Chapter 8
Frequency Modulation (FM)

Contents

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FM was invented and commercialized after AM. Its main advantage is that it is more resistant to additive noise than AM.

Instantaneous Frequency

The *instantaneous frequency* of $\cos\theta(t)$ is

$$\omega(t) = \frac{d}{dt}\theta(t) \qquad (1)$$

Motivational Example

Let $\theta(t) = \omega_c t$. The instantaneous frequency of $s(t) = \cos\omega_c t$ is $\dfrac{d}{dt}\omega_c t = \omega_c$.

FM Signal for Message $m(t)$

The instantaneous frequency of an FM wave with carrier frequency ω_c for a baseband message $m(t)$ is

$$\omega(t) = \omega_c + k_\omega m(t) \qquad (2)$$

FM Signal Definition (cont.)

where k_ω is a positive constant called the *frequency sensitivity*.

An oscillator whose frequency is controlled by its input $m(t)$ in this manner is called a *voltage controlled oscillator*.

The angle of the FM signal, assuming the value is 0 at $t = 0$, is

$$\theta(t) = \int_0^t \omega(\tau)\,d\tau = \omega_c t + \theta_m(t) \tag{3}$$

where

$$\theta_m(t) = k_\omega \int_0^t m(\tau)\,d\tau \tag{4}$$

is the carrier phase deviation caused by $m(t)$. The FM signal generated by $m(t)$ is

$$s(t) = A_c \cos[\omega_c t + \theta_m(t)] \tag{5}$$

Discrete-Time FM Modulator

A discrete-time approximation to the FM wave can be obtained by replacing the integral by a sum. The approximate phase angle is

$$\theta(nT) = \sum_{k=0}^{n-1} \omega(kT)T = \omega_c nT + \theta_m(nT) \qquad (6)$$

