

The first integral on the right-hand-side is over $k+n$ complete periods of the sine and is always zero. The second integral is over $|k-n|$ periods and is always zero also. Using identity (5) gives

$$\int_{t_0}^{t_0+T_0} \sin n\omega_0 t \sin k\omega_0 t \, dt = \frac{1}{2} \int_{t_0}^{t_0+T_0} \cos[(n-k)\omega_0 t] \, dt - \frac{1}{2} \int_{t_0}^{t_0+T_0} \cos[(n+k)\omega_0 t] \, dt \tag{17}$$

Both integrals on the right-hand-side are zero except when $n = k$. Then the first integral is $T_0/2$. Therefore

$$\boxed{b_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin n\omega_0 t \, dt} \quad \text{for} \quad n \geq 1 \tag{18}$$

The dot product of the three dimensional vectors $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ is $\mathbf{A} \cdot \mathbf{B} = a_x b_x + a_y b_y + a_z b_z$. Give two functions $x(t)$ and $y(t)$, the integral

$$\int_{t_0}^{t_0+T_0} x(t)\overline{y(t)}\, dt$$

is abstractly similar to the dot product. Two vectors are orthogonal if their dot product is zero. It was shown above that the integrals of $\cos n\omega_0 t \cos k\omega_0 t$ and $\sin n\omega_0 t \sin k\omega_0 t$ over an interval of length T_0 are always zero for $k \neq n$. Also the integral of $\sin n\omega_0 t \cos k\omega_0 t$ over an interval of length T_0 is zero for all n and k Therefore, these sines and cosines can be considered to be orthogonal basis vectors in an infinite dimensional vector space. The Fourier coefficients are the coordinates of the function $x(t)$ with respect to the basis vectors in this infinite dimensional space.

Even though we have derived formulas for the a_n's and b_n's this does not prove that the series converges to $x(t)$ because the sines and cosines may not be a rich enough set of functions. Fortunately, it can be shown that a rather mild set of conditions know as the Dirichlet conditions guarantee the series converges point wise to the function except at discontinuities. All the real-world functions we encounter satisfy these conditions. See Oppenheim and Willsky[1] for these conditions. Also, if the magnitude squared of $x(t)$ integrated over one fundamental period is finite, it can be shown that the series converges to $x(t)$ in the sense that the integral over one fundamental period of the squared magnitude of the error between $x(t)$ and the series is converges to zero. This is known as convergence in the mean-square error sense.

Even if a function is not periodic, the Fourier series will converge to the function over the interval of integration $(t_0, t_0 + T_0)$ and will extend periodically outside this interval.

EXAMPLE 1 Symmetric Square Wave

Let $x(t)$ be the symmetric square wave show by the dashed purple lines in Figure 1. The formula for one period of this square wave centered about the

[1] A.V. Oppenheim and A.S. Willsky, *Signals and Systems*, 2nd Edition, Prentice Hall, 1996, pp. 197–198.

origin is

$$x(t) = \begin{cases} -A & \text{for} \quad -T_0/2 < t < 0 \\ A & \text{for} \quad 0 < t < T_0/2 \end{cases} \tag{19}$$

The average or DC value of this signal is

$$\frac{a_0}{2} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, dt = 0 \tag{20}$$

For $n \geq 1$

$$a_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cos n\omega_0 t\, dt = 0 \quad \text{for} \quad n \geq 1 \tag{21}$$

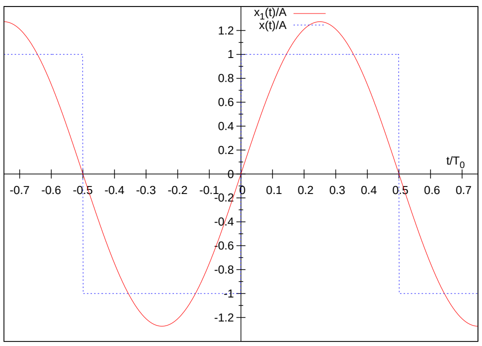
This is true because function $x(t)$ is odd and $\cos n\omega_0$ is even, so the product $x(t) \cos n\omega_0 t$ is odd and the integral of this product symmetrically about the origin is zero. Using the facts that $\omega_0 T_0 = 2\pi$ and $x(t)$ and $\sin n\omega_0 t$ are both odd so that their product is even, the coefficients of the sine terms are

$$\begin{aligned} b_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \sin n\omega_0 t\, dt = \frac{4}{T_0} \int_{0}^{T_0/2} x(t) \sin n\omega_0 t\, dt = \frac{4}{T_0} \int_{0}^{T_0/2} A \sin n\omega_0 t\, dt \\ &= \frac{4A}{T_0} \left[\frac{-\cos n\omega_0 t}{n\omega_0} \right] \Bigg|_0^{T_0/2} = \frac{2A}{n\pi}(1 - \cos n\pi) = \frac{2A}{n\pi}[1 - (-1)^n] \\ &= \begin{cases} \dfrac{4A}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \end{aligned} \tag{22}$$

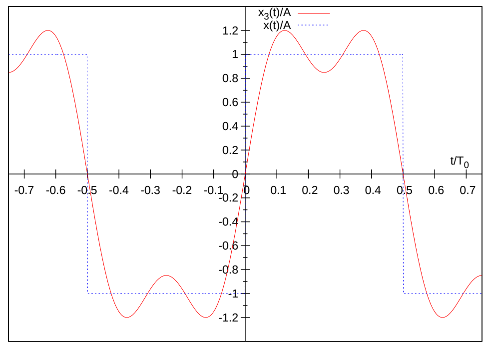
Using these coefficients, the Fourier series for the square wave can be written as

$$x(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)\omega_0 t \tag{23}$$

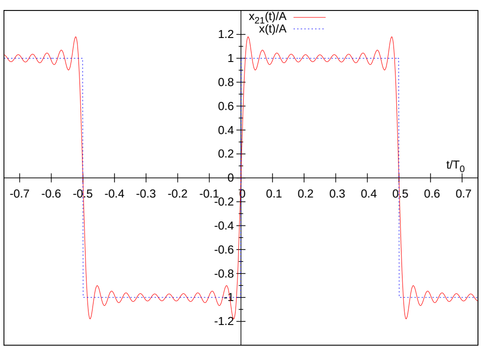
Approximations to $x(t)$ when the sum is truncated at the $N = 2n - 1$ harmonic or $n = (N + 1)/2$ for $N = 1, 3, 21$ and 41 are shown in Figure 1. Notice that there is a significant overshoot at a jump even as N becomes larger. It can be shown that the jump remains at close to 9% of the height of the jump as N increases. This is known as Gibbs' phenomenon. The



