

# Chapter 7

## Single-Sideband Modulation (SSB) and Frequency Translation

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# Chapter 7

## Single-Sideband Modulation (SSB) and Frequency Translation

Remember that an AM signal

$$s(t) = A_c[1 + k_a m(t)] \cos \omega_c t$$

has the Fourier transform

$$S(\omega) = A_c \pi \delta(\omega + \omega_c) + A_c \pi \delta(\omega - \omega_c) \\ + \frac{A_c}{2} k_a M(\omega + \omega_c) + \frac{A_c}{2} k_a M(\omega - \omega_c)$$

- The spectral components in the AM signal equal distances above and below the carrier frequency contain identical information because they are complex conjugates of each other.
- The portion above the carrier frequency is called the *upper sideband* and the portion below the *lower sideband*.

## SSB Modulation by DSBSC-AM and Filtering

- In *single-sideband* (SSB) modulation only the upper sideband or the lower sideband is transmitted. Thus, SSB modulation requires half the bandwidth of AM or DSBSC-AM modulation.
- We will assume that the baseband message signal  $m(t)$  is band limited with a cutoff frequency  $W$  which is less than the carrier frequency  $\omega_c$ . Then the required channel bandwidth for an SSB signal is  $W$ .

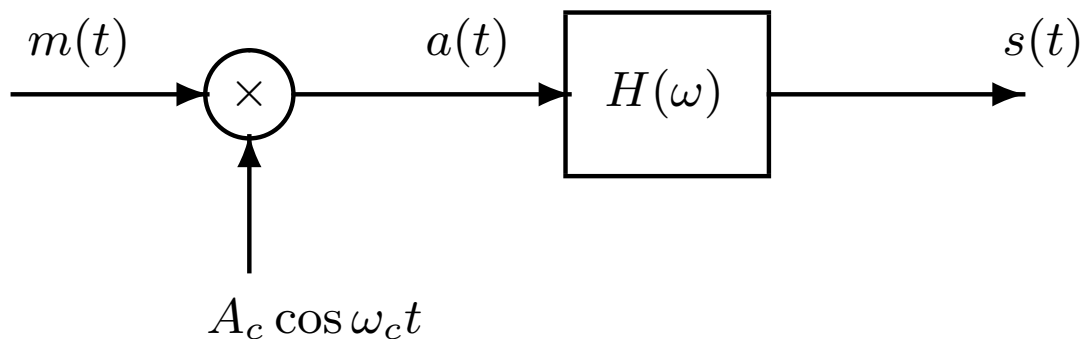


Figure 1: SSB Modulator Using DSBSC-AM and Filtering

## SSB Modulation by DSBSC-AM and Filtering (cont.)

First the DSBSC-AM signal

$$a(t) = A_c m(t) \cos \omega_c t$$

is formed which has the transform

$$A(\omega) = 0.5A_c M(\omega - \omega_c) + 0.5A_c M(\omega + \omega_c)$$

and is centered at the carrier frequency  $\omega_c$ .

Then  $H(\omega)$  selects the desired sideband. Upper sideband modulation uses the highpass filter

$$H_u(\omega) = \begin{cases} 1 & \text{for } |\omega| > \omega_c \\ 0 & \text{elsewhere} \end{cases}$$

and the lower sideband SSB modulation uses the lowpass filter

$$H_l(\omega) = \begin{cases} 1 & \text{for } |\omega| < \omega_c \\ 0 & \text{elsewhere} \end{cases}$$

## Representing SSB Signals in Terms of Hilbert Transforms

Let the baseband message be  $m(t)$  and its Hilbert transform  $\hat{m}(t)$ . The pre-envelope of the SSB signal has the transform

$$\begin{aligned} S_+(\omega) &= 2S(\omega)u(\omega) \\ &= 2A(\omega)H(\omega)u(\omega) \\ &= A_cM(\omega - \omega_c)H(\omega)u(\omega) \end{aligned}$$

and the transform of its complex envelope is

$$\tilde{S}(\omega) = S_+(\omega + \omega_c) = A_cM(\omega)H(\omega + \omega_c)u(\omega + \omega_c)$$

### Upper Sideband Case

Substituting  $H_u(\omega)$  for  $H(\omega)$  gives

$$\begin{aligned} \tilde{S}(\omega) &= A_cM(\omega)u(\omega) = 0.5A_cM(\omega)(1 + \text{sign } \omega) \\ &= 0.5A_cM(\omega)[1 + j(-j\text{sign } \omega)] \\ &= 0.5A_cM(\omega) + j0.5A_c\hat{M}(\omega) \end{aligned}$$

## Hilbert Transform Representation (cont. 1)

The complex envelope is

$$\tilde{s}(t) = 0.5A_c[m(t) + j\hat{m}(t)]$$

Therefore, the SSB signal can be expressed as

$$\begin{aligned} s(t) &= \Re\{\tilde{s}(t)e^{j\omega_c t}\} \\ &= 0.5A_c\Re\{[m(t) + j\hat{m}(t)]e^{j\omega_c t}\} \\ &= 0.5A_c m(t) \cos \omega_c t - 0.5A_c \hat{m}(t) \sin \omega_c t \end{aligned}$$