where

$$\theta_m(nT) = k_\omega T \sum_{k=0}^{n-1} m(kT) \qquad (7)$$

The total carrier angle can be computed recursively by the formula

$$\theta(nT) = \theta((n-1)T) + \omega_c T + k_\omega T m((n-1)T) \quad (8)$$

The resulting FM signal sample is

$$s(nT) = A_c \cos\theta(nT) \qquad (9)$$

Single Tone FM Modulation

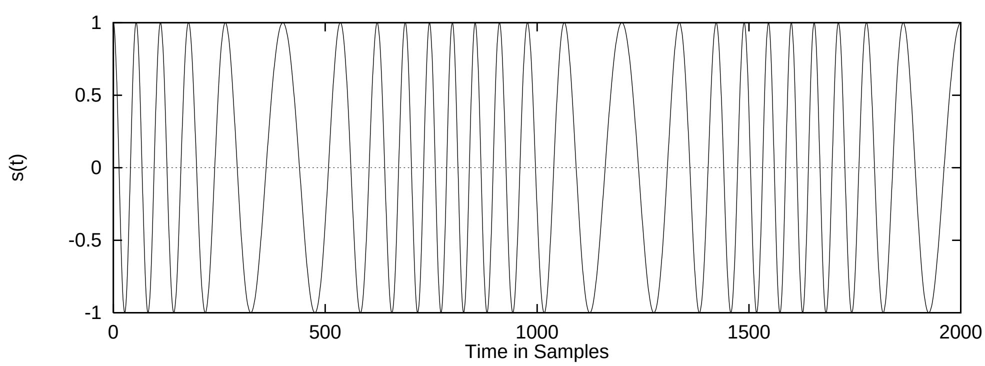
Let $m(t) = A_m \cos \omega_m t$. Then

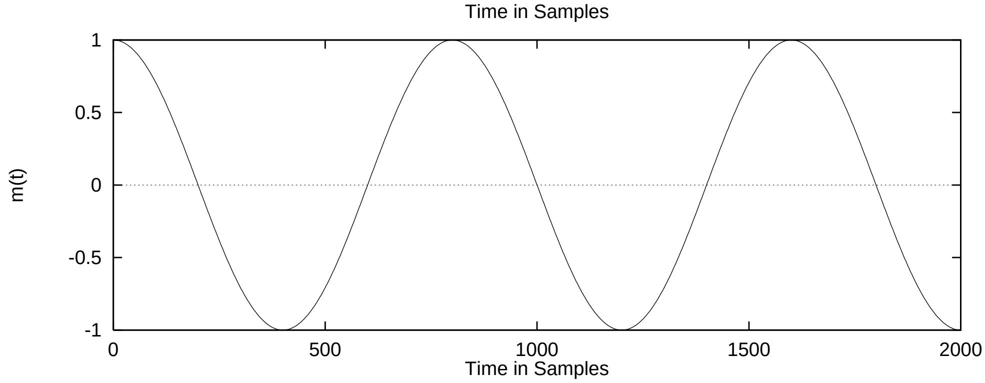
$$s(t) = A_c \cos\left(\omega_c t + \frac{k_\omega A_m}{\omega_m} \sin \omega_m t\right) \qquad (10)$$

The *modulation* index is defined as

$$\beta = \frac{k_\omega A_m}{\omega_m} = \frac{\text{peak frequency deviation}}{\text{modulating frequency}} \qquad (11)$$

Example: $f_c = 1$ kHz, $f_m = 100$ Hz,
$f_s = 80$ kHz, $\beta = 5$





Single Tone FM (cont.)

It can be shown that $s(t)$ has the series exansion

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[(\omega_c + n\omega_m)t] \qquad (12)$$

where $J_n(\beta)$ is the n-th order Bessel function of the first kind. These functions can be computed by the series

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}x\right)^{n+2m}}{m!(n+m)!} \qquad (13)$$

Clearly, the spectrum of the FM signal is much more complex than that of the AM signal.

- There are components at the infinite set of frequencies $\{\omega_c + n\omega_m;\ n = -\infty, \cdots, \infty\}$
- The sinusoidal component at the carrier frequency has amplitude $J_0(\beta)$ and can actually become zero for some β.

Narrow Band FM Modulation

The case where $|\theta_m(t)| \ll 1$ for all t is called *narrow band* FM. Using the approximations $\cos x \simeq 1$ and $\sin x \simeq x$ for $|x| \ll 1$, the FM signal can be approximated as:

$$\begin{aligned} s(t) &= A_c \cos[\omega_c t + \theta_m(t)] \\ &= A_c \cos\omega_c t \cos\theta_m(t) - A_c \sin\omega_c t \sin\theta_m(t) \\ &\simeq A_c \cos\omega_c t - A_c \theta_m(t) \sin\omega_c t \qquad (14) \end{aligned}$$

or in complex notation

$$s(t) \simeq A_c \Re e\left\{e^{j\omega_c t}[1 + j\theta_m(t)]\right\} \qquad (15)$$

This is similar to the AM signal except that the discrete carrier component $A_c \cos\omega_c t$ is 90° out of phase with the sinusoid $A_c \sin\omega_c t$ multiplying the phase angle $\theta_m(t)$. The spectrum of narrow band FM is similar to that of AM.

The Bandwidth of an FM Signal

The following formula, known as *Carson's rule* is often used as an estimate of the FM signal bandwidth:

$$B_T = 2(\Delta f + f_m) \quad \text{Hz} \tag{16}$$

where Δf is the peak frequency deviation

and f_m is the maximum baseband message frequency component.