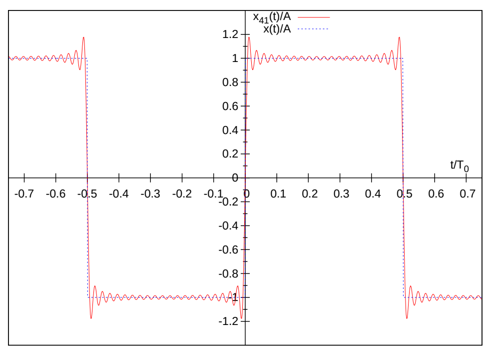
(a) N = 1



(b) N = 3



(c) N = 21



(d) N = 41

Figure 1: Some Truncated Fourier Series Approximations to a Square Wave

frequency of the ripples increases but the ripples move closer to the jump and decay more quickly away from the jump as N increases.

$\blacksquare$

We saw in class that

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\,\cos[\theta - \arctan(b/a)] \tag{24}$$

Therefore, the Fourier series can also be expressed as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2}\,\cos[n\omega_0 t - \arctan(b_n/a_n)] \tag{25}$$

This form shows the amplitude and phase shift of each harmonic.

2 The Complex Exponential Form of the Fourier Series

We will now see that the real form of the Fourier series can be converted into a more compact form that is a sum of scaled complex exponentials at multiples of the fundamental frequency. Remember that

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j} \tag{26}$$

Using these identities, the real form Fourier series becomes

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2} + b_n \frac{e^{jn\omega_o t}-e^{jn\omega_0 t}}{2} \tag{27}$$

The series can be rearranged by collecting the terms involving $e^{jn\omega_0 t}$ and $e^{-jn\omega_0 t}$ resulting in

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \frac{a_n + jb_n}{2} e^{-jn\omega_0 t} \tag{28}$$

Now define the new coefficient c_n as

$$c_n = \frac{a_n - jb_n}{2} = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)(\cos n\omega_0 t - j\sin n\omega_0 t)\,dt$$

$$= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t}\, dt \tag{29}$$

Notice that

$$\frac{a_n + jb_n}{2} = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{jn\omega_0 t}\, dt = c_{-n} \quad \text{and} \quad \frac{a_0}{2} = c_0 \tag{30}$$

Therefore (28) can be collapsed into the following single sum:

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}} \tag{31}$$

where

$$\boxed{c_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t}\, dt} \tag{32}$$

It is interesting to observe that the complex exponentials in the series are orthogonal, that is,

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{j(n-m)\omega_0 t}\, dt = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases} \tag{33}$$

Therefore, the Fourier series can be thought of as the representation of $x(t)$ in an infinite dimensional vector space where the basis vectors are the complex exponentials and the coordinates are the c_n Fourier coefficients.

The coefficients of the real form of the series can be found from the coefficients of the complex form by adding and subtracting c_n and c_{-n} to get

$$a_n = c_n + c_{-n} \quad \text{and} \quad jb_n = c_{-n} - c_n \tag{34}$$

If $x(t)$ is real, then $c_n = \overline{c_{-n}}$. In polar form $c_n = |c_n| e^{j\theta_n}$ so $c_{-n} = |c_n| e^{-j\theta_n}$. The Fourier series then can be written as

$$x(t) = c_0 + \sum_{n=1}^{\infty} |c_n| \left(e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)} \right) = c_0 + \sum_{n=1}^{\infty} (2|c_n|) \cos(n\omega_0 t + \theta_n) \tag{35}$$

By comparison with the real form of the series or by direct computation, it follows that $2|c_n| = (a_n^2 + b_n^2)^{1/2}$ and $\theta_n = -\arctan(b_n/a_n)$.

EXAMPLE 2 Pulse Train

One period of a periodic pulse train, $x(t)$, is shown in Figure 2. The coeffi-

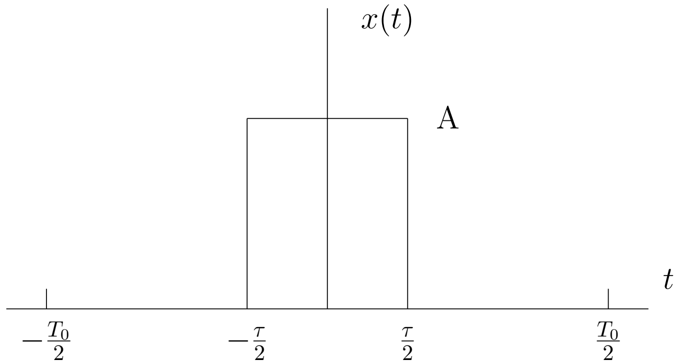


Figure 2: A Periodic Pulse Train

cients for the complex exponential form of the Fourier series are

$$x_n = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A e^{-jn\omega_0 t}\, dt = \frac{A}{T_0} \int_{-\tau/2}^{\tau/2} \cos n\omega_0 t - j \sin n\omega_0 t\, dt \tag{36}$$

The integral of $\sin n\omega_0$ symmetrically about the origin is zero. The function $\cos n\omega_0 t$ is even. Using these two facts gives

$$x_n = \frac{2A}{T_0} \int_{0}^{\tau/2} \cos n\omega_0 t\, dt = \frac{2A}{T_0 n\omega_0} \sin n\omega_0 \tau/2 = A \frac{\tau}{T_0} \frac{\sin n\pi \frac{\tau}{T_0}}{n\pi \frac{\tau}{T_0}} \tag{37}$$

