Il passe, entre duex êtres que se rencontrent pour la première fois, d'étranges secrets de vie et de mort; et bien d'autres secrets qui n'ont pas encore de nom, mais qui s'emparent immédiatement de notre attitude, de nos regards et de notre visage.

M. Maeterlinck
"Les Avertis" du "Tresor des Humbles"

CHAPTER V

EQUIVALENCE

A. Summary

Through the use of various transformations on the canonical state-variable equations one can generally find all canonical equation representations for a given transfer function. When the realizations are minimal this occurs through nonsingular transformations on the state. When it is a question of nonminimal equivalents, decompositions involving the "encirclement" of controllable and observable portions result.

B. Minimal Equivalents

Given a transfer function matrix $\underset{\sim}{T}(p)$ which is rational and having $\underset{\sim}{T}(\infty)$ well defined we have seen in the last chapter how to obtain a canonical set of state variable equations

$$\dot{\underset{\sim}{s}} = \underset{\sim}{A}\underset{\sim}{s} + \underset{\sim}{B}\underset{\sim}{u} \qquad \text{(I-11a)}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\underset{\sim}{s} + \underset{\sim}{D}\underset{\sim}{u} \qquad \text{(I-11b)}$$

such that the state has minimal dimension, δ, and with

$$\underset{\sim}{T}(p) = \underset{\sim}{D} + \underset{\sim}{C}(p\underset{\sim}{1}_{\delta}-\underset{\sim}{A})^{-1}\underset{\sim}{B} \qquad \text{(I-11d)}$$

One problem of equivalence, and that which we treat here, is that of finding other realizations $\underset{\sim}{R} = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$, perhaps nonminimal, for which the above equations are true. Here in fact we will find all such realizations. However we first show how to find all minimal realizations.

Let us consider as on hand two minimal realizations $\underset{\sim}{R} = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$ and $\hat{\underset{\sim}{R}} = \{\hat{\underset{\sim}{A}}, \hat{\underset{\sim}{B}}, \hat{\underset{\sim}{C}}, \underset{\sim}{D}\}$ of a given transfer function $n \times m$ matrix $\underset{\sim}{T}(p)$. We define the observability, $\underset{\sim}{P}$ and $\hat{\underset{\sim}{P}}$, matrices and controllability, $\underset{\sim}{Q}$ and $\hat{\underset{\sim}{Q}}$, matrices as before, Eq. (IV-11); then we find

$$\underset{\sim}{S}_r = \tilde{\underset{\sim}{P}}\underset{\sim}{Q} = \tilde{\hat{\underset{\sim}{P}}}\hat{\underset{\sim}{Q}} \qquad \text{(V-1a)}$$

We also recall that $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ have δ rows and are of rank δ, in which case $\underset{\sim}{Q}\tilde{\underset{\sim}{Q}}$, $\underset{\sim}{P}\tilde{\underset{\sim}{P}}$, and the same expressions in terms of $\hat{\underset{\sim}{P}}$ and $\hat{\underset{\sim}{Q}}$, are $\delta \times \delta$ nonsingular matrices. If we premultiply $\underset{\sim}{S}_r$ by $\underset{\sim}{P}$ we obtain

$$\underset{\sim}{Q} = [(\underset{\sim}{P}\tilde{\underset{\sim}{P}})^{-1}\underset{\sim}{P}\tilde{\hat{\underset{\sim}{P}}}]\hat{\underset{\sim}{Q}} = \underset{\sim}{T}\hat{\underset{\sim}{Q}} \qquad \text{(V-1b)}$$

which serves to define the transformation matrix $\underset{\sim}{T} = (\underset{\sim}{P}\tilde{\underset{\sim}{P}})^{-1}\underset{\sim}{P}\tilde{\hat{\underset{\sim}{P}}}$ which is nonsingular by the fact that

$$\underset{\sim}{S}_r = \tilde{\underset{\sim}{P}}\underset{\sim}{T}\hat{\underset{\sim}{Q}} = \tilde{\hat{\underset{\sim}{P}}}\hat{\underset{\sim}{Q}}$$

has rank δ and $\underset{\sim}{T}$ is $\delta \times \delta$. Postmultiplying both sides of this latter by the transpose of $\hat{\underset{\sim}{Q}}$ gives, on cancellation of the nonsingular matrix $\hat{\underset{\sim}{Q}}\tilde{\hat{\underset{\sim}{Q}}}$,

$$\tilde{\hat{\underset{\sim}{P}}} = \tilde{\underset{\sim}{P}}\underset{\sim}{T} \qquad \text{(V-1c)}$$

Since the first m columns in $\underset{\sim}{Q}$ are $\underset{\sim}{B}$ we conclude from Eq. (V-1b) that $\underset{\sim}{B} = \underset{\sim}{T}\hat{\underset{\sim}{B}}$. Likewise since the first n rows of $\underset{\sim}{P}$ are $\underset{\sim}{C}$ we have from Eq. (V-1c) that $\hat{\underset{\sim}{C}} = \underset{\sim}{C}\underset{\sim}{T}$. The canonical state variable equations are then

$$\dot{\underset{\sim}{s}} = \underset{\sim}{A}\underset{\sim}{s} + \underset{\sim}{T}\hat{\underset{\sim}{B}}\underset{\sim}{u} \qquad \dot{\hat{\underset{\sim}{s}}} = \hat{\underset{\sim}{A}}\hat{\underset{\sim}{s}} + \hat{\underset{\sim}{B}}\underset{\sim}{u} \qquad \text{(V-1d)}$$

$$\underset{\sim}{y} = \hat{\underset{\sim}{C}}\underset{\sim}{T}^{-1}\underset{\sim}{s} + \underset{\sim}{D}\underset{\sim}{u} \qquad \underset{\sim}{y} = \hat{\underset{\sim}{C}}\hat{\underset{\sim}{s}} + \underset{\sim}{D}\underset{\sim}{u} \qquad \text{(V-1e)}$$

It is then reasonable that the identification

$$s = T\hat{s} \tag{V-1f}$$

should be made, in which case $T^{-1}AT\hat{s} = \hat{A}\hat{s}$. As any initial state is allowed we can cancel the $\hat{s}$ to conclude that any two minimal realizations are related through a nonsingular transformation by the relationships

$$\hat{A} = T^{-1}AT, \qquad \hat{B} = T^{-1}B, \qquad \hat{C} = CT \tag{V-2a}$$

In other words, any two minimal realizations are given one in terms of the other through Eqs. (V-2a) where in fact

$$T = (P\tilde{P})^{-1}P\tilde{\hat{P}} \tag{V-2b}$$

By letting T run through all nonsingular $\delta\times\delta$ matrices we obtain all minimal realizations from a given one.

We comment that previously we checked, at Eq. (I-11e), that this transformation, Eq. (V-2a), does leave the transfer function invariant.

As an example let us reconsider the Brune section of Chapter I for which

$$\dot{s} = \begin{bmatrix} 0 & -g_2/c_2 \\ g_2/c_1 & 0 \end{bmatrix} s + \begin{bmatrix} & 1 \\ g_1-g_2 & 0 \end{bmatrix} u \tag{I-9g}$$

$$y = \begin{bmatrix} -1/c_1 & 0 \\ 0 & (g_1-g_2)/c_2 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{I-9h}$$

If it is desired to have a skew-symmetric A matrix we can set $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ and examine the set of (nonlinear in t_{ij}) entries in

$T^{-1}AT$ such that the result is skew-symmetric. We find a suitable T as

$$T = \begin{bmatrix} \sqrt{c_1} & -\sqrt{c_1} \\ \sqrt{c_2} & \sqrt{c_2} \end{bmatrix} \qquad \text{(V-3a)}$$

Thus, we find

$$\hat{A} = T^{-1}AT = \frac{g_2}{\sqrt{c_1 c_2}}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{B} = T^{-1}B = \frac{1}{2}\begin{bmatrix} \frac{g_1-g_2}{\sqrt{c_1 c_2}} & \frac{1}{\sqrt{c_1}} \\ \frac{g_1-g_2}{\sqrt{c_1 c_2}} & \frac{-1}{\sqrt{c_2}} \end{bmatrix}$$

$$\hat{C} = CT = \begin{bmatrix} -1/\sqrt{c_1} & 1/\sqrt{c_1} \\ (g_1-g_2)/\sqrt{c_2} & (g_1-g_2)/\sqrt{c_2} \end{bmatrix} \qquad \text{(V-3b)}$$

which we know yields an equivalent structure to the original Brune section.

C. Controllability and Observability

In order to proceed to nonminimal equivalents it is necessary to introduce the concepts of controllability and observability which we have already seen enter into the theory of equivalence through the matrices P and Q.

To be somewhat precise we say that an initial state $s_1(t_o)$ is controllable if there exists a finite time t_1 and an input $u(t)$, $t_o \le t \le t_1$, such that $s_1(t) = 0$ for $t \ge t_1$. That is, such that the state can be brought to zero (which is also the origin of the state space). By beginning on a trajectory of a controllable state starting at t_o we see that later values of time yield controllable initial states and hence we can work with controllable states $s(t)$ in which case we can decompose the state space into the set of controllable states and those which are not, the uncontrollable states (this requires also letting t_1 tend to infinity).

On the other hand an initial state $\underset{\sim}{s}_1(t_o)$ is <u>observable</u> if there exists a finite time t_1 and a zero input output $\underset{\sim}{y}(t)$, $t_o \le t \le t_1$, such that a knowledge of $\underset{\sim}{y}(t)$ determines $\underset{\sim}{s}_1(t_o)$. Again we extend the concept to all times and hence can decompose the set of states into those which are observable and nonobservable.

Unfortunately the background concepts needed to derive useful results from these definitions are rather complicated so we will state some of the results omitting to some extent noncrucial proofs. As background we recall that a vector $\underset{\sim}{x}$ is in the nullspace of a matrix $\underset{\sim}{M}$ if $\underset{\sim}{M}\underset{\sim}{x} = \underset{\sim}{0}$. Considering the time-invariant case, a state $\underset{\sim}{s}(t_o)$ is controllable if it is not in the null-space of [1, p. 409]

$$\underset{\sim}{W}(t_o,t_1) = \int_{t_o}^{t_1} e^{\underset{\sim}{A}(t_o-t)}\underset{\sim}{B}\tilde{\underset{\sim}{B}}e^{\tilde{\underset{\sim}{A}}(t-t_o)}dt \qquad \text{(V-4a)}$$

Likewise a state is observable if it is not in the null-space of

$$\underset{\sim}{M}(t_o,t_1) = \int_{t_o}^{t_1} e^{\tilde{\underset{\sim}{A}}(t_o-t)}\tilde{\underset{\sim}{C}}\underset{\sim}{C}e^{\underset{\sim}{A}(t-t_o)}dt \qquad \text{(V-4b)}$$

One can see the validity of this latter, for example, by noting that the zero input-output is

$$\underset{\sim}{y}(t) = \underset{\sim}{C}e^{\underset{\sim}{A}(t-t_o)}\underset{\sim}{s}(t_o)$$

If we multiply by $\exp[\tilde{\underset{\sim}{A}}(t_o-t)]\tilde{\underset{\sim}{C}}$ and integrate we have

$$\int_{t_o}^{t_1} e^{\tilde{\underset{\sim}{A}}(t_o-t)}\tilde{\underset{\sim}{C}}\underset{\sim}{y}(t)dt = \underset{\sim}{M}(t_o,t_1)\underset{\sim}{s}(t_o)$$

from which $\underset{\sim}{s}(t_o)$ can be determined if it is not in the null-space of $\underset{\sim}{M}(t_o,t_1)$.

From the similarity in form and associated statements of $\underset{\sim}{W}$ and $\underset{\sim}{M}$ we see that the controllability and observability properties of

$$\dot{\underset{\sim}{s}} = \underset{\sim}{A}\underset{\sim}{s} + \underset{\sim}{B}\underset{\sim}{u} \qquad \text{(I-11a)}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\underset{\sim}{s} + \underset{\sim}{D}\underset{\sim}{u} \qquad \text{(I-11b)}$$

are respectively the observability and controllability properties of the transposed system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{C}\underset{\sim}{u}_t \qquad \text{(V-5a)}$$

$$\underset{\sim}{y}_t = \underset{\sim}{B}\underset{\sim}{x} + \underset{\sim}{D}\underset{\sim}{u}_t \qquad \text{(V-5b)}$$

This result is customarily referred to as the principle of system <u>duality</u> and essentially means that we need to consider only one of the two concepts (controllability or observability) as independent.

Actually the matrices $\underset{\sim}{M}$ and $\underset{\sim}{W}$ are rather difficult to work with and have been only introduced to obtain the principle of duality which links the concepts. Equivalent results are expressed in terms of the observability and controllability matrices

$$\underset{\sim}{P} = [\underset{\sim}{C}, \underset{\sim}{A}\underset{\sim}{C}, \ldots, \underset{\sim}{A}^{k-1}\underset{\sim}{C}], \qquad \underset{\sim}{Q} = [\underset{\sim}{B}, \underset{\sim}{A}\underset{\sim}{B}, \ldots, \underset{\sim}{A}^{k-1}\underset{\sim}{B}] \qquad \text{(V-6)}$$

where k is the order of $\underset{\sim}{A}$. Thus the set of controllable (initial) states is the space spanned by the columns of $\underset{\sim}{Q}$ while the set of non-observable states is perpendicular to the space spanned by the columns of $\underset{\sim}{P}$ [2, pp. 500, 504]. These criterion are easier to apply, as compared to those for $\underset{\sim}{M}$ and $\underset{\sim}{W}$. We note that if $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ have rank k then all states are controllable and observable; in this situation it is actually true that the realization is minimal, $k = o$ (as $\underset{\sim}{S}_r$ of Eq. (IV-12) has rank k).

D. <u>Nonminimal Equivalents</u>

At this point we can turn to the general result. From two sections previous we know how to find all minimal equivalents so we are interested in the cases where the dimension k of the state is larger than the minimum size σ. Such can occur when there are either uncontrollable or nonobservable states present, or both. Consequently it is convenient to partition the state vector $\underset{\sim}{s}$ into various canonical components, as

$$\underset{\sim}{s}_c = [\underset{\sim}{s}^{cn}, \underset{\sim}{s}^{co}, \underset{\sim}{s}^{un}, \underset{\sim}{s}^{uo}] \qquad \text{(V-6)}$$

where the superscript indices have the following meaning:

c: controllable

o: observable

u: uncontrollable

n: nonobservable

Thus, for example, $\underset{\sim}{s}^{uo}$ is the set of uncontrollable but observable states.

To accompany the partition of the states we can partition a given realization to obtain the canonical equations in the form

$$\begin{bmatrix} \dot{\underset{\sim}{s}}^{cn} \\ \dot{\underset{\sim}{s}}^{co} \\ \dot{\underset{\sim}{s}}^{un} \\ \dot{\underset{\sim}{s}}^{uo} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{A}_{12} & \underset{\sim}{A}_{13} & \underset{\sim}{A}_{14} \\ \underset{\sim}{A}_{21} & \underset{\sim}{A}_{22} & \underset{\sim}{A}_{23} & \underset{\sim}{A}_{24} \\ \underset{\sim}{A}_{31} & \underset{\sim}{A}_{32} & \underset{\sim}{A}_{33} & \underset{\sim}{A}_{34} \\ \underset{\sim}{A}_{41} & \underset{\sim}{A}_{42} & \underset{\sim}{A}_{43} & \underset{\sim}{A}_{44} \end{bmatrix} \begin{bmatrix} \underset{\sim}{s}^{cn} \\ \underset{\sim}{s}^{co} \\ \underset{\sim}{s}^{un} \\ \underset{\sim}{s}^{uo} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{B}_1 \\ \underset{\sim}{B}_2 \\ \underset{\sim}{B}_3 \\ \underset{\sim}{B}_4 \end{bmatrix} \underset{\sim}{u} \qquad \text{(V-7a)}$$

$$\underset{\sim}{y} = [\underset{\sim}{C}_1 \; \underset{\sim}{C}_2 \; \underset{\sim}{C}_3 \; \underset{\sim}{C}_4] \begin{bmatrix} \underset{\sim}{s}^{cn} \\ \underset{\sim}{s}^{co} \\ \underset{\sim}{s}^{un} \\ \underset{\sim}{s}^{uo} \end{bmatrix} + \underset{\sim}{D}\underset{\sim}{u} \qquad \text{(V-7b)}$$

In order to have the state $\underset{\sim}{s}$ partitioned in the form given by Eq. (V-6) generally requires that a transformation be performed upon the state. But once such a partition has been performed we see from the physical meaning of controllability and observability that $\underset{\sim}{B}_3$, $\underset{\sim}{B}_4$, $\underset{\sim}{C}_1$, $\underset{\sim}{C}_3$ are zero. Also since there should be no way for the input to couple to the uncontrollable states, $\underset{\sim}{A}_{31}$, $\underset{\sim}{A}_{32}$, $\underset{\sim}{A}_{41}$ and $\underset{\sim}{A}_{42}$ are also zero. Since also the nonobservable states should not be seen at the output even after coupling through observable states we find $\underset{\sim}{A}_{21}$, $\underset{\sim}{A}_{23}$ and

$\underset{\sim}{A}_{43}$ are also zero. Thus we can obtain the decomposition

$$\begin{bmatrix} \dot{\underset{\sim}{s}}^{cn} \\ \dot{\underset{\sim}{s}}^{co} \\ \dot{\underset{\sim}{s}}^{un} \\ \dot{\underset{\sim}{s}}^{uo} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{A}_{12} & \underset{\sim}{A}_{13} & \underset{\sim}{A}_{14} \\ \underset{\sim}{0} & \underset{\sim}{A}_{22} & \underset{\sim}{0} & \underset{\sim}{A}_{24} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{A}_{33} & \underset{\sim}{A}_{34} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{A}_{44} \end{bmatrix} \begin{bmatrix} \underset{\sim}{s}^{cn} \\ \underset{\sim}{s}^{co} \\ \underset{\sim}{s}^{un} \\ \underset{\sim}{s}^{uo} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{B}_1 \\ \underset{\sim}{B}_2 \\ \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix} \underset{\sim}{u} \qquad \text{(V-8a)}$$

$$\underset{\sim}{y} = [\underset{\sim}{0} \; \underset{\sim}{C}_2 \; \underset{\sim}{0} \; \underset{\sim}{C}_4] \begin{bmatrix} \underset{\sim}{s}^{cn} \\ \underset{\sim}{s}^{co} \\ \underset{\sim}{s}^{un} \\ \underset{\sim}{s}^{uo} \end{bmatrix} + \underset{\sim}{D}\underset{\sim}{u} \qquad \text{(V-8b)}$$

Equations (V-8) give a canonical form for realizations of a given transfer function $\underset{\sim}{T}(p)$ when the state has nonminimal size. They can be obtained from any other realization by a transformation $\underset{\sim}{T}_c$ applied to the state $\underset{\sim}{s}$ [3, p. 172]

$$\underset{\sim}{s} = \underset{\sim}{T}_c \underset{\sim}{s}_c \qquad \text{(V-8c)}$$

To actually find $\underset{\sim}{T}_c$ there are fixed procedures, but we remark that the dimensions of the four subcomponents of $\underset{\sim}{s}_c$ can be determined from $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ in which case one can solve for $\underset{\sim}{T}_c$ by hunting for a canonical realization $R_c = \{\underset{\sim}{A}_c, \underset{\sim}{B}_c, \underset{\sim}{C}_c, \underset{\sim}{D}\}$, that is, one of the form of Eqs. (V-8a,b), in terms of a given one $R = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$ by applying the result

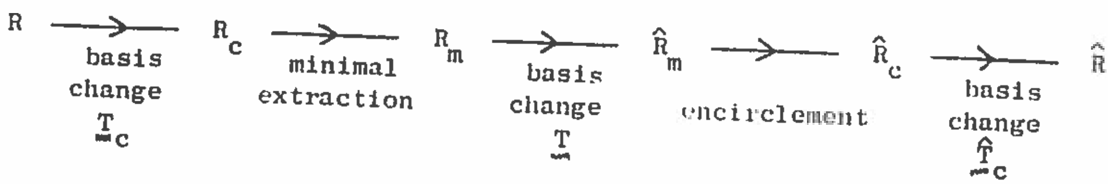
$$\underset{\sim}{T}_c \underset{\sim}{A}_c = \underset{\sim}{A}\underset{\sim}{T}_c, \qquad \underset{\sim}{T}_c \underset{\sim}{B}_c = \underset{\sim}{B}, \qquad \underset{\sim}{C}_c = \underset{\sim}{C}\underset{\sim}{T}_c \qquad \text{(V-8d)}$$

which holds since Eq. (V-8c) is valid.

The important point to observe is that only the matrices of a minimal realization $R_m = \{\underset{\sim}{A}_{22}, \underset{\sim}{B}_2, \underset{\sim}{C}_2, \underset{\sim}{D}\}$ enter into $\underset{\sim}{T}(p)$, that is since

$$\underset{\sim}{T}(p) = \underset{\sim}{D} + \underset{\sim}{C}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B} = \underset{\sim}{D} + \underset{\sim}{C}_2(p\underset{\sim}{1}_m - \underset{\sim}{A}_{22})^{-1}\underset{\sim}{B}_2 \qquad \text{(V-9)}$$

the other nonzero entries of R_c are completely arbitrary. Thus, given any minimal realization we can find all other realizations, nonminimal or not, by "encircling" the minimal one arbitrarily but as required by Eqs. (V-8a,b) and then transforming by arbitrary (nonsingular) $\underset{\sim}{T}_c$ as required by Eq. (V-8d). This being the case we can derive any realization $\hat{R}$ from any other R as shown in Fig. V-1 [4].



Equivalence for Two Realizations R and $\hat{R}$

Figure V-1

Of most practical interest to us is the derivation of nonminimal realizations from minimal ones. Since we can readily find a minimal realization the procedure of encirclement is convenient for taking a given transfer function $\underset{\sim}{T}(p)$ and finding all realizations. Note that Eq. (V-9) shows that minimal realizations have all state components controllable and observable.

As an example, the circuit of Fig. V-2 has

$$\dot{s} = -s+u \tag{V-10a}$$

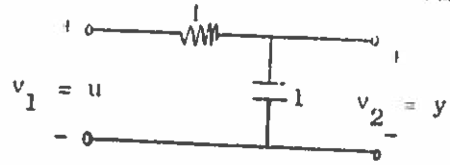
$$y = s \tag{V-10b}$$

If for some reason one were to want a configuration using two capacitors, perhaps to be used jointly for some other purpose, but with only observable portions one could proceed from

$$\begin{bmatrix} \dot{s}^{co} \\ \dot{s}^{uo} \end{bmatrix} = \begin{bmatrix} -1 & \alpha \\ 0 & \beta \end{bmatrix} \begin{bmatrix} s^{co} \\ s^{uo} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{V-10c}$$

$$y = [1 \quad \gamma] \begin{bmatrix} s^{co} \\ s^{uo} \end{bmatrix} \tag{V-10d}$$

One can easily check that these two sets of canonical equations yield the same transfer function. To obtain the most general realization of the type required one next can apply the transformation of Eq. (V-8d).



Example Circuit

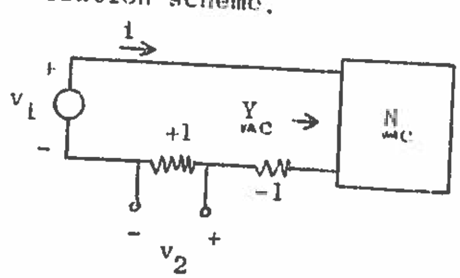
Figure V-2

In Eqs. (V-10) we comment that α, β, γ are arbitrary constants. However, if $\gamma = 0$ then s^{uo} is not observable so that there is some further constraint placed on the arbitrariness of the matrix $\underset{\sim}{C}_1$; this constraint we believe remains to be determined but should be expressable in terms of the observability matrix $\underset{\sim}{P}$.

From Section III-d) we know that for $u = v_1$ and $y = i$ the equations of Eq. (V-10) can be physically realized by loading a circuit realization of the coupling admittance matrix

$$\underset{\sim}{Y}_c = \begin{bmatrix} 0 & -1 & -\gamma \\ 1 & 1 & -\alpha \\ 0 & 0 & -\beta \end{bmatrix} \quad \text{(V-10e)}$$

in two unit capacitors. To obtain the output as a voltage one can then insert a resistor and its negative in series with the source to convert $y = i$ to $y = v_2$, as shown in Fig. V-3. Such gives an alternate but not too practical realization scheme.



Augmentation to Convert to Voltage Output

Figure V-3

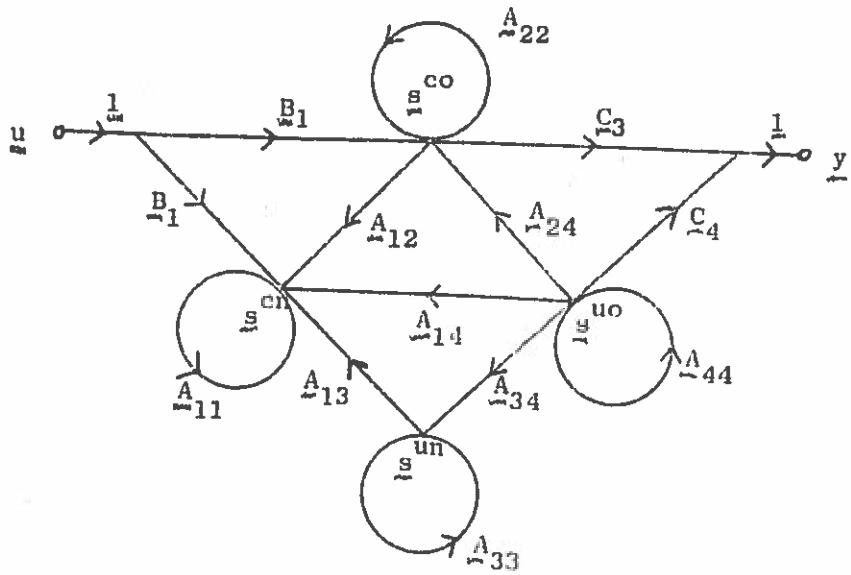
E. Discussion

Given one set of canonical state variable equations we have shown in this chapter how to find all others possessing the same transfer function. Since we have previously seen how to find one realization, in fact minimal, from a transfer function, we are now in a position to find all state variable realizations from a given transfer function. In some sense then we have found all equivalents.

However in another sense we have not completed the picture since we have not shown how to find all physical circuits yielding a given set of canonical equations. To be sure there are several since, for example, we can give an analog simulation or we can synthesize a resistive coupling network