### Lower Sideband Case

The transform of the complex envelope is

$$\begin{aligned} \tilde{S}(\omega) &= A_c M(\omega) u(-\omega) \\ &= 0.5A_c M(\omega)(1 - \text{sign } \omega) \\ &= 0.5A_c M(\omega)[1 - j(-j\text{sign } \omega)] \\ &= 0.5A_c M(\omega) - j0.5A_c \hat{M}(\omega) \end{aligned}$$

Therefore, the complex envelope is

$$\tilde{s}(t) = 0.5A_c[m(t) - j\hat{m}(t)]$$

## Hilbert Transform Representation (cont. 2)

The corresponding SSB signal is

$$\begin{aligned} s(t) &= \Re\{\tilde{s}(t)e^{j\omega_c t}\} \\ &= 0.5A_c m(t) \cos \omega_c t + 0.5A_c \hat{m}(t) \sin \omega_c t \end{aligned}$$

## Single-Sideband Modulator Using a Hilbert Transform

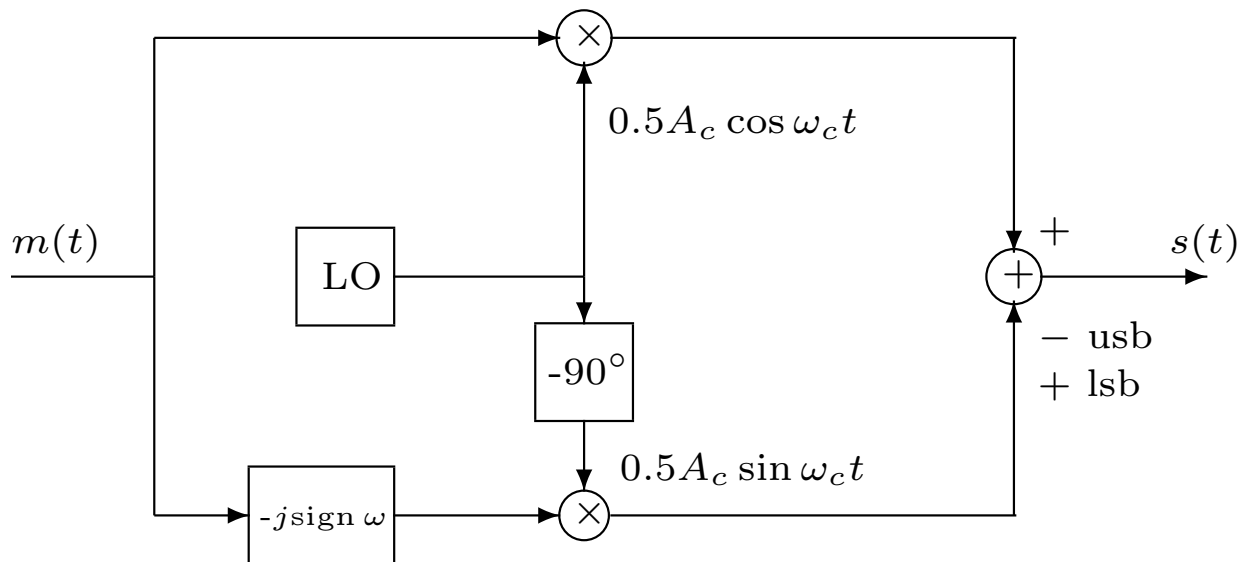


Figure 2: A Single-Sideband Modulator Block Diagram



## Another Approach to the SSB Signal Representation

Let the baseband message have transform  $M(\omega)$ . An example is shown in Figure 3. Its pre-envelope is

$$m_+(t) = m(t) + j\hat{m}(t)$$

which has the transform

$$M_+(\omega) = 2M(\omega)u(\omega)$$

The upper-sideband SSB signal pre-envelope is

$$s_+(t) = m_+(t)e^{j\omega_c t}$$

which has the transform

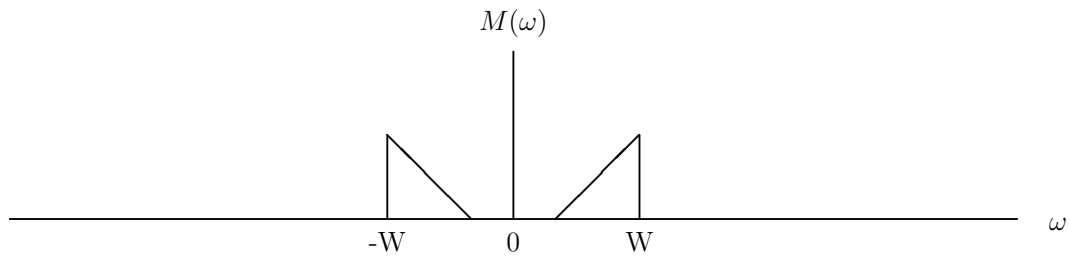
$$S_+(\omega) = M_+(\omega - \omega_c)$$