The quotient τ/T_0 is called the *duty factor* of the pulse train. The fundamental frequency for the pulse train is $f_0 = 1/T_0$. The frequency corresponding to coefficient x_n is $n/T_0 = nf_0$. If nf_0 is replaced by f in (37), the nulls of $\sin \pi\tau f$ occur when $\pi\tau f = k\pi$ or $f = k/\tau$ where k is an integer. The first null occurs at $1/\tau$ which is the reciprocal of the pulse width. Using the original integral with $n = 0$ or by using L'Hospital's rule it follows that $x_0 = A\tau/T_0$ so the DC value is the pulse amplitude multiplied by the duty factor. The function

$$g(f) = \frac{\tau}{T_0} \frac{\sin \pi\tau f}{\pi\tau f} \tag{38}$$

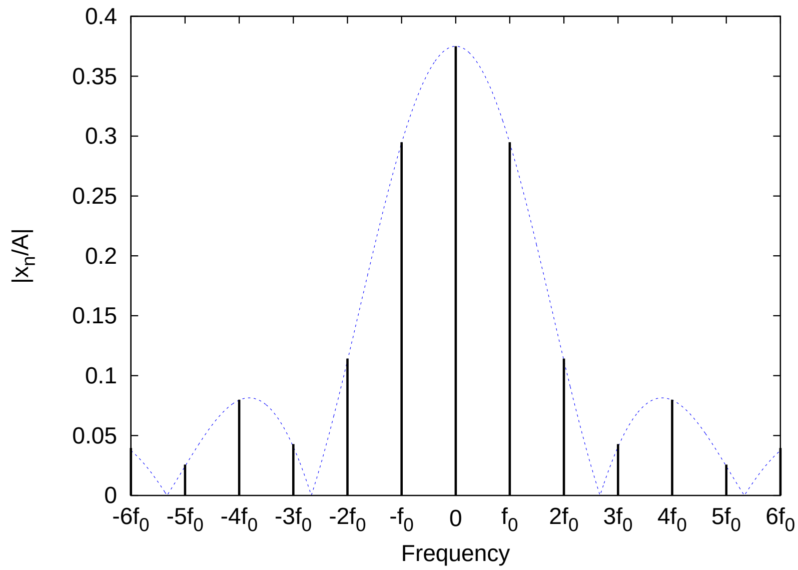


Figure 3: Magnitudes of the Fourier Coefficients for the Pulse Train with $\tau/T_0 = 0.375$.

is plotted in Figure 3 as the purple dashed line for $\tau/T_0 = 0.375$ or, equivalently, $\tau = 0.375/f_0$. Then the nulls occur at $k/\tau = kf_0/0.375$. The Fourier coefficients x_n/A are samples of $g(f)$ for $f = nf_0$. Some of the magnitudes of these coefficients are plotted as the vertical lines in Figure 3.

If T_0 is fixed and τ decreases, the null at $1/\tau$ increases. So, as intuitively expected, the signal bandwidth increases as the pulse becomes shorter. On the other hand, when τ is fixed and T_0 increases (or the fundamental frequency f_0 decreases), the harmonic frequencies become closer together. We will see soon that when T_0 becomes infinite so that $x(t)$ is a single isolated pulse at the origin, the plot of the coefficients becomes a continuum which is called the Fourier transform of the signal. ▬

EXAMPLE 3 Period Train of Impulse Functions

Let $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$. The complex exponential Fourier series coefficients are

$$x_n = \frac{1}{T_0} \int\limits_{T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t}\, dt = \frac{1}{T_0} \quad \text{for all } n \tag{39}$$

Therefore

$$x(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \tag{40}$$

The Fourier coefficients for this signal all have the same value $1/T_0$, so its bandwidth is infinite. $\blacksquare$

3 Fourier Series for Signals with Special Symmetries

3.1 Even Signals

An even signal has the property that $x(t) = x(-t)$ for all t. Remember that the coefficients of the cosine terms for the real form of the series are

$$a_n = \frac{2}{T_0} \int\limits_{-T_0/2}^{T_0/2} x(t) \cos n\omega_0 t\, dt \tag{41}$$

The product $x(t)\cos n\omega_0 t$ is even since both factors are even. Therefore these coefficients can also be computed as

$$a_n = \frac{4}{T_0} \int\limits_{0}^{T_0/2} x(t) \cos n\omega_0 t\, dt \tag{42}$$

The formula for the coefficients of the sine terms is

$$b_n = \frac{2}{T_0} \int\limits_{-T_0/2}^{T_0/2} x(t) \sin n\omega_0 t\, dt \tag{43}$$

The product $x(t)\sin n\omega_0 t$ is odd since $x(t)$ is even and $\sin n\omega_0 t$ is odd. Therefore the integral of this product symmetrically about the origin must be zero and $b_n = 0$ for all n. Thus the Fourier series is just a sum of cosine terms plus the DC value. These terms are all even functions. An even function cannot contain any odd components, so no sine terms can be present.

The Fourier coefficients for the complex exponential form of the series are

$$c_n = a_n/2 = \frac{2}{T_0}\int\limits_0^{T_0/2} x(t)\cos n\omega_0 t\, dt \tag{44}$$

because $b_n = 0$. When x(t) is real, c_n is real. For real or complex signals $c_n = c_{-n}$.

3.2 Odd Signals

3.3 Half Wave Symmetry

A signal $x(t)$ has half wave symmetry if $x(t+\frac{T_0}{2}) = -x(t)$ for all t. The symmetric square wave is an example of a signal with half wave symmetry. It will now be shown that signals of this type only have odd harmonics. The Fourier coefficients are

$$\begin{aligned}
x_n &= \frac{1}{T_0}\int\limits_0^{T_0} x(t)e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int\limits_0^{T_0/2} x(t)e^{-jn\omega_0 t}\, dt + \frac{1}{T_0}\int\limits_{T_0/2}^{T_0} x(t)e^{-jn\omega_0 t}\, dt \\
&= \frac{1}{T_0}\int\limits_0^{T_0/2} x(t)e^{-jn\omega_0 t}\, dt + \frac{1}{T_0}\int\limits_0^{T_0/2} x\left(t+\frac{T_0}{2}\right) e^{-jn\omega_0\left(t+\frac{T_0}{2}\right)}\, dt \\
&= \frac{1}{T_0}\int\limits_0^{T_0/2} x(t)e^{-jn\omega_0 t} + x\left(t+\frac{T_0}{2}\right) e^{-jn\omega_0 t}e^{-jn\omega_0 T_0/2}\, dt \\
&= \frac{1}{T_0}\int\limits_0^{T_0/2} x(t)\left(1-e^{-jn\omega_0 T_0/2}\right)e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int\limits_0^{T_0/2} x(t)\left[1-e^{-jn\pi}\right]e^{-jn\omega_0 t}\, dt \\
&= \frac{1}{T_0}\int\limits_0^{T_0/2} x(t)\left[1-(-1)^n\right]e^{-jn\omega_0 t}\, dt
\end{aligned}$$

$$= \begin{cases} \dfrac{2}{T_0} \displaystyle\int_0^{T_0/2} x(t) e^{-jn\omega_0 t}\, dt & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \tag{45}$$