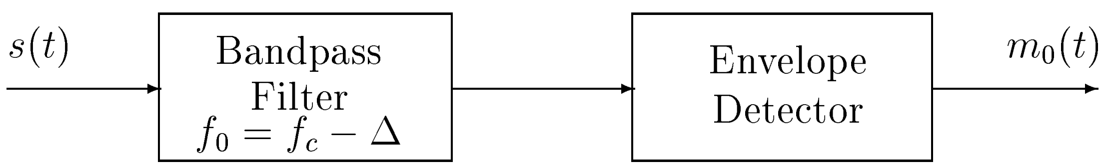
Example

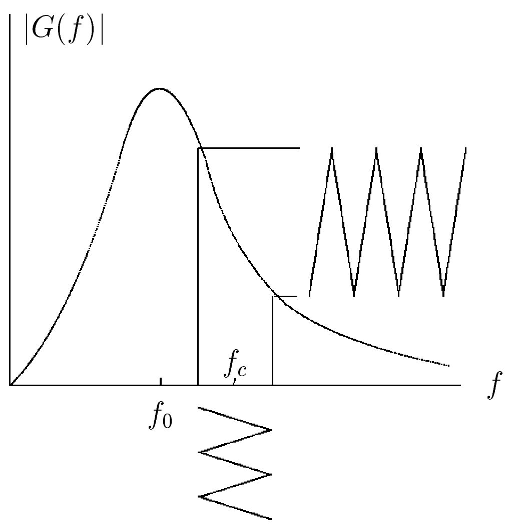
Commercial FM signals use a peak frequency deviation of $\Delta f = 75$ kHz and a maximum baseband message frequency of $f_m = 15$ kHz. Carson's rule estimates the FM signal bandwidth as $B_T = 2(75 + 15) = 180$ kHz which is six times the 30 kHz bandwidth that would be required for AM modulation.

FM Demodulation by a Frequency Discriminator

A frequency discriminator is a device that converts a received FM signal into a voltage that is proportional to the instantaneous frequency of its input without using a local oscillator and, consequently, in a noncoherent manner.

An Elementary Discriminator





Elementary FM Discriminator (cont.)

- When the instantaneous frequency changes slowly relative to the time-constants of the filter, a *quasi-static* analysis can be used.
- In quasi-static operation the filter output has the same instantaneous frequency as the input but with an envelope that varies according to the amplitude response of the filter at the instantaneous frequency.
- The amplitude variations are then detected with an envelope detector like the ones used for AM demodulation.

An FM Discriminator Using the Pre-Envelope

When $\theta_m(t)$ is small and band-limited so that $\cos\theta_m(t)$ and $\sin\theta_m(t)$ are essentially band-limited signals with cutoff frequencies less than ω_c, the pre-envelope of the FM signal is

$$s_+(t) = s(t) + j\hat{s}(t) = A_c e^{j(\omega_c t + \theta_m(t))} \tag{17}$$

The angle of the pre-envelope is

$$\varphi(t) = \arctan[\hat{s}(t)/s(t)] = \omega_c t + \theta_m(t) \tag{18}$$

The derivative of the phase is

$$\frac{d}{dt}\varphi(t) = \frac{s(t)\dfrac{d}{dt}\hat{s}(t) - \hat{s}(t)\dfrac{d}{dt}s(t)}{s^2(t) + \hat{s}^2(t)} = \omega_c + k_\omega m(t) \tag{19}$$

which is exactly the instantaneous frequency. This can be approximated in discrete-time by using FIR filters to form the derivatives and Hilbert transform. Notice that the denominator is the squared envelope of the FM signal.

Discriminator Using the Pre-Envelope (cont.)

This formula can also be derived by observing

$$\begin{aligned}
\frac{d}{dt}s(t) &= \frac{d}{dt}A_c\cos[\omega_c t + \theta_m(t)] \\
&= -A_c[\omega_c + k_\omega m(t)]\sin[\omega_c t + \theta_m(t)]
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\hat{s}(t) &= \frac{d}{dt}A_c\sin[\omega_c t + \theta_m(t)] \\
&= A_c[\omega_c + k_\omega m(t)]\cos[\omega_c t + \theta_m(t)]
\end{aligned}$$

so

$$\begin{aligned}
s(t)\frac{d}{dt}\hat{s}(t) - \hat{s}(t)\frac{d}{dt}s(t) &= A_c^2[\omega_c + k_\omega m(t)] \\
&\quad \times \{\cos^2[\omega_c t + \theta_m(t)] + \sin^2[\omega_c t + \theta_m(t)]\} \\
&= A_c^2[\omega_c + k_\omega m(t)] \qquad (20)
\end{aligned}$$

The bandwidth of an FM discriminator must be at least as great as that of the received FM signal which is usually much greater than that of the baseband message. This limits the degree of noise reduction that can be achieved by preceding the discriminator by a bandpass receive filter.