The transmitted SSB signal is

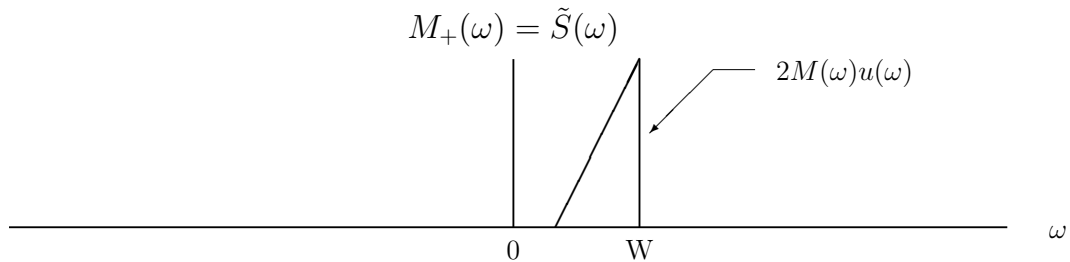
$$s(t) = \Re\{s_+(t)\} = \frac{s_+(t) + \overline{s_+(t)}}{2}$$

which has the transform

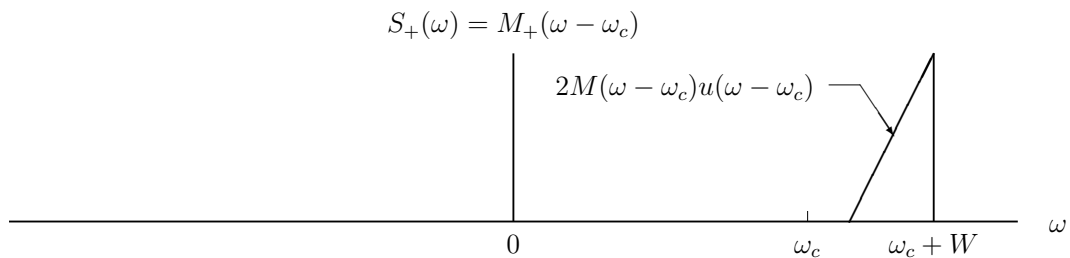
$$S(\omega) = 0.5S_+(\omega) + 0.5\overline{S_+(-\omega)}$$



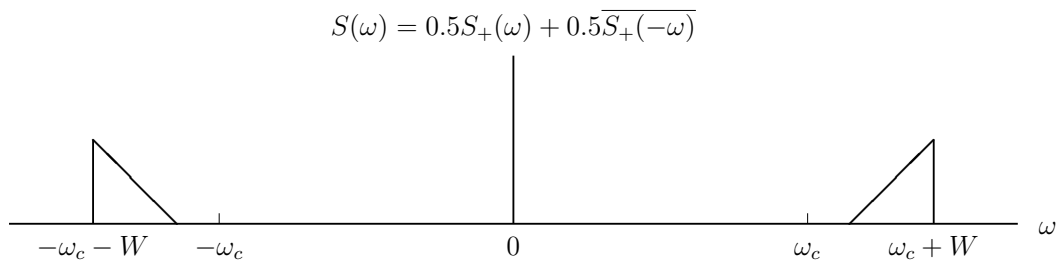
(a) Fourier Transform of the Baseband Message



(b) Fourier Transform of the Pre-envelope of the Baseband Message



(c) Fourier Transform of the Pre-envelope of the Upper-Sideband Transmitted Signal



(d) Fourier Transform of the Upper-Sideband Transmitted Signal

Figure 3: Signal Fourier Transforms in Steps for Generating an Upper-Sideband SSB Signal

## Coherent Demodulation of SSB Signals

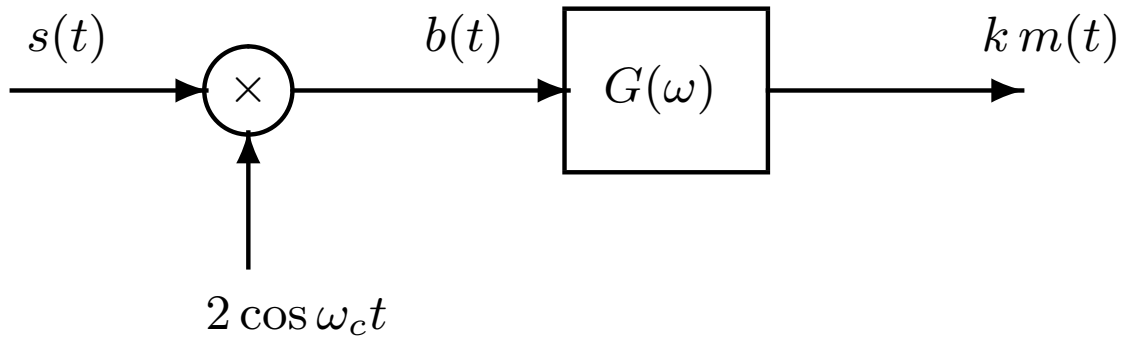


Figure 4: An SSB Demodulator

First the received signal is multiplied by a locally generated replica of the carrier signal.