4 Fourier Series for Some Simple Operations on Periodic Signals

4.1 Linearity

Let $x(t)$ and $y(t)$ be periodic signals each with period T_0 and complex exponential Fourier series coefficients x_n and y_n, respectively. Let $z(t) = c_1 x(t) + c_2 y(t)$. Then by the linearity of integrals it follows that the Fourier coefficients for $z(t)$ are $z_n = c_1 x_n + c_2 y_n$.

4.2 The Delay Theorem

Let the periodic signal $x(t)$ with period T_0 have the Fourier series expansion

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

Then the delayed signal $y(t) = x(t-d)$ is

$$y(t) = x(t-d) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 (t-d)} = \sum_{n=-\infty}^{\infty} \left(x_n e^{-jn\omega_0 d} \right) e^{jn\omega_0 t} \tag{46}$$

Thus the Fourier coefficients for $y(t)$ are

$$y_n = x_n e^{-jn\omega_0 d} \tag{47}$$

Notice that $|y_n| = |x_n|$ but the delay adds a phase shift of $-n\omega_o d$ to the coefficients of $x(t)$. This phase shift varies linearly with the frequency $n\omega_0$.

4.3 Differentiation

Let $x(t)$ be a periodic signal with period T_0 and complex exponential Fourier series coefficients x_n. Then its derivative with respect to time is

$$y(t) = \frac{d}{dt}x(t) = \frac{d}{dt}\sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} (jn\omega_0 x_n)e^{jn\omega_0 t} \tag{48}$$

Therefore, the coefficients for the derivative are

$$y_n = jn\omega_0 x_n \tag{49}$$

Notice that differentiation removes the DC value since $y_0 = 0$. It emphasizes the high frequency harmonics by a factor increasing linearly with the harmonic frequency $n\omega_0$. In addition, the factor j adds a 90 degree phase shift to each component.

4.4 Integration

Let x(t) be a periodic signal with period T_0, Fourier coefficients x_n, and average value $x_0 = 0$. Consider the signal

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \tag{50}$$

If x_0 were not zero then this integral would increase with t without bound and $y(t)$ would not be periodic. Integrating the complex exponential Fourier series for $x(t)$ gives

$$y(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{x_n}{jn\omega_0} e^{jn\omega_0 \tau}\, d\tau + K \tag{51}$$

where K is a constant of integration. Since the sum does not contain the $n = 0$ term, K must be the average value y_0 of $y(t)$. From the sum it is clear that the Fourier coefficients of $y(t)$ are

$$y_n = \frac{x_n}{jn\omega_0} \quad \text{for} \quad n \neq 0 \quad \text{and } y_0 = \text{average value of } y(t) \tag{52}$$

Integration emphasizes the low frequency components and attenuates the high frequency ones by a factor inversely proportional to frequency. The $1/j$ term adds a -90 degree phase shift to each harmonic.