A Discriminator Using the Complex Envelope

The complex envelope is

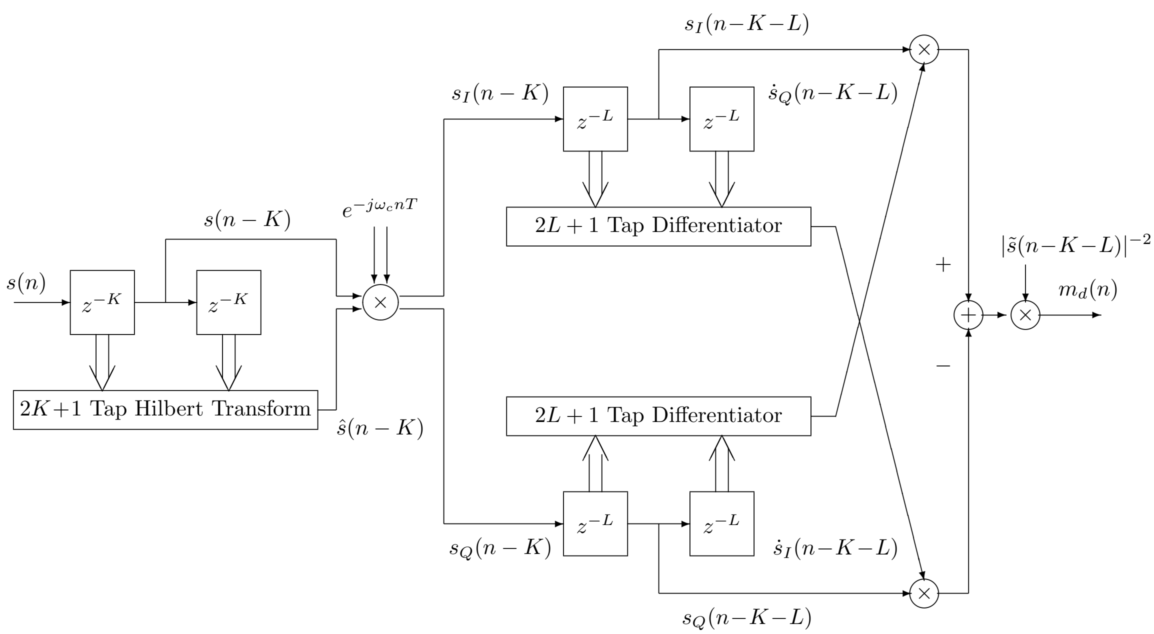
$$\tilde{s}(t) = s_+(t)e^{-j\omega_c t} = s_I(t) + j\, s_Q(t) = A_c e^{j\theta_m(t)} \qquad (21)$$

