**Vector Space (VS)** contains objects called **vectors** \((x, y, z, \ldots)\) and objects called **scalars** \((\alpha, \beta, \gamma, \ldots)\) where the laws of addition of vectors and multiplication of vectors by scalars apply.

1. **Addition of two vectors** \(x + y = z\)
2. **Multiplication of a vector by a scalar** \(\alpha \cdot x = z\)

Note that it is up to us to define addition and multiplication, so long as the result of addition and multiplication is also a vector within the same linear vector space. In other words, we do not generate new types of objects that are not contained in the original space by performing these two operations. Any objects that follow these two laws are considered to be vectors (but not necessarily "linear", which we discuss below).

**Vectors that reside in Linear Vector Space (LVS)** possess the following properties.

1. **Commutative** \(x + y = y + x\)
2. **Associative** \((x + y) + z = x + (y + z)\)
3. **Distributive** \(\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y\)
4. **Unique zero vector** \(x + 0 = 0 + x = x\) do not confuse with a zero **scalar** \(0 = x + (-x)\)

**Examples of linear vector space.**

- **Arrows** \(\longleftrightarrow \uparrow \longrightarrow\) \(x=\quad y=\uparrow\)
  1. Graphically define what addition means.

- **Columns of n real numbers.**
  \[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \]
  1. Define what addition means.

- **Define what multiplication by a scalar means.**
  \[ \alpha \cdot \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \alpha \cdot z_1 \\ \alpha \cdot z_2 \\ \vdots \\ \alpha \cdot z_n \end{bmatrix} \]
- What if addition and multiplication are defined as?

\[
\begin{align*}
x + y &= \begin{bmatrix}
x_1 + y_1 \\
x_2 + y_2 \\
\vdots \\
x_n + y_n
\end{bmatrix} \\
\alpha \cdot x &= \begin{bmatrix}
\alpha \cdot x_1 \\
\alpha \cdot x_2 \\
\vdots \\
\alpha \cdot x_n
\end{bmatrix}
\end{align*}
\]

- Columns of n complex numbers with the usual definition of addition and multiplication by a scalar.
- Rows of n complex numbers (ditto).
- \(m \times n\) real numbers arranged in a rectangular fashion (ditto).

1. Define what addition of two vectors means.

\[
\begin{align*}
x + y &= \begin{bmatrix}
x_1 + z_1 \\
x_2 + z_2 \\
\vdots \\
x_m + z_m
\end{bmatrix}
\end{align*}
\]

2. Define what multiplication of a vector by a scalar means.

\[
\begin{align*}
\alpha \cdot x &= \begin{bmatrix}
\alpha \cdot x_1 \\
\alpha \cdot x_2 \\
\vdots \\
\alpha \cdot x_m
\end{bmatrix}
\end{align*}
\]

- What if addition and multiplication are defined as? And the zero vector is?

\[
\begin{align*}
x + y &= \begin{bmatrix}
x_1 + y_1 + 2 \\
x_2 + y_2 - 1
\end{bmatrix} \\
\alpha \cdot x &= \begin{bmatrix}
\alpha \cdot x_1 + 2 \cdot (\alpha - 1) \\
\alpha \cdot x_2 - (\alpha - 1)
\end{bmatrix}
\end{align*}
\]

- logarithm-like definition
- semi-logarithm-like definition
- shift-like definition
• All real-valued continuous functions \( f(t) \) in \( 0 \leq t \leq 1 \)
\[
x = f(t) \quad y = g(t)
\]

1. Define what addition means. \( x + y = \varphi + g \)

2. Define what multiplication by a scalar means. \( \alpha \cdot x = \alpha \cdot f(t) \)

• All real-valued, continuous, and twice differentiable functions \( \phi(t) \) that satisfy the following ODE.
\[
\frac{d^2}{dt^2} \phi(t) + \phi(t) = 0
\]

Suppose \( \phi_1(t) \) and \( \phi_2(t) \) are two members of this LVS.