EXAMPLE 4 Fourier Series for a Triangular Wave

The differentiation and integration theorems along with the train of impulses

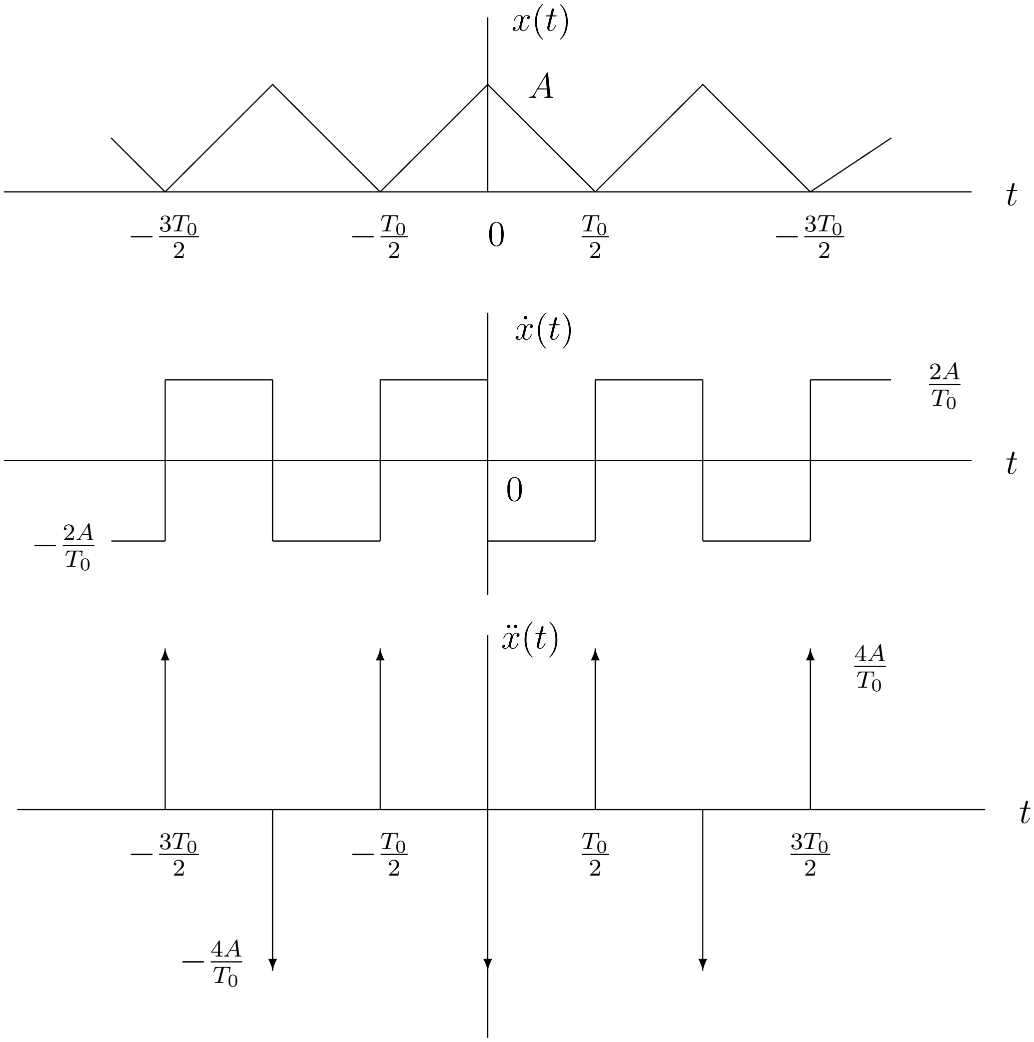


Figure 4: Triangular Wave

example can be used to easily find the Fourier series for signals consisting of polynomial segments. The method is to differentiate the signal repeatedly until a sum of impulse trains arises. The Fourier series for this sum of impulse trains is easily found from the single impulse train example and the delay theorem. The resulting coefficients are then divided by $jn\omega_0$ raised to the number of derivatives used to get to the sum of impulse trains. Finally, the $n = 0$ coefficient must be set equal to the average value of the original signal. To illustrate this procedure consider the triangular wave shown in Figure 4. An equation for $\ddot{x}(t)$ is

$$\ddot{x}(t) = -\frac{4A}{T_0} \sum_{n=-\infty}^{\infty} \delta(t - nT_0) + \frac{4A}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(t - nT_0 - \frac{T_0}{2}\right) \tag{53}$$

Notice that the second sum on the right hand side is the first sum delayed by $T_0/2$ and with a sign change. Using the delay theorem and the result of Example 3, the Fourier series coefficients for $\ddot{x}(t)$ are $\ddot{x}_0 = 0$ since its average is zero and

$$\ddot{x}_n = -\frac{4A}{T_0}\left(\frac{1}{T_0} - \frac{1}{T_0}\, e^{-jn\omega_0 T_0/2}\right) = -\frac{4A}{T_0^2}\,[1 - (-1)^n] \quad \text{for } n \neq 0 \tag{54}$$

(The double dots above the coefficients do not mean "take the second derivative of the coefficient." They are an abuse of notation and simply indicate that the coefficients are for the second derivative of $x(t)$.) The average value of $\dot{x}(t)$ is zero, so its Fourier coefficients are $\dot{x}_0 = 0$ and

$$\dot{x}_n = \frac{\ddot{x}_n}{jm\omega_0} = \frac{-\frac{4A}{T_0^2}\,[1 - (-1)^n]}{jn\omega_0} \quad \text{for } n \neq 0 \tag{55}$$

The average value of $x(t)$ is

$$x_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, dt = \frac{A}{2} \tag{56}$$

and the coefficients for $n > 0$ are

$$x_n = \frac{\ddot{x}_n}{(jn\omega_0)^2} = \frac{-4A[1 - (-1)^n]}{(jn\omega_0 T_0)^2} = \frac{A[1 - (-1)^2]}{(n\pi)^2} \quad \text{for } n \neq 0 \tag{57}$$

More explicitly

$$x_n = \begin{cases} A/2 & \text{for } n = 0 \\ 0 & \text{for } n \text{ even} \\ \dfrac{2A}{n^2\pi^2} & \text{for } n \text{ odd} \end{cases} \tag{58}$$

If the average value $A/2$ is subtracted from $x(t)$ the resulting signal is a triangular wave centered about the x-axis and has half wave symmetry. Therefore it would be expected that the even harmonics are all zero except for the DC term. The triangular wave is smoother than the symmetric square wave, so its Fourier coefficients decay faster, like $1/n^2$ rather than $1/n$. ▬

4.5 Multiplication of Two Periodic Signals with the Same Period

Let $x(t)$ and $y(t)$ be two signals with the same period and Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} \quad y(t) = \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_0 t}$$

The product of these signals $z(t) = x(t)y(t)$ has the Fourier coefficients

$$z_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k} = x_n * y_n \tag{59}$$

It is interesting to see that discrete-time convolution arises in situations other than finding the output of a discrete-time LTI system by convolving the input with the impulse response.

Proof:
The product can be written as

$$z(t) = x(t)y(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t} y(t) \tag{60}$$

so the Fourier coefficients for $z(t)$ are

$$z_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)y(t)e^{-jn\omega_0 t}\, dt = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t} y(t) e^{-jn\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} x_k \, \frac{1}{T_0} \int\limits_{t_0}^{t_0+T_0} y(t) e^{-j(n-k)\omega_0 t} \, dt = \sum_{k=-\infty}^{\infty} x_k y_{n-k} \tag{61}$$

4.6 Fourier Coefficients for the Complex Conjugate of a Signal

Let x(t) be a signal with the Fourier coefficients x_n. Let $y(t) = \overline{x(t)}$. The formula for computing x_n is

$$x_n = \frac{1}{T_0} \int\limits_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} \, dt \tag{62}$$

Using the facts that the conjugate of an integral is the integral of the conjugate of the integrand, and the conjugate of a product is the product of the conjugates of the factors gives

$$\bar{x}_n = \frac{1}{T_0} \int\limits_{t_0}^{t_0+T_0} \overline{x(t)} e^{jn\omega_0 t} \, dt \quad \text{so} \quad \bar{x}_{-n} = \frac{1}{T_0} \int\limits_{t_0}^{t_0+T_0} \overline{x(t)} e^{-jn\omega_0 t} \, dt \tag{63}$$

Therefore, the Fourier coefficients for $y(t) = \overline{x(t)}$ are

$$y_n = \bar{x}_{-n} \tag{64}$$

In words, conjugation turns the spectrum around backwards and conjugates the terms.