The angle of the complex envelope is

$$\tilde{\varphi}(t) = \arctan[s_Q(t)/s_I(t)] = \theta_m(t) \qquad (22)$$

The derivative of the phase is

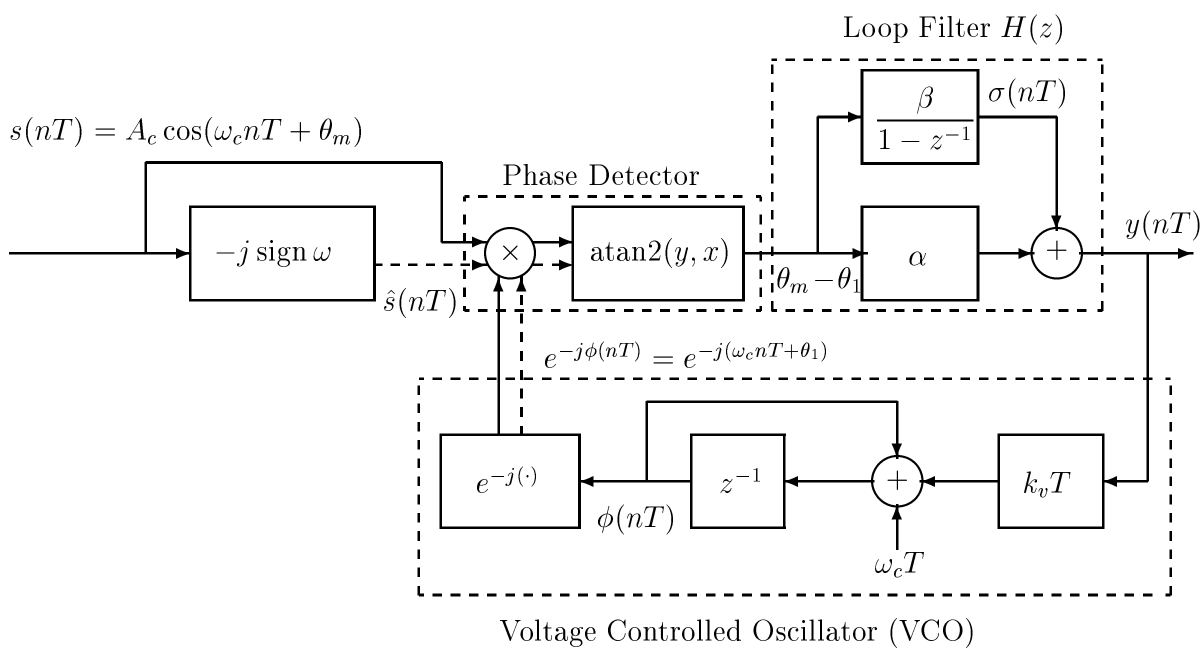
$$\frac{d}{dt}\tilde{\varphi}(t) = \frac{s_I(t)\frac{d}{dt}s_Q(t) - s_Q(t)\frac{d}{dt}s_I(t)}{s_I^2(t) + s_Q^2(t)} = k_\omega m(t) \qquad (23)$$



Discrete-Time Discriminator Realization

Using a Phase-Locked Loop for FM Demodulation

A device called a *phase-locked loop* (PLL) can be used to demodulate an FM signal with better performance in a noisy environment than a frequency discriminator. The block diagram of a discrete-time version of a PLL is shown in the figure below.



PLL Analysis

The PLL input shown in the figure is the noisless FM signal

$$s(nT) = A_c \cos[\omega_c nT + \theta_m(nT)] \tag{24}$$

This input is passed through a Hilbert transform filter to form the pre-envelope

$$s_+(nT) = s(nT) + j\hat{s}(nT) = A_c e^{j[\omega_c nT + \theta_m(nT)]} \tag{25}$$

The pre-envelope is multiplied by the output of the voltage controlled oscillator (VCO) block. The input to the z^{-1} block is the phase of the VCO one sample into the future which is

$$\phi((n+1)T) = \phi(nT) + \omega_c T + k_v T y(nT) \tag{26}$$

Starting at $n = 0$ and iterating the equation, it follows that

$$\phi(nT) = \omega_c nT + \theta_1(nT) \tag{27}$$

PLL Analysis (cont. 1)

where

$$\theta_1(nT) = \theta(0) + k_v T \sum_{k=0}^{n-1} y(kT) \qquad (28)$$

The VCO output is

$$v(nT) = e^{-j\phi(nT)} = e^{-j[\omega_c nT + \theta_1(nT)]} \qquad (29)$$

The multiplier output is

$$p(nT) = A_c e^{j[\theta_m(nT) - \theta_1(nT)]} \qquad (30)$$

The phase error can be computed as

$$\theta_m(nT) - \theta_1(nT) = \arctan\left[\frac{\Im m\{p(nT)\}}{\Re e\{p(nT)\}}\right] \qquad (31)$$

This is shown in the figure as being computed by the C library function atan2(y,x) which is a four quadrant arctangent giving angles between $-\pi$ and π. The block consisting of the multiplier and arctan function is called a *phase detector*.

