

Proof of the Hilbert Transform Lowpass/Highpass Theorem

Let $m(t)$ be a lowpass signal with Fourier transform $M(\omega)$ which is zero for $|\omega| > W$. Let $c(t)$ be a highpass signal with Fourier transform $C(\omega)$ which is zero for $|\omega| < W$. Then the Hilbert transform of the signal $s(t) = m(t)c(t)$ is $\hat{s}(t) = m(t)\hat{c}(t)$ where $\hat{c}(t)$ is the Hilbert transform of $c(t)$.

Proof

According to the product theorem for Fourier transforms, the transform of $m(t)c(t)$ is the following convolution of transforms:

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M(\lambda)C(\omega - \lambda) d\lambda = \frac{1}{2\pi} \int_{-W}^W M(\lambda)C(\omega - \lambda) d\lambda \quad (1)$$

The Fourier transform of $\hat{s}(t)$ is

$$\hat{S}(\omega) = S(\omega)(-j \operatorname{sign} \omega) = \frac{1}{2\pi} \int_{-W}^W M(\lambda)C(\omega - \lambda) d\lambda (-j \operatorname{sign} \omega) \quad (2)$$

The inverse Fourier transform of $\hat{S}(\omega)$ is

$$\begin{aligned} \hat{s}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{S}(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-W}^W M(\lambda)C(\omega - \lambda) d\lambda (-j \operatorname{sign} \omega) e^{j\omega t} d\omega \end{aligned} \quad (3)$$

The factor $(-j \operatorname{sign} \omega) e^{j\omega t}$ can be taken inside the integral with respect to λ . Then the order of integration can be changed and (3) can be written as

$$\begin{aligned} \hat{s}(t) &= \frac{1}{2\pi} \int_{-W}^W M(\lambda) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega - \lambda) (-j \operatorname{sign} \omega) e^{j\omega t} d\omega \right] d\lambda \\ &= \frac{1}{2\pi} \int_{-W}^W M(\lambda) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega - \lambda) (-j \operatorname{sign} \omega) e^{j(\omega - \lambda)t} d\omega \right] e^{j\lambda t} d\lambda \end{aligned} \quad (4)$$

An example for $C(\omega)$, $C(\omega - \lambda)$ as a function of ω for a fixed $\lambda \in (-W, W)$, $\operatorname{sign}(\omega)$, and $\operatorname{sign}(\omega - \lambda)$ are shown in Figure 1.

From the figures it can be seen that the edge of $C(\omega - \lambda)$ at $W + \lambda$ moves from $-2W$ to 0 as λ varies from $-W$ to W and the edge at $-W + \lambda$ moves from 0 to $2W$ as λ varies from