Multiplying the formulas for upper and lower sideband SSB signals by  $2 \cos \omega_c t$  yields

$$\begin{aligned} b(t) &= A_c m(t) \cos^2 \omega_c t \mp A_c \hat{m}(t) \sin \omega_c t \cos \omega_c t \\ &= 0.5 A_c m(t) + 0.5 A_c m(t) \cos 2\omega_c t \\ &\quad \mp 0.5 A_c \hat{m}(t) \sin 2\omega_c t \end{aligned}$$

## Coherent Demodulation (cont.)

Observe that

- $0.5A_c m(t)$  is the desired component.
- $0.5A_c m(t) \cos 2\omega_c t$  and  $0.5A_c \hat{m}(t) \sin 2\omega_c t$  have spectra centered about  $2\omega_c$ .

The components around  $2\omega_c$  are removed by the lowpass filter  $G(\omega)$  with cutoff frequency  $W$ .

In practice, the demodulator shown above should be preceded by a *receive* bandpass filter that passes  $s(t)$  and eliminates out-of-band noise.

### Frequency Domain Analysis of Operation

Remember that  $b(t) = s(t)2 \cos \omega_c t$ . So

$$B(\omega) = S(\omega + \omega_c) + S(\omega - \omega_c)$$

This translates the sidebands around  $\pm\omega_c$  down to baseband and forms  $M(\omega)$  which is the desired term and also translates them up to  $\pm 2\omega_c$  which are the terms removed by the lowpass filter.

## SSB Demodulator Using a Hilbert Transform

First take the Hilbert transform of  $s(t)$  and form the pre-envelope

$$\begin{aligned} s_+(t) &= s(t) + j\hat{s}(t) = \tilde{s}(t)e^{j\omega_c t} \\ &= 0.5A_c[m(t) \pm j\hat{m}(t)]e^{j\omega_c t} \end{aligned}$$

where the plus sign is for upper sideband and the minus sign is for lower sideband modulation.

Multiplying the pre-envelope by  $e^{-j\omega_c t}$  generates the complex envelope

$$\tilde{s}(t) = s_+(t)e^{-j\omega_c t} = 0.5A_c[m(t) \pm j\hat{m}(t)]$$

Taking the real part of the complex envelope gives

$$\begin{aligned} 0.5A_c m(t) &= \Re\{s_+(t)e^{-j\omega_c t}\} \\ &= \Re[s(t) + j\hat{s}(t)][\cos \omega_c t - j \sin \omega_c t] \\ &= s(t) \cos \omega_c t + \hat{s}(t) \sin \omega_c t \end{aligned}$$

which is proportional to the desired signal.

## **SSB Demodulator Using a Hilbert Transform (cont.)**

This demodulator requires taking a Hilbert transform but does not require filtering out terms at twice the carrier frequency.

The modulator shown on Slide 7-6 is also a block diagram for a demodulator that implements the formula at the bottom of the previous slide if the input  $m(t)$  is replaced by the received signal  $s(t)$ , the cosine and sine amplitudes are set to 1, and the plus sign is chosen at the output adder.

In practice, the demodulator would be preceded by an bandpass filter that passes the signal components and rejects out-of-band noise.

### **Need for a Pilot Tone**

These two demodulators assume that the receiver has perfect knowledge of the received carrier frequency and phase. Unfortunately, this

## SSB Demodulation (cont.)

### Using a Pilot Tone

information cannot be derived by a system like the Costas loop because the SSB signal is the sum of an *inphase* component  $m(t) \cos \omega_c t$  and a *quadrature* component  $\hat{m}(t) \sin \omega_c t$ .

A standard approach to solving this problem is to add a small sinusoidal component called a *pilot tone* whose frequency is not in the SSB signal band and has a known relationship to the carrier frequency. The pilot tone frequency is often chosen to be the carrier frequency when the baseband message signal has no DC components. The receiver can then generate a local carrier reference by using a narrow bandwidth bandpass filter to select the pilot tone and possibly following this filter by a phase-locked loop.

## Frequency Translation

Reasons for needing frequency translation:

- To place the signal spectrum in an allocated channel.
- Several messages can be multiplexed together by shifting them to non-overlapping adjacent spectral bands and transmitting the sum of the resulting signals. This is called *frequency division multiplexing* (FDM).
- To correct for carrier frequency offsets caused by oscillator inaccuracies or Doppler shifts.

Let  $s(t)$  be a bandpass signal with the frequency  $\omega_0$  somewhere in its passband. The problem is to translate the spectrum so that  $\omega_0$  is moved to  $\omega_1 = \omega_0 + \Delta\omega$ .