1. Addition: \( \phi_1(t) + \phi_2(t) \) satisfies the given ODE and is a member of the LVS.
\[
\frac{d^2}{dt^2} \left( \phi_1(t) + \phi_2(t) \right) + \left( \phi_1(t) + \phi_2(t) \right) = \frac{d^2}{dt^2} \phi_1(t) + \phi_1(t) + \frac{d^2}{dt^2} \phi_2(t) + \phi_2(t) = 0 + 0 = 0 \quad \text{... ok}
\]

2. Multiplication by a scalar: \( \alpha \cdot \phi_1(t) \) satisfies the given ODE.
\[
\frac{d^2}{dt^2} \left( \alpha \cdot \phi(t) \right) + \left( \alpha \cdot \phi(t) \right) = \alpha \cdot \left( \frac{d^2}{dt^2} \phi(t) + \phi(t) = 0 \right) = \alpha \cdot 0 = 0 \quad \text{... ok}
\]

Non-Examples of linear vector space.

• All real-valued, continuous, and twice differentiable functions \( \phi(t) \) that satisfy the following ODE.
\[
\frac{d^2}{dt^2} \phi(t) + \phi(t) = 1
\]

\( \phi_1(t) + \phi_2(t) \) does not satisfy the given ODE.
\[
\frac{d^2}{dt^2} \left( \phi_1(t) + \phi_2(t) \right) + \left( \phi_1(t) + \phi_2(t) \right) = 2 = 1
\]

\( \alpha \cdot \phi_1(t) \) does not satisfy the given ODE, either.
\[
\frac{d^2}{dt^2} \left( \alpha \cdot \phi(t) \right) + \left( \alpha \cdot \phi(t) \right) = \alpha \cdot \left( \frac{d^2}{dt^2} \phi(t) + \phi(t) = 1 \right) = \alpha \cdot 1 = \alpha = 1
\]

• All polynomials of degree \( n \) with real-coefficients.
\[
P_n(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \quad a_n = 0
\]
\[
Q_n(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n \quad b_n = 0
\]

Addition \( P_n(t) + Q_n(t) = (a_0 + b_0) + (a_1 + b_1) t + (a_2 + b_2) t^2 + \ldots + (a_n + b_n) t^n \)

Because it is possible that \( a_n + b_n = 0 \)

the result of addition could be a polynomial of degree less than \( n \).

• What about all polynomials of degree \( n \) or less with real-coefficients?
Linear independence. Definition: A set of n vectors $x_1, x_2, ..., x_n$ in LVS are linearly independent if whenever
\[ \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0 \]
we have $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$.
Thus, if $x_1, x_2, ..., x_n$ are not linearly independent, they are linearly dependent. In other words, we cannot construct a new linearly independent vector $x_j$ by linearly combining the remaining linearly independent vectors.

"Proof". Suppose $x_1, x_2, ..., x_n$ are linearly independent & some $\alpha_i \neq 0$ (say, $\alpha_1 \neq 0$) in the following equation
\[ \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0 \]
We put this term on the LHS and move the remaining terms to RHS.
\[ \alpha_1 x_1 = -\alpha_2 x_2 - ... - \alpha_n x_n \]
Since $\alpha_1 \neq 0$, we divide by $\alpha_1$.
\[ x_1 = \frac{1}{\alpha_1} (\alpha_2 x_2 + ... + \alpha_n x_n) \]
and this means we can express $x_1$ as a linear combination of other $x_i$s. $\longrightarrow$ $x_1$ is linearly dependent on other $x_i$s.

Dimension. Definition: A LVS has dimension $n$ if there is a positive integer $n$ such that the LVS contains $n$ linearly independent vectors but does not contain $n+1$ linearly independent vectors.

Examples.
- Arrows in a plane: dimension=2
- Columns of $n$ real numbers: dimension=$n$.
- $m \times n$ real numbers arranged in a rectangular fashion: dimension=$m \cdot n$.
- All real-valued, continuous, and twice differentiable functions $\phi(t)$ that satisfy the following ODE.
  \[ \frac{d^2}{dt^2} \phi(t) + \phi(t) = 0 \]
  dimension=$2$
- All polynomials of degree $n$ or less with real-coefficients: dimension=$n+1$
- All real-valued continuous functions $f(t)$ in $0 \leq t \leq 1$
  dimension=$\infty$

Existence and uniqueness of vector representation. Let $g_1, g_2, ..., g_n$ be a basis (i.e., any set of $n$ linearly independent vectors) in LVS of finite dimension $n$. For any vector $x$, there exist a unique set of scalars $\alpha_1, \alpha_2, ..., \alpha_n$ such that
\[ x = \alpha_1 g_1 + \alpha_2 g_2 + ... + \alpha_n g_n = \sum_{j=1}^{n} \alpha_j g_j \]
Note that given a set of basis, the representation of a given vector is unique. However, the choice of basis is not unique.
Examples.