4.7 Average Power and Parseval's Theorem

The average power for a periodic signal $x(t)$ with period T_0 and Fourier coefficients x_n is

$$P = \frac{1}{T_0} \int\limits_{t_0}^{t_0+T_0} |x(t)|^2 \, dt = \frac{1}{T_0} \int\limits_{t_0}^{t_0+T_0} x(t)\overline{x(t)} \, dt \tag{65}$$

In (61) let $y(t) = \overline{x(t)}$, so $y_k = \bar{x}_{-k}$ and $y_{-k} = \bar{x}_k$. Also let $n = 0$. Then (61) becomes

$$z_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |x_k|^2 = P \tag{66}$$

This is often referred to as Parseval's Theorem. Using the same approach, it is easy to show that the cross-correlation between two sequences is

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\overline{y(t)} \, dt = \sum_{k=-\infty}^{\infty} x_k \bar{y}_k \tag{67}$$

Let a_n and b_n be the Fourier coefficients for the real form series and assume that $x(t)$ is a real signal. Remember that

$$x_n = \frac{a_n - jb_n}{2} \quad \text{and} \quad x_{-n} = \bar{x}_n$$

so that

$$|x_n|^2 = |x_{-n}|^2 = \frac{a_n^2 + b_n^2}{4} \tag{68}$$

Then

$$\begin{aligned} P &= \sum_{n=-\infty}^{-1} |x_n|^2 + x_0^2 + \sum_{n=1}^{\infty} |x_n|^2 = x_0^2 + 2\sum_{n=1}^{\infty} |x_n|^2 \\ &= \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2} \end{aligned} \tag{69}$$

Remember that the amplitude of the nth harmonic in the real form series is $\sqrt{a_n^2 + b_n^2}$. Therefore the power associated with that harmonic is $(a_n^2 + b_n^2)/2$.

In summary, the total power in a periodic signal is the sum of the powers of the individual harmonics. There are no cross terms between different harmonics.

5 Minimum Mean-Square Error Approximation

Let $x(t)$ be a periodic signal with the complex exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

Suppose we want to best approximate $x(t)$ by a finite term trigonometric sum of the form

$$x_N(t) = \sum_{n=-N}^{N} \alpha_n e^{jn\omega_0 t} \tag{70}$$

To proceed we must define what we mean by "best." There are many choices for "best." We will define "best" as selecting the coefficients $\{\alpha_n\}$ to minimize the average of the integral of the magnitude squared of the error between $x(t)$ and $x_N(t)$ over one period. This is called the *mean-square error* or MSE. We could simply truncate the Fourier series for $x(t)$ and choose the α_n's as the Fourier coefficients, but is that the best choice? The instantaneous error is

$$\begin{aligned}\epsilon(t) = x(t) - x_N(t) &= \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} - \sum_{n=-N}^{N} \alpha_n e^{jn\omega_0 t} \\ &= \sum_{n=-N}^{N} (x_n - \alpha_n) e^{jn\omega_0 t} + \sum_{|n|>N} x_n e^{jn\omega_0 t}\end{aligned} \tag{71}$$

Thus $\epsilon(t)$ is a Fourier series with coefficients $\epsilon_n = x_n - \alpha_n$ for $-N \leq n \leq N$ and x_n for $|n| > N$. According to Parseval's theorem, the MSE is

$$\mathcal{E}^2 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |\,\epsilon(t)|^2 \, dt = \sum_{n=-N}^{N} |x_n - \alpha_n|^2 + \sum_{|n|>N} |x_n|^2 \tag{72}$$

The coefficients x_n are determined by the Fourier series for $x(t)$. They are fixed and we cannot change them. Therefore, $\mathcal{E}^2$ is minimized by selecting $\alpha_n = x_n$ for $-N \leq n \leq N$. Thus the best choice for the α_n's is always the Fourier coefficients x_n for $-N \leq n \leq N$ for every value of the truncation limit N. With this choice, the minimum MSE is

$$\min_{\{\alpha_n\}} \mathcal{E}^2 = \sum_{|n|>N} |x_n|^2 = \sum_{n=-\infty}^{\infty} |x_n|^2 - \sum_{n=-N}^{N} |x_n|^2$$

and by using Parseval's theorem for the first integral on the right

$$= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 \, dt - \sum_{n=-N}^{N} |x_n|^2 \tag{73}$$

The integral of $|x(t)|^2$ can be directly computed from the given $x(t)$. A desired MSE can be achieved by increasing N and incrementing the sum of $|x_n|^2$ until the desired value is reached. It is interesting to observe that increasing N does not change the best coefficients found for smaller N. They are always the Fourier series coefficients. Another observation is that minimum MSE, (73), decreases to zero as N increases as a result of Parseval's theorem.

EXAMPLE 5 Pulse Train (continued)

Consider the periodic pulse train of Example 2. By direct calculation using $x(t)$ and then Parseval's theorem

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt = A^2 \frac{\tau}{T_0} = \sum_{n=-\infty}^{\infty} \left[A \, \frac{\tau}{T_0} \, \frac{\sin n\pi \dfrac{\tau}{T_0}}{n\pi \dfrac{\tau}{T_0}} \right]^2 \tag{74}$$

The sum of squared Fourier coefficients can be truncated at increasing N to find the value of N for a desired MSE. ∎

6 Fourier Series and LTI Systems

Consider an LTI system with the impulse response $h(t)$. If the input is the sinusoid $x(t) = Ce^{j(\omega t + \beta)}$ the output is

$$y(t) = \int_{-\infty}^{\infty} h(\tau) C e^{j[\omega(t-\tau)+\beta]} \, d\tau = Ce^{j(\omega t+\beta)} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} \, d\tau = Ce^{j(\omega t+\beta)} H(\omega) \tag{75}$$

where on renaming τ to t

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt \tag{76}$$

The function $H(\omega)$ is called the *frequency response* of the system. Shortly we will see that $H(\omega)$ is the *Fourier transform* of the impulse response. Thus the output is a sinusoid at the same frequency as the input but scaled by the complex number $H(\omega)$. In polar form $H(\omega) = A(\omega)e^{j\theta(\omega)}$ where

$$A(\omega) = |H(\omega)| \quad \text{and} \quad \theta(\omega) = \arg H(\omega) \tag{77}$$

The function $A(\omega)$ is called the *amplitude response* of the system and $\theta(\omega)$ is called the *phase response* of the system. The amplitude response in dB is $\alpha(\omega) = 20\log_{10} A(\omega)$. Thus the output can be written as

$$\begin{aligned} y(t) &= A(\omega)Ce^{j[\omega t+\beta+\theta(\omega)]} \\ &= A(\omega)C\cos[\omega t+\beta+\theta(\omega)] + jA(\omega)C\sin[\omega t+\beta+\theta(\omega)] \end{aligned} \tag{78}$$

Thus when the input to a system is a sinusoid at some frequency, the output is a sinusoid at the same frequency but the amplitude is scaled by the amplitude response and the phase is shifted by the phase response of the system. Sinusoids are called the *eigenfunctions* of LTI systems in analogy to the eigenvectors of a square matrix.