The first step is to form the pre-envelope

$$s_+(t) = s(t) + j\hat{s}(t)$$



## Frequency Translation (cont. 1)

The corresponding Fourier transform is

$$S_+(\omega) = 2S(\omega)u(\omega)$$

The next step is to multiply by a complex exponential with frequency  $\Delta\omega$  to get

$$\begin{aligned} r_+(t) &= s_+(t)e^{j\Delta\omega t} \\ &= [s(t) + j\hat{s}(t)][\cos \Delta\omega t + j \sin \Delta\omega t] \end{aligned}$$

which has the transform

$$R_+(\omega) = S_+(\omega - \Delta\omega)$$

This translates the original spectrum to the right by  $\Delta\omega$  and moves the value at  $\omega_0$  to the frequency  $\omega_1 = \omega_0 + \Delta\omega$ .

Taking the real part of  $r_+(t)$  gives the following formula for the translated signal:

$$r(t) = s(t) \cos \Delta\omega t - \hat{s}(t) \sin \Delta\omega t$$

## Frequency Translation (cont. 2)

The real part of  $r_+(t)$  can also be expressed as

$$r(t) = [r_+(t) + \bar{r}_+(t)]/2$$

so its Fourier transform is

$$\begin{aligned} R(\omega) &= [R_+(\omega) + \bar{R}_+(-\omega)]/2 \\ &= S(\omega - \Delta\omega)u(\omega - \Delta\omega) \\ &\quad + \bar{S}(-\omega - \Delta\omega)u(-\omega - \Delta\omega) \end{aligned}$$

The figure on Slide 7-6 is also the block diagram for a frequency translator if

1. the input  $m(t)$  is replaced by the bandpass signal  $s(t)$ ,
2. the frequency  $\omega_c$  is replaced by  $\Delta\omega$ ,
3.  $0.5A_c$  is replaced by 1,
4. the negative sign is used at the output adder.

## Frequency Translation (cont. 3)

Notice that the formula for computing  $r(t)$  from  $s(t)$  and  $\hat{s}(t)$  above can be used even when the passband of the translated signal overlaps that of the original signal.

To do this using real signals would require a double conversion process where the signal is

1. first shifted to a non-overlapping band by multiplying by  $\cos \omega_3 t$ ,
2. one sideband of this modulated signal is selected with a highpass filter,
3. and then another modulation is performed with the appropriate carrier frequency and the signal in the desired band is selected with a filter.

This is generally not as convenient for DSP applications.

## Laboratory Experiments

Initialize the DSP and codec as in Chapter 2. Use a sampling rate of 16 kHz.

### Making an SSB Modulator

- Write a program to implement the SSB modulator shown in Slide 7-6. Implement both the upper and lower sideband modulators. Take the message samples  $m(nT)$  from the A/D converter. Send the modulated signal samples to the D/A converter. Implement the Hilbert transform FIR filter in C or as an assembly language function that is called from C. Use the carrier frequency  $f_c = 4$  kHz and an amplitude  $A_c$  that scales the output samples appropriately for the codec.
- Attach the signal generator to the DSK input and set it to generate a 2 kHz sine wave.