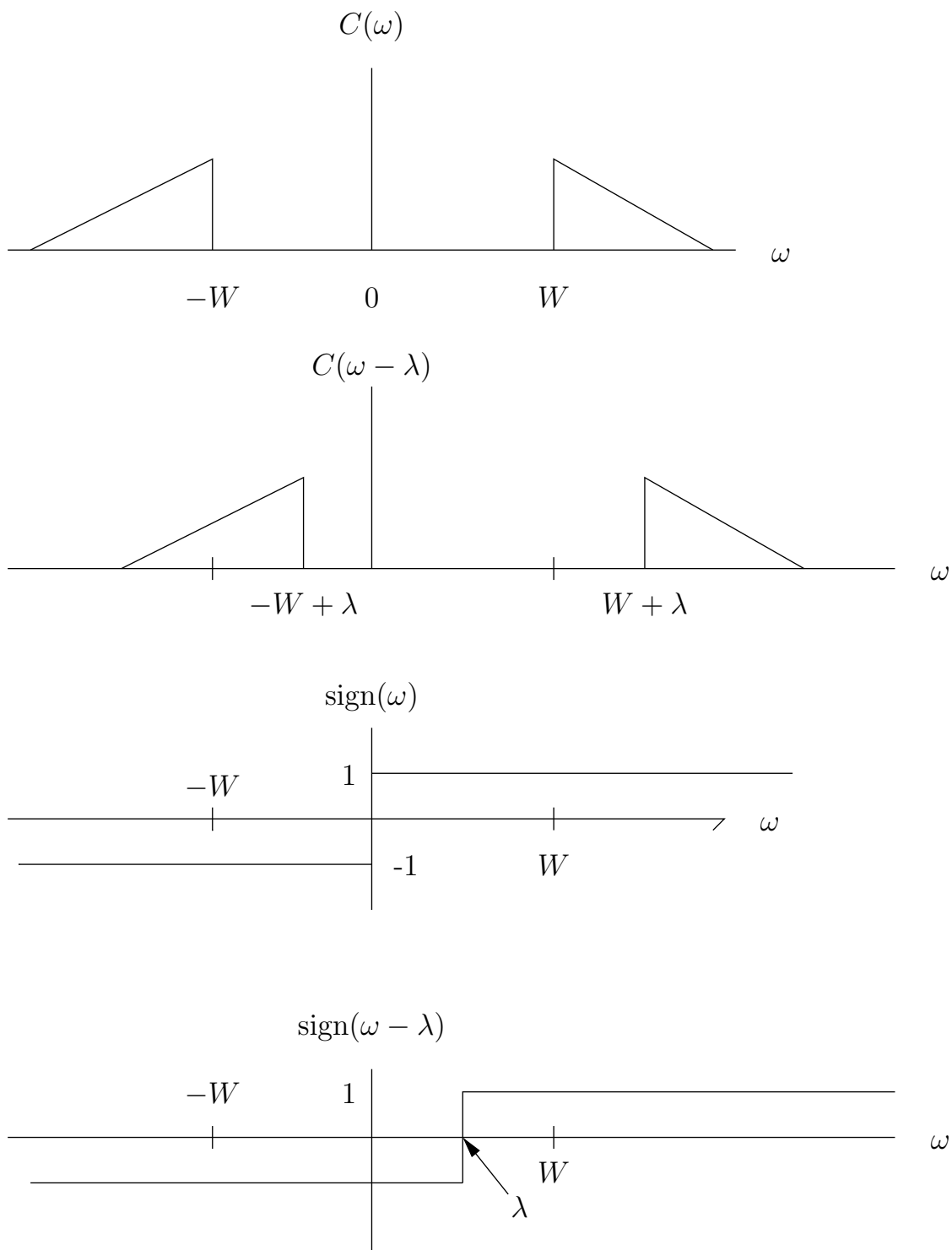


Figure 1: Example of a Highpass Signal Fourier Transform $C(\omega)$

$-W$ to W . These edges never cross 0 where $\text{sign}(\omega)$ makes its transition. Therefore, $\text{sign}(\omega)$ can be replaced by $\text{sign}(\omega - \lambda)$ and the product $C(\omega - \lambda) \text{sign}(\omega - \lambda) = C(\omega - \lambda) \text{sign}(\omega)$ for $-W < \lambda < W$. Using this result in (4) gives

$$s(t) = \frac{1}{2\pi} \int_{-W}^W M(\lambda) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega - \lambda) (-j \text{sign}(\omega - \lambda)) e^{j(\omega - \lambda)t} d\omega \right] e^{j\lambda t} d\lambda \quad (5)$$

For each value of the outer integrand λ the value of $\omega - \lambda$ varies between $-\infty$ and ∞ , so the integral in the rectangular brackets is the Hilbert transform $\hat{c}(t)$ which can be taken out of the integral with respect to λ . The resulting integral with respect to λ is inverse Fourier transform of $M(\lambda)$ which is $m(t)$. Therefore,

$$\hat{s}(t) = m(t) \hat{c}(t) \quad (6)$$