The output when $h(t)$ is real is

$$\begin{aligned} y(t) &= Ce^{j(\omega t+\beta)} * h(t) = C[\cos(\omega t+\beta) + j\sin(\omega t+\beta)] * h(t) \\ &= C\cos(\omega t+\beta) * h(t) + jC\sin(\omega t+\beta) * h(t) \end{aligned} \tag{79}$$

Both convolutions on the right hand side result in real signals. Therefore, comparing (78) and (79) we see that the output resulting from the input $C\cos(\omega t+\beta)$ is

$$\Re e\{y(t)\} = C\cos(\omega t+\beta) * h(t) = A(\omega)C\cos[\omega t+\beta+\theta(\omega)] \tag{80}$$

Thus for a real impulse response and input sine wave at frequency ω, the output is again a real sine wave at frequency ω scaled in amplitude by $A(\omega)$ and phase shifted by $\theta(\omega)$.

If the input to an LTI system with impulse response $h(t)$ is a periodic signal $x(t)$ with the Fourier series $x(t) = \sum_{-\infty}^{\infty} x_n e^{jn\omega_0 t}$, then by superposition the output will be

$$y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) x_n e^{jn\omega_0 t} \tag{81}$$

EXAMPLE 6

Consider a system with the impulse response $h(t) = e^{-at}u(t)$ with $a > 0$. The frequency response is

$$H(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t}\, dt = \frac{1}{j\omega + a} \tag{82}$$

The amplitude and phase responses are

$$A(\omega) = |H(\omega)| = \frac{1}{\sqrt{\omega^2 + a^2}} \quad \text{and} \quad \theta(\omega) = \arg H(\omega) = -\arctan(\omega/a) \tag{83}$$

This is an elementary lowpass filter. The amplitude response has a maximum at $\omega = 0$ and decreases monotonically as $|\omega|$ increases. The output when the input is a periodic signal with complex exponential Fourier series coefficients $\{x_n\}$ is

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{1}{jn\omega_0 + a}\, x_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{(n\omega_0)^2 + a^2}}\, x_n e^{j[n\omega_0 t + \theta(n\omega_0)]} \tag{84}$$

■

7 Speed of Convergence

Some idea about the speed of convergence of Fourier series can be obtained by assuming the signal consists of polynomial segment. Suppose $x(t)$ must be differentiated K times until impulse trains first appear. We have seen that the Fourier coefficients for an impulse train are all the same constant. The integration theorem states that the Fourier coefficients of $x(t)$ are those of the Kth derivative divided by $(jn\omega_0)^K$ except for x_0 which is the average value of $x(t)$. Therefore, the Fourier coefficients of $x(t)$ decrease like $1/n^K$. The more derivatives required to get to impulses, the faster the Fourier coefficients decrease with n. The signal $x(t)$ must be smoother when K is larger. This agrees with our intuition that smoother signals have less high frequency content.

8 Effect of Truncating a Fourier Series by a Coefficient Window

Let $x(t)$ be a periodic signal with the complex exponential Fourier series coefficients x_n. A periodic signal $y(t)$ is formed with the Fourier coefficients $y_n = x_n b_n$ for some desired sequence b_n. Typically, $b_n = 0$ for $|n| > N$ and truncates the Fourier series. The sequence b_n is called a *coefficient window* and forming the product is called *coefficient windowing.* Let the artificial periodic time signal formed from b_n be

$$b(t) = \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t} \tag{85}$$

The series for $y(t)$ is

$$y(t) = \sum_{n=-\infty}^{\infty} x_n b_n e^{jn\omega_0 t} \tag{86}$$

Replacing x_n by the integral to compute it gives

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(\tau) e^{-jn\omega_0 \tau}\, d\tau \; b_n e^{jn\omega_0 t} \\ &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(\tau) \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 (t-\tau)}\, d\tau = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(\tau) b(t-\tau)\, d\tau \end{aligned} \tag{87}$$

The resulting integral is called the *periodic convolution* of $x(t)$ and $b(t)$.

EXAMPLE 7 Rectangular Window

As usual, let $x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$. Let $b_n = 1$ for $-L \le n \le L$ and 0 for

$n > |L|$. The windowed signal is

$$y(t) = \sum_{n=-\infty}^{\infty} x_n b_n e^{jn\omega_0 t} = \sum_{n=-L}^{L} x_n e^{jn\omega_0 t} \tag{88}$$

which is just the original series truncated below $-L$ and above L. There are $N = 2L+1$ terms in the truncated series. The window time function is

$$b(t) = \sum_{n=-L}^{L} e^{jn\omega_0 t} = e^{-jL\omega_0 t}(1 + e^{j\omega_0 t} + \cdots + e^{j2L\omega_0 t}) \tag{89}$$

Using the formula for the sum of a geometric series we find that

$$b(t) = e^{-jL\omega_0 t}\, \frac{e^{j(2L+1)\omega_0 t} - 1}{e^{j\omega_0 t} - 1} \tag{90}$$

Factoring $e^{j(2L+1)\omega_0 t/2}$ from the numerator and $e^{j\omega_0 t/2}$ from the denominator and dividing both by $2j$ gives

$$\begin{aligned} b(t) &= e^{-jL\omega_0 t}\, \frac{e^{j(2L+1)\omega_0 t/2}}{e^{j\omega_0 t/2}}\, \frac{\left(e^{j(2L+1)\omega_0 t/2} - e^{-j(2L+1)\omega_0 t/2}\right)/(2j)}{\left(e^{j\omega_0 t/2} - e^{-j\omega_0 t/2}\right)/(2j)} \\ &= \frac{\sin N\omega_0 t/2}{\sin \omega_0 t/2} \quad \text{where} \quad N = 2L+1 \end{aligned} \tag{91}$$

From the original sum, (89), or by L'Hospital's rule we see that $b(0) = N$. It has period T_0 so $b(kT_0) = N$ for any integer k. The numerator has zeros at $N\omega_0 t/2 = m\pi$ for integer m or at $t = m2\pi/(N\omega_0) = mT_0/N$. Thus $b(mT_0/N) = 0$ except when m is a multiple of N where it has a peak of value N.