## Making an SSB Modulator (cont. 1)

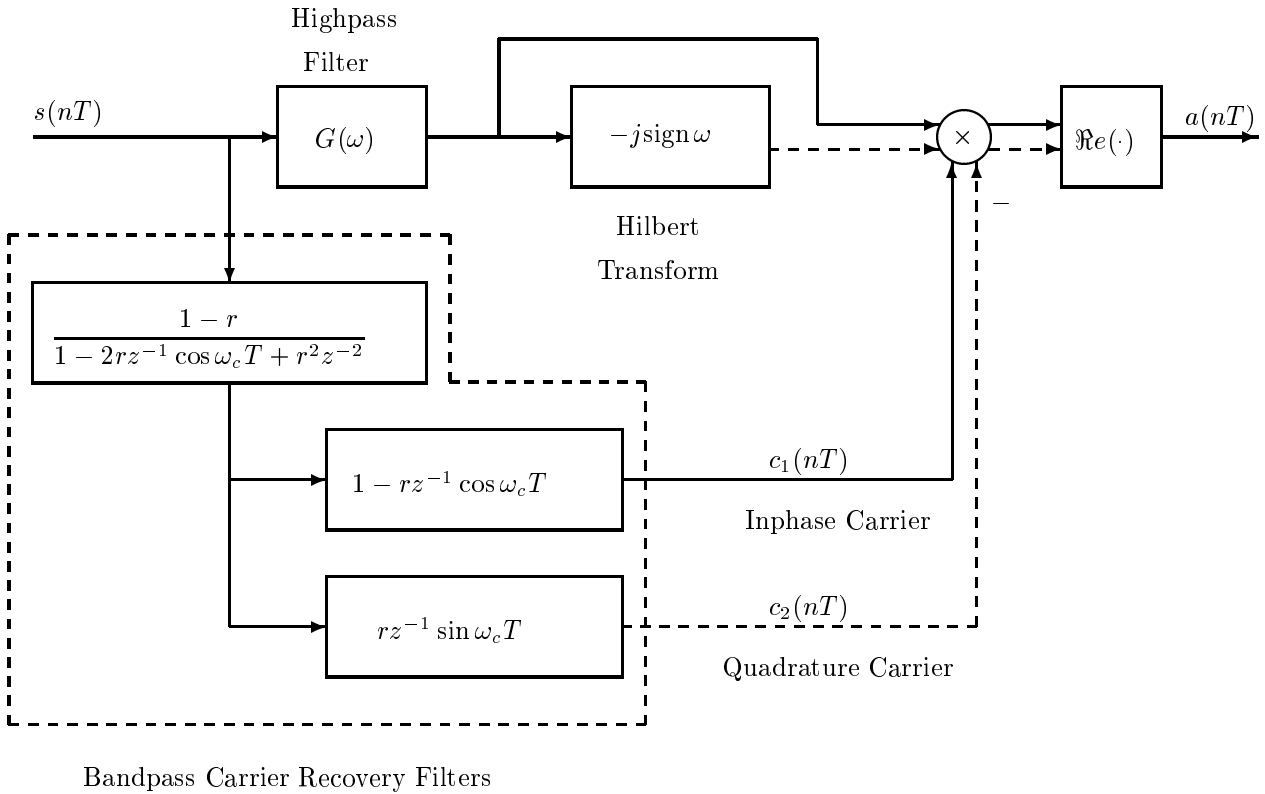
- Determine the theoretical formulas for the transmitted SSB signals for both the upper and lower sideband cases with the input  $m(t) = A_m \cos 2\pi 2000 t$ .
- Observe the signals generated by your modulator on the scope for both cases and compare them with the theoretical ones.
- Vary the frequency of  $m(t)$  from 0 to 4 kHz and observe  $s(t)$  on the scope. Report what happens to the frequency of  $s(t)$  for both the upper and lower sideband modulators.
- Next add a pilot tone  $p(t) = A_p \cos 2\pi f_c t$  to the SSB output signal to provide a carrier reference for the demodulator you will make next. Use your judgement in choosing the value for  $A_p$ .

## Coherent Demodulator for an SSB Signal

One possible demodulator is shown on Slide 7-21.

### Extracting the Message Sideband

- In the upper sideband case, the input signal  $s(nT)$  is passed through a highpass or bandpass filter  $G(\omega)$  that rejects the pilot tone but passes the upper sideband.
- In the lower sideband case,  $G(\omega)$  should be a lowpass or bandpass filter that passes the lower sideband and rejects the pilot tone.
- A notch filter could also be used to reject the pilot tone.
- In practice a bandpass receive filter would be used before the demodulator to reject out-of-band noise.



Using a Pilot Tone in SSB Demodulation

**Figure 5: Block Diagram of an Upper Sideband Demodulator Using a Pilot Tone**

## Coherent Demodulator (cont.)

### Extracting the Pilot Tone

The portion enclosed by dotted lines is a pair of bandpass filters that extract replicas of the pilot tone and its  $-90^\circ$  phase shift. The transfer functions of these two filters are

$$B_1(z) = \frac{(1-r)(1-rz^{-1}\cos\omega_c T)}{1-2rz^{-1}\cos\omega_c T+r^2z^{-2}}$$

and

$$B_2(z) = \frac{(1-r)rz^{-1}\sin\omega_c T}{1-2rz^{-1}\cos\omega_c T+r^2z^{-2}}$$

The denominators of these filters have the factorization

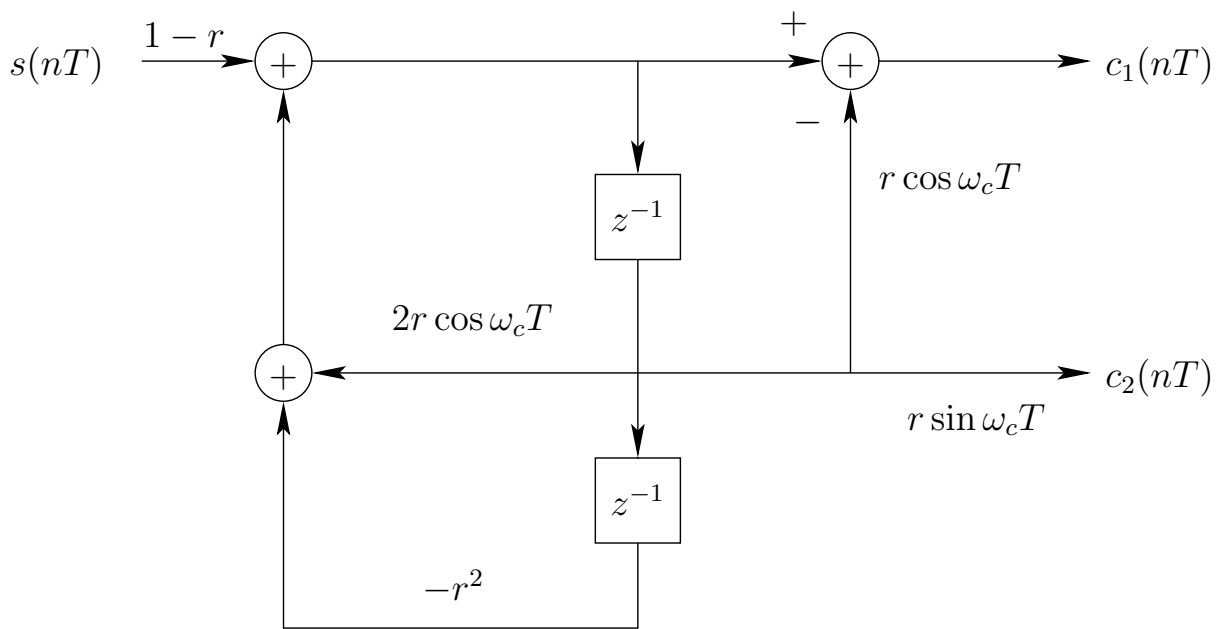
$$\begin{aligned} 1-2rz^{-1}\cos\omega_c T+r^2z^{-2} \\ = (1-re^{j\omega_c T}z^{-1})(1-re^{-j\omega_c T}z^{-1}) \end{aligned}$$