PLL Analysis (cont. 2)

A less accurate, but computationally simpler, estimate of the phase error when the error is small is

$$\begin{aligned}\Im m\{p(nT)\} &= \hat{s}(nT)\cos[\omega_c nT + \theta_1(nT)] \\ &\quad -s(nT)\sin[\omega_c nT + \theta_1(nT)] \quad (32) \\ &= A_c \sin[\theta_m(nT) - \theta_1(nT)] \\ &\simeq A_c[\theta_m(nT) - \theta_1(nT)] \quad (33)\end{aligned}$$

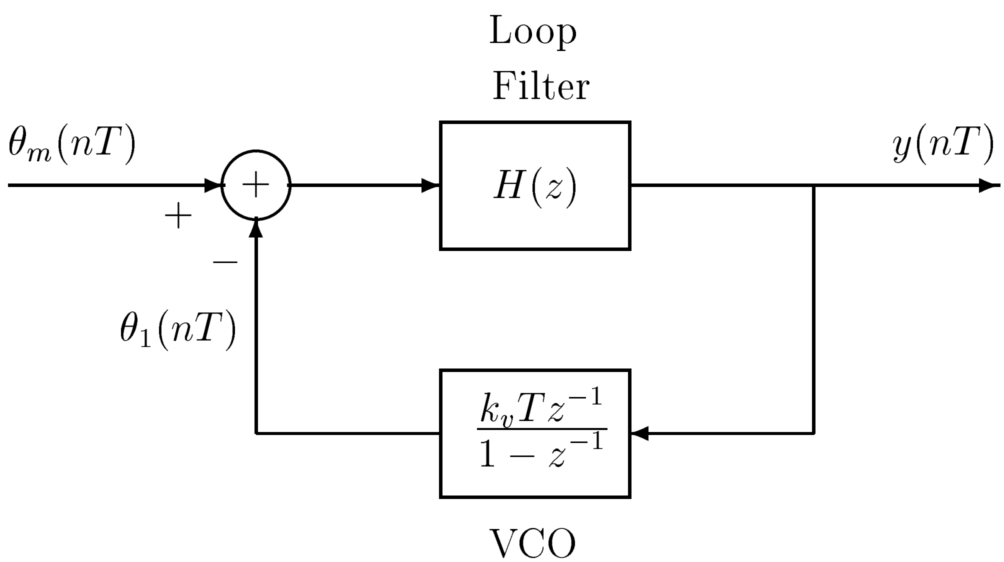
The phase detector output is applied to the loop filter which has the transfer function

$$H(z) = \alpha + \frac{\beta}{1 - z^{-1}} = (\alpha + \beta)\frac{1 - \frac{\alpha}{\alpha+\beta} z^{-1}}{1 - z^{-1}} \quad (34)$$

The accumulator portion of the loop filter which has the output $\sigma(nT)$ enables the loop to track carrier frequency offsets with zero error. It will be shown shortly that the output $y(nT)$ of the loop filter is an estimate of the transmitted message $m(nT)$.

Linearized Model for PLL

The PLL is a nonlinear system because of the characteristics of the phase detector. If the discontinuities in the arctangent are ignored, the PLL can be represented by the linearized model shown in the following figure.