The area under $b(t)/T_0$ over the interval $-T_0/2 < t < T_0/2$ is easily found by integrating (89). The only term that is not zero is the integral over the $n = 0$ term which is 1. Therefore,

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} b(t)\, dt = 1 \tag{92}$$

In Example 3 we saw that

$$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \tag{93}$$

Therefore

$$\lim_{L\to\infty} \frac{1}{T_0}\, b(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \tag{94}$$

A plot of $b(t)$ for $N = 11$ is shown in Figure 5.

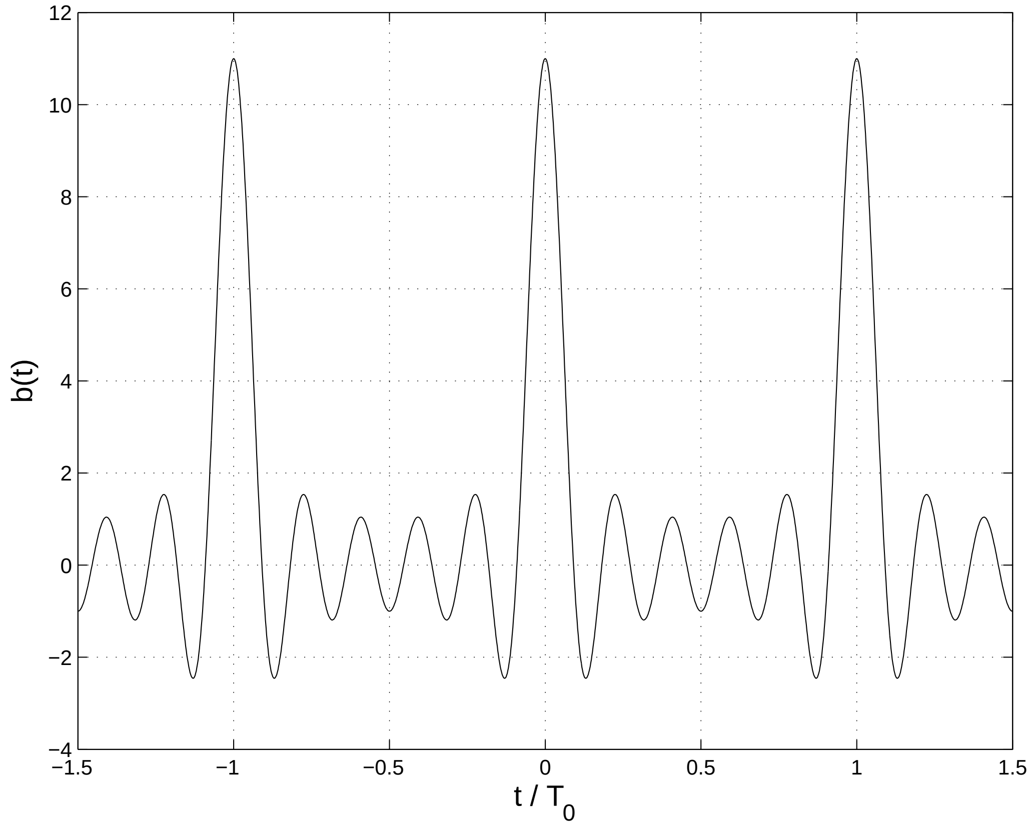


Figure 5: $b(t)$ for $N = 11$

Suppose $-T_0/2 < t < T_0/2$ and $t_0 = -T_0/2$ in (87). Then

$$y(t) = \frac{1}{T_0} \int_{T_0/2}^{T_0/2} x(\tau) b(t - \tau)\, d\tau \tag{95}$$

The window $b(t - \tau)$ has its peak at $\tau = t$ and $y(t)$ is essentially an average of the values of $x(\tau)$ in the vicinity of $\tau = t$. As N increases, the main lobe of $b(t-\tau)$ gets narrower and taller and $y(t)$ depends more and more on values of $x(\tau)$ in the vicinity of $\tau = t$. In the limit as N becomes infinite, $b(t-\tau)$ becomes the impulse $T_0\delta(t-\tau)$. Therefore, the periodic convolution gives $\lim_{N\to\infty} y(t) = x(t)$ at times where $x(t)$ is continuous. Since $b(t-\tau)$ is symmetric about $\tau = t$ the periodic convolution converges at a jump in $x(t)$ to the average of the values of $x(t)$ just to the left and right of the jump. This value is a point halfway up the jump.

In Example 1 we observed that the truncated series overshoots the jumps and has ripples that decrease away from the jumps. This is called *Gibbs' phenomenon*. The cause of the ripples can be seen by looking at the periodic convolution. Suppose $x(\tau)$ has a jump at $\tau = t_1$. The ripples occur as the lobes of $b(t-\tau)$ slide by the jump at $\tau = t_1$ as t is varied. When $t = t_1$ the main lobe of $b(t_1-\tau)$ is centered at the jump and the integral converges to a point halfway up the jump. The peak of the first ripple near the jump occurs T_0/N away from the jump when all the area of the main lobe of the window function is included. It can be shown[2] that the peak overshoot remains close to 8.95% of the height of the jump as N increases. This is true even for N as small as 31. Methods for reducing the amplitude of the ripples by using other truncation windows are discussed in courses on digital signal processing. ▬

[2]S.A. Tretter, *Discrete-Time Signal Processing*, John Wiley & Sons, 1976, pp. 227–230.