Thus the filter poles are at

$$z = re^{\pm j\omega_c T}$$



## Implementing the Pilot Tone Extraction Filters



The coefficients  $2r \cos \omega_c T$ ,  $-r^2$ ,  $r \cos \omega_c T$ , and  $r \sin \omega_c T$  should be pre-computed in your program before the demodulation loop.

## Coherent Demodulator (cont.)

The quantity  $r$  is a number slightly less than 1 and controls the bandwidth of the filters. The closer it is to 1, the narrower the bandwidth.

Let the signal input to the multiplier be the pre-envelope  $v(nT) = v_1(nT) + jv_2(nT)$ . Then, the demodulator output is

$$\begin{aligned} a(nT) &= \Re\{v(nT)[c_1(nT) - jc_2(nT)]\} \\ &= v_1(nT)c_1(nT) + v_2(nT)c_2(nT) \end{aligned}$$

## Theory Explaining the Pilot Tone Filters

Consider the single complex pole filter

$$H(z) = \frac{1 - r}{1 - re^{j\omega_c T} z^{-1}}$$

where  $0 < r < 1$ . The frequency response of this filter is

$$H^*(\omega) = H(e^{j\omega T}) = \frac{1 - r}{1 - re^{j(\omega_c - \omega)T}}$$

## Pilot Tone Filter Theory (cont.)

Its squared amplitude response is

$$\begin{aligned} |H^*(\omega)|^2 &= \frac{(1-r)^2}{[1 - re^{j(\omega_c - \omega)T}][1 - re^{-j(\omega_c - \omega)T}]} \\ &= \frac{(1-r)^2}{1 + r^2 - 2r \cos(\omega_c - \omega)T} \end{aligned}$$

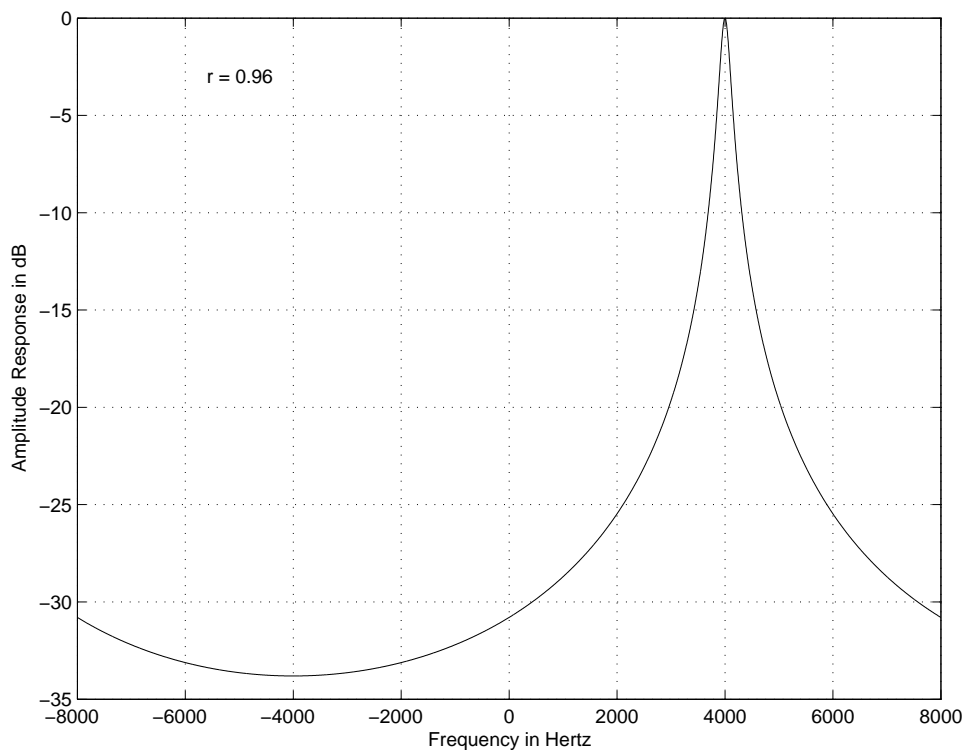


Figure 6: Amplitude Response of the Single Pole Complex Filter for  $r = 0.96$

## Pilot Tone Filter Theory (cont.)

The frequency response in polar form is

$$H^*(\omega) = A(\omega)e^{j\theta(\omega)}$$

where

$$A(\omega) = |H^*(\omega)| \quad \text{and} \quad \theta(\omega) = \arg H^*(\omega)$$