The transfer function for the linearized PLL is

$$\begin{aligned} L(z) = \frac{Y(z)}{\Theta_m(z)} &= \frac{H(z)}{1+H(z)\frac{k_v T z^{-1}}{1-z^{-1}}} \\ &= \frac{(1-z^{-1})(\alpha+\beta-\alpha z^{-1})}{1-[2-(\alpha+\beta)k_v T]z^{-1}+(1-\alpha k_v T)z^{-2}} \end{aligned} \quad (35)$$

Proof that the PLL is an FM Demodulator

At low frequencies, which corresponds to $z \simeq 1$, $L(z)$ can be approximated by

$$L(z) \simeq \frac{z-1}{k_v T} \tag{36}$$

Thus

$$Y(z) \simeq \Theta_m(z)\frac{z-1}{k_v T} \tag{37}$$

and in the time-domain

$$y(nT) \simeq \frac{\theta_m((n+1)T) - \theta_m(nT)}{k_v T} \tag{38}$$

Using the formula on slide 8-3 for θ_m gives

$$y(nT) \simeq \frac{k_\omega}{k_v} m(nT) \tag{39}$$

This last equation demonstrates that the PLL is an FM demodulator under the appropriate conditions.

Comments on PLL Performance

- The frequency response of the linearized loop has the characteristics of a band-limited differentiator.
- The loop parameters must be chosen to provide a loop bandwidth that passes the desired baseband message signal but is as small as possible to suppress out-of-band noise.
- The PLL performs better than a frequency discriminator when the FM signal is corrupted by additive noise. The reason is that the bandwidth of the frequency discriminator must be large enough to pass the modulated FM signal while the PLL bandwidth only has to be large enough to pass the baseband message. With wideband FM, the bandwidth of the modulated signal can be significantly larger than that of the baseband message.

Bandwidth of FM PLL vs. Costas Loop

The PLL described in this experiment is very similar to the Costas loop presented in Chapter 6 for coherent demodulation of DSBSC-AM. However, the bandwidth of the PLL used for FM demodulation must be large enough to pass the baseband message signal, while the Costas loop is used to generate a stable carrier reference signal so its bandwidth should be very small and just wide enough to track carrier drifts and allow a reasonable acquisition time.

Laboratory Experiments for Frequency Modulation

Initialize the DSK as before and use a 16 kHz sampling rate for these experiments.

Chapter 8, Experiment 1
Making an FM Modulator

Make an FM modulator using equations (8) and (9) on slide 8-3.

1. Use the carrier frequency $f_c = 1000$ Hz.
2. Set the signal generator to output a baseband message, $m(t)$, which is a sine wave with amplitude 1 volt and frequency 100 Hz. Connect this signal to the left channel of the codec.
3. In your DSK program, read message samples from the left channel of the codec and convert them to floating-point values.
4. Try $k_\omega = 0.2$ in your program.

Experiment 8.1
FM Modulator (cont. 1.)

5. Remember to limit the carrier angle to the range $[0, 2\pi)$.

6. Send the FM modulated message samples to the left codec output channel and observe the time signal on the oscilloscope. Remember to scale the samples to use a large portion of the dynamic range of the DAC. The signal should resemble the figure on Slide 8-4.

7. Also, use the FFT capability of the oscilloscope to see the signal spectrum.

8. Vary k_ω and observe the resulting time signals and spectra. You can vary k_ω in your program or you can change the message amplitude on the signal generator.

Chapter 8, Experiment 2

Spectrum of an FM Signal

1. Set the signal generator to FM modulate an $f_c = 4$ kHz sinusoidal carrier with an $f_m = 100$ Hz sine wave by doing the following steps:
   (a) Make sure the signal type is set to a sine wave.
   (b) Press the blue “SHIFT” button and then the “AM/FM” button.
   (c) Set the carrier frequency by pressing the “FREQ” button and setting the frequency to 4 kHz.
   (d) Set the modulation frequency by pressing the “RATE” button and setting it to 100 Hz.

Experiment 8.2
FM Spectrum (cont. 1.)

(e) Adjust the modulation index by pressing the "SPAN" button and setting a value. The displayed value is related to, but not, the modulation index β.

2. Connect the FM output signal to the oscilloscope and observe the resulting waveforms as you vary the frequency deviation.

3. Use the FFT function of the oscilloscope to observe the spectrum of the FM signal by performing the following steps:

   (a) Turn off the input channels to disable the display of the time signals.