- Arrows in a plane: \( g_1 \rightarrow g_2 \) (where \( g_1 \) and \( g_2 \) are not necessarily orthogonal nor unit length.
  
  We can express any arrow \( x \) in a plane as a linear combination of \( g_1 \) and \( g_2 \).

\[
x = \alpha_1 g_1 + \alpha_2 g_2
\]

- Columns of \( n \) real numbers. One choice of basis vectors.

\[
g_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad g_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\]

We can express any column of \( n \) numbers \( x \) as a linear combination of \( g_1, g_2, \ldots, g_n \).

\[
x = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n
\]

Another possibility of basis vectors.

\[
g_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad g_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\]

- 4 real numbers arranged in a square fashion.

\[
g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad g_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

We can express any 4 numbers arranged in a square fashion as a linear combination of \( g_1, g_2, g_3, \) and \( g_4 \).

\[
x = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \alpha_4 g_4 = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

Another possibility of basis vectors.

\[
g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad g_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

- All polynomials of degree \( n \) or less with real-coefficients.

\[
g_1 = 1, \quad g_2 = t, \quad g_3 = t^2, \ldots, \quad g_{n+1} = t^n
\]

We can express any polynomial of degree \( n \) or less as a linear combination of \( g_1, g_2, \ldots, g_n \).

\[
p = \alpha_0 g_1 + \alpha_2 g_2 + \alpha_3 t^2 + \ldots + \alpha_{n+1} t^n
\]

Another possibility of basis vectors.

\[
g_1 = 1, \quad g_2 = 1 + t, \quad g_3 = 1 + t + t^2, \ldots, \quad g_{n+1} = 1 + t + t^2 + \ldots + t^n
\]

Another possibility of basis vectors.

\[
g_1 = 1, \quad g_2 = 1 + t, \quad g_3 = t^2, \ldots, \quad g_{n+1} = t^n - 1 + t^n
\]

Another possibility of basis vectors (Lagrange polynomials of degree \( n \)).

\[
g_1 = \mathbb{L}_0 (t), \quad g_2 = \mathbb{L}_1 (t), \quad g_3 = \mathbb{L}_2 (t), \ldots, \quad g_{n+1} = \mathbb{L}_n (t)
\]
• All real-valued, continuous, and twice differentiable functions $\phi(t)$ that satisfy the following ODE.

$$\frac{d^2}{dt^2}\phi(t) + \phi(t) = 0$$

$g_1(t) = \sin(t)$

$g_2(t) = \cos(t)$

We can express any solution to the above ODE as a linear combination of $g_1$ and $g_2$.

$$\phi(t) = \alpha_1 \cdot \sin(t) + \alpha_2 \cdot \cos(t)$$

A random function has dimension=$\infty$. However, those that satisfy the given ODE have dimension=2, because we cannot find another (a 3rd) linearly independent function.

• All real-valued continuous functions $f(t)$ in $0 \leq t \leq 1$

  Question: can we find basis?
  Answer: Since basis is defined for finite dimension $n$ and since a continuous function may have $n=\infty$, (a finite number of) basis is not defined for this vector space.
Change of Basis. Let \( g_1, g_2, \ldots, g_n \) and \( g'_1, g'_2, \ldots, g'_n \) be two basis. We find how representations in two different sets of basis are related.

Express the 2nd set of basis \( g' \) in terms of the 1st set of basis \( g \)

\[
\begin{align*}
g'_1 &= a_{11}g_1 + a_{12}g_2 + \cdots + a_{1n}g_n \\
g'_2 &= a_{21}g_1 + a_{22}g_2 + \cdots + a_{2n}g_n \\
&\vdots \\
g'_n &= a_{n1}g_1 + a_{n2}g_2 + \cdots + a_{nn}g_n
\end{align*}
\]

\[g'_i = \sum_{j=1}^{n} a_{ji}g_j \quad \text{for} \quad i = 1, 2, \ldots, n\]

We can represent any vector \( x \) in LVS as a linear combination of either set of basis \( g \) or \( g' \).