According to sinusoidal steady-state analysis, when the input to  $H(z)$  is  $x(n) = e^{j\omega nT}$ , the output is

$$y(n) = H^*(\omega)e^{j\omega nT} = A(\omega)e^{j[\omega nT + \theta(\omega)]}$$

The real and imaginary parts of  $y(n)$  are

$$\Re\{y(n)\} = A(\omega) \cos[\omega nT + \theta(\omega)]$$

and

$$\Im\{y(n)\} = A(\omega) \sin[\omega nT + \theta(\omega)]$$

Notice that the imaginary part lags the real part by 90 degrees.

## Pilot Tone Filter Theory (cont.)

Now let  $r$  be close to 1 and the filter input be

$$x(n) = \cos \omega_c nT = 0.5e^{j\omega_c nT} + 0.5e^{-j\omega_c nT}$$

Then the filter output is

$$y(n) = 0.5H^*(\omega_c)e^{j\omega_c nT} + 0.5H^*(-\omega_c)e^{-j\omega_c nT}$$

$H^*(\omega_c) = 1$ . When  $r$  is close to 1,  $H^*(-\omega_c)$  is small. Thus

$$y(n) \simeq 0.5e^{j\omega_c nT} = 0.5 \cos \omega_c nT + j0.5 \sin \omega_c nT$$

$H(z)$  can also be written as

$$\begin{aligned} H(z) &= \frac{(1-r)(1-re^{-j\omega_c T}z^{-1})}{(1-re^{j\omega_c T}z^{-1})(1-re^{-j\omega_c T}z^{-1})} \\ &= \frac{(1-r)(1-rz^{-1}\cos\omega_c T)}{1-2rz^{-1}\cos\omega_c T+r^2z^{-2}} \\ &\quad + j \frac{(1-r)(rz^{-1}\sin\omega_c T)}{1-2rz^{-1}\cos\omega_c T+r^2z^{-2}} \\ &= B_1(z) + j B_2(z) \end{aligned}$$

## Pilot Tone Filter Theory (cont.)

Notice that  $B_1(z)$  and  $B_2(z)$  both have real coefficients so that when the input is  $x(n) = \cos \omega_c nT$ , the real part of the output is  $x(n)$  passed through  $B_1(z)$  and the imaginary part is  $x(n)$  passed through  $B_2(z)$ . That is

$$y_1(n) = \Re\{y(n)\} \simeq 0.5 \cos \omega_c nT$$

and

$$y_2(n) = \Im\{y(n)\} \simeq 0.5 \sin \omega_c nT$$

The impulse responses of the pilot tone filters are

$$b_1(n) = (1 - r)r^n \cos(\omega_c nT)u(n)$$

and

$$b_2(n) = (1 - r)r^n \sin(\omega_c nT)u(n)$$

## Coherent Demodulator (cont.)

### Theoretical Exercise

1. Prove that at the carrier frequency  $\omega_c$  and when  $r$  is very close to 1, the transfer functions of the pilot tone extraction filters are approximately

$$B_1(e^{j\omega_c T}) \simeq 0.5$$

and

$$B_2(e^{j\omega_c T}) \simeq -0.5j$$

2. By trial and error, choose a value of  $r$  that gives roughly a 25 Hz 3 dB bandwidth.

### Experimental Demodulator Exercises

Perform the following tasks:

1. Add to your modulator program a section to implement the demodulator discussed above and shown on Slide 7-21. The filter,  $G(\omega)$ , can be an FIR or IIR filter. Pipe the samples

## Experimental Demodulator Exercises (cont.)

generated by your modulator code directly to your demodulator code internally within the DSP. Write the modulator output samples to the left channel of the codec and write the demodulator output samples to the right channel of the codec.

2. Check that the demodulator is working properly by observing the output on the oscilloscope as you vary the message frequency.
3. (Optional) When your demodulator is working, send the modulator samples to the left channel of the codec. Connect the left channel analog output to the left channel input. Demodulate the left channel input and



## Experimental Demodulator Exercises (cont.)

write the demodulated output samples to the right channel output. Observe the modulated signal (left channel output) and demodulated signal (right channel output) on the scope.

This will add some phase shift and, possibly, attenuation to the modulator output signal as a result of the properties of the analog path in the hardware.