   (b) Press "Math."

   (c) Under the oscilloscope display screen,

      i. Set "Operator" to FFT.

      ii. Set "Source 1" to your input channel.

      iii. Set "Span" to 2.00 kHz.

Experiment 8.2
FM Spectrum (cont. 2.)

iv. Set "Center" to 4.00 kHz.

v. Use the "Horizontal" knob at the top left of the control knob section to set the "FFT Resolution" to "763 mHz" (0.763 Hz) or "381 mHz" (0.381 Hz).

Note: You can turn off the FFT by pressing "Math" again.

4. Watch the amplitude of the 4 kHz carrier component on the scope as the modulation index is increased from 0. Remember that this component should be proportional to $J_0(\beta)$.

5. Increase the modulation index slowly from 0 until the carrier component becomes zero for the first and second times and record the displayed SPAN values. Compare these displayed values with the theoretical values of β for the first two zeros of $J_0(\beta)$.

Experiment 8.2 FM Spectrum (cont. 3)

You can generate values of the Bessel function by using the series expansion given on Slide 8-5 or with MATLAB.

6. Plot the theoretical power spectra for a sinusoidally modulated FM signal with $\beta = 2$, 5, and 10. Compare them with the spectra observed on the oscilloscope.

Chapter 8, Experiment 3

FM Demodulation Using a Frequency Discriminator

- Write a C program that implements the frequency discriminator described on Slide 8-12. Assume that:
  - the carrier frequency is 4 kHz,
  - the baseband message is band limited with a cutoff frequency of 500 Hz,
  - the sampling rate is 16 kHz.

Experiment 8.3 Discriminator Implementation (cont. 1)

Use REMEZ87.EXE, WINDOW.EXE, or MATLAB to design the Hilbert transform and FIR differentiation filters. Use enough taps to approximate the desired Hilbert transform frequency response well from 1200 to 6800 Hz. Try a differentiator bandwidth extending from 0 to 8000 Hz. WINDOW.EXE gives good differentiator designs. (**Be sure to match the delays of your filters in your implementation.**)

- Synchronize the sample processing loop with the transmit ready flag (XRDY) of McBSP1. Read samples from the ADC, apply them to your discriminator, and write the output samples to the DAC.

Experiment 8.3 Discriminator Implementation (cont. 2)

- Use the signal generator to create a sinusoidally modulated FM signal as you did for the FM spectrum measurement experiments. Attach the signal generator to the DSK line input and observe your demodulated signal on the oscilloscope to check that the program is working.
- Modify your program to add Gaussian noise to the input samples and observe the discriminator output as you increase the noise variance. Listen to the noisy output with the PC speakers. Does the performance degrade gracefully as the noise gets larger?

Chapter 8, Experiment 4
Using a Phase-Locked Loop for FM Demodulation

Implement a PLL like the one shown on Slide 8-13 to demodulate a sinusoidally modulated FM signal with the same parameters used previously in this experiment. Let $\alpha = 1$ and choose β to be a factor of 100 or more smaller than α.

- Compute and plot the amplitude response of the linearized loop using the equation (35) on slide 8-17 for different loop parameters until you find a set that gives a reasonable response.
- Theoretically compute and plot the time response of the linearized loop to a unit step input for your selected set of parameters by iterating a difference equation corresponding to the transfer function.

Experiment 8.4 PLL Demodulator (cont.)

- Write a C program to implement the PLL. Test this demodulator by connecting an FM signal from the signal generator to the DSK line input and observing the DAC output on the oscilloscope.
- See if your PLL will track carrier frequency offsets by changing the carrier frequency on the signal generator slowly and observing the output. See how large an offset your loop will track. Observe any differences in behavior when you change the carrier frequency smoothly and slowly or make step changes.
- Modify your program to add Gaussian noise to the input samples and observe the demodulated output as the noise variance increases. How does the quality of the demodulated output signal compare with that of the frequency discriminator at the same SNR.