In terms of the first set of basis:

\[x = \sum_{j=1}^{n} \xi_j g_j\]

In terms of the second set of basis:

\[
x = \sum_{j=1}^{n} \xi'_j g'_j = \sum_{j=1}^{n} \xi'_j \left( \sum_{k=1}^{n} a_{kj}g_k \right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \xi'_j a_{kj} g_k \\
= \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \xi'_j a_{kj} \right) g_k \quad \text{← compare} \quad \rightarrow \quad x = \sum_{j=1}^{n} \xi_j g_j
\]

Change of basis formula:

\[
\xi_j = \sum_{k=1}^{n} \xi'_k a_{jk} \quad j = 1, 2, \ldots, n
\]
The above derivation in compact "matrix" notation (which most vector analysis books avoid because we end up with a column or a row of vectors $g_1, g_2,$ etc, instead of the usual column of scalar numbers). Some despise mixing two different types of vectors on the same page.

Express the $i$th $g'$ in terms of $g$ (with basis arranged in a row).

$$g'_i = \begin{pmatrix} g_1 & g_2 & \ldots & g_n \end{pmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{ni} \end{bmatrix} \text{ for } i=1,2,..,n$$

Express $g'$ in terms of $g$

$$\begin{pmatrix} g'_1 & g'_2 & \ldots & g'_n \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \ldots & g_n \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

$$g' = g \cdot A \text{ where } g = \begin{pmatrix} g_1 & g_2 & \ldots & g_n \end{pmatrix} \text{ and } A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

We can represent any vector $x$ in LVS as a linear combination of either basis $g$ or $g'$.

In terms of the first set of basis:

$$x = \begin{pmatrix} g_1 & g_2 & \ldots & g_n \end{pmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = g \cdot \xi \text{ where } \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$$

In terms of the second set of basis:

$$x = \begin{pmatrix} g'_1 & g'_2 & \ldots & g'_n \end{pmatrix} \begin{bmatrix} \xi'_1 \\ \xi'_2 \\ \vdots \\ \xi'_n \end{bmatrix} = g' \cdot \xi' = (g \cdot A) \cdot \xi' = g \cdot (A \cdot \xi') \leftrightarrow \text{ compare } \rightarrow x = g \cdot \xi$$

Comparing the last line yields **Change of basis formula**: $\xi = A^{-1} \cdot \xi'$.
Other less compact ways of expressing Change of basis formula

\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{bmatrix}
\begin{bmatrix}
\xi_1' \\
\xi_2' \\
\vdots \\
\xi_n' \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\xi_1' \\
\xi_2' \\
\vdots \\
\xi_n' \\
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\end{bmatrix}
\]

\[
\xi_i = (a_{1i} \ a_{2i} \ \cdots \ a_{ni}) \cdot \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\end{bmatrix}
\text{ for } i=1,2,\ldots,n
\]

Express ith \(g'\) in terms of \(g\) (with basis arranged in a column -- not recommended).

\[
g'_i = (a_{1i} \ a_{2i} \ \cdots \ a_{ni}) \cdot \begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n \\
\end{bmatrix}
\text{ for } i=1,2,\ldots,n
\]

Express \(g'\) in terms of \(g\)

\[
\begin{bmatrix}
g'_1 \\
g'_2 \\
\vdots \\
g'_n \\
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n \\
\end{bmatrix}
\]

\[g' = A^T \cdot g\]

where

\[
\begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n \\
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{bmatrix}
\begin{bmatrix}
g'_1 \\
g'_2 \\
\vdots \\
g'_n \\
\end{bmatrix}
\]

We can represent any vector \(x\) in LVS as a linear combination of either basis \(g\) or \(g'\).

In terms of the first set of basis:

\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\end{bmatrix} =
\begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n \\
\end{bmatrix} \cdot \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\end{bmatrix}
\]

In terms of the second set of basis:

\[
\begin{bmatrix}
\xi'_1 \\
\xi'_2 \\
\vdots \\
\xi'_n \\
\end{bmatrix} =
\begin{bmatrix}
g'_1 \\
g'_2 \\
\vdots \\
g'_n \\
\end{bmatrix} \cdot \begin{bmatrix}
\xi'_1 \\
\xi'_2 \\
\vdots \\
\xi'_n \\
\end{bmatrix}
\]

Comparing the last line yields Change of basis formula:

\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\end{bmatrix} = A \cdot \begin{bmatrix}
\xi'_1 \\
\xi'_2 \\
\vdots \\
\xi'_n \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\xi'_1 \\
\xi'_2 \\
\vdots \\
\xi'_n \\
\end{bmatrix} = A^{-1} \cdot \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\end{bmatrix}
\]