Estimate reaction rate parameters via linear regression. (Problem 3.3 of Shuler & Kargi)
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\[
\begin{array}{c}
E + S \rightleftharpoons ES \rightarrow E + P \\
\end{array}
\]

Given kinetic constants: 
\[ k_1 := 10^9 \text{ M}^{-1}\text{s}^{-1} \quad k_i := 4.4 \times 10^4 \text{ s}^{-1} \quad k_2 := 10^3 \text{ s}^{-1} \]

Find Michaelis-Menten constant: 
\[ K_m = \frac{k_i + k_2}{k_1} \quad K_m = 4.5 \times 10^{-5} \text{ M} \]

Given initial enzyme conc: 
\[ E_0 := 10^6 \text{ M} \]
Find maximum reaction rate: 
\[ v_m = k_2 E_0 \quad v_m = 0.001 \text{ M}\cdot\text{s}^{-1} \]

Given initial substrate conc. 
\[ s_0 := 10^{-3} \text{ M} \]
Find initial reaction rate: 
\[ v_0 := v_m s_0 \quad v_0 = 9.569 \times 10^{-4} \text{ M}\cdot\text{s}^{-1} \]

When \( K_m \) is small, the reaction rate is approximately 0th order wrt s. 
\[ v_0 := v_m \quad v_0 = 0.001 \text{ M}\cdot\text{s}^{-1} \]

**Beyond the Initial Rate.**

Integrating \( \frac{ds}{dt} = \frac{v_m s}{K_m + s} \) gives 
\[ s_0 - s = K_m \ln \left( \frac{s}{s_0} \right) = v_m t \]

An analytical expression of \( s(t) \) does not exist.
The numerical solution starting with an initial guess of \( s := s_0 \)

Given 
\[ s_0 - s = K_m \ln \left( \frac{s}{s_0} \right) = v_m t \quad s(t) := \text{Find}(s) \]

Example. 
\[ s(1) = 1.025 \times 10^{-4} \]

When \( K_m \) is small, the reaction rate is approximately 0th order, and the above expression is:
\[ s_0 - s = v_m t \]

The product concentration is 
\[ p(t) := s_0 - s(t) \]
\[ t := 0, 0.01, 1 \]

Since \( K_m << s_0 \), straight lines describe the changes of s and p with time quite well.
**Substrate Half-Life.** The length of time it takes for s to drop from $s = s_0$ to $s = \frac{s_0}{2}$

(Copy $s_0/2$ into memory, mark s in the following eqn, choose [Symbolic]|Substitute for Variable])

$$s_0 - s - K_m \ln \left( \frac{s}{s_0} \right) = \frac{v_m t}{2}$$

Doing so gives:

$$\frac{1}{2} s_0 + K_m \ln(2) = \frac{v_m t}{2}$$

Thus, $t_{\text{half}} = \frac{s_0 + 2 \cdot K_m \ln(2)}{2 \cdot v_m}$  
$t_{\text{half}} = 0.531 \text{ s}$

When $K_m$ is small, the half life is approximately: $t_{\text{half}} = \frac{s_0}{2 \cdot v_m}$

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**Enzyme Deactivation.**

With 1st-order enzyme deactivation, we have

$$\frac{ds}{dt} = v(m) = \frac{v_m \exp(-k_d t) \cdot s}{K_m + s}$$

Integrating

$$s_0 - s - K_m \ln \left( \frac{s}{s_0} \right) = \frac{v_m}{k_d} \left( 1 - \exp(-k_d t) \right)$$

Again, an analytical expression of $s(t)$ does not exist. The following defines how $s$ changes with time in the presence of enzyme deactivation.

$s := s_0$  
Given $s_0 - s - K_m \ln \left( \frac{s}{s_0} \right) = \frac{v_m}{k_d} \left( 1 - \exp(-k_d t) \right)$

Half-life:

$$\frac{1}{2} s_0 + K_m \ln(2) = \frac{v_m}{k_d} \left( 1 - \exp(-k_d t_{\text{half}}) \right)$$

With enzyme deactivation, it now takes twice as long to reduce the substrate level to one half of the initial value, compared to that without enzyme deactivation. Find the deactivation time constant $\tau_d = 1/k_d$.

We first substitute into the above half-life expression

$$t_{\text{half}} = \frac{s_0 + 2 \cdot K_m \ln(2)}{v_m}$$

$$\frac{1}{2} s_0 + K_m \ln(2) = \frac{v_m}{k_d} \left( 1 - \exp\left( -2 \cdot k_d \cdot t_{\text{half}} \right) \right)$$

Alternatively, take the ratio of the half-life expression /wo enzyme deactivation and that /w enzyme deactivation. Doing so gives:

$$t_{\text{half}} = \frac{1}{k_d} \left( 1 - \exp\left( -k_d \cdot t_{\text{half}} \right) \right) \rightarrow k_d \cdot t_{\text{half}} = 1 - \exp\left( -2 \cdot k_d \cdot t_{\text{half}} \right) \rightarrow \text{ Let } x = k_d \cdot t_{\text{half}}$$

$x := 1$  
Given $x = 1 - \exp(-2 \cdot x)$  
$x := \text{Find}(x)$  
$k_d := \frac{x}{t_{\text{half}}}$  
$\tau_d := \frac{1}{k_d}$  
$\tau_d = 0.667 \text{ s}$
With an enzyme that follows the Michaelis-Menten kinetics and deactivates, complete conversion of the substrate is not possible. At $t \to \infty$, we have:

\[
s_{\text{limit}} := s_0
\]

Given

\[
\frac{s_0 - s_{\text{limit}}}{K_m} = \frac{v}{k_d}
\]

Find $s_{\text{limit}} = 0.377 \times s_0$

\[
t := 0, 0.01 \ldots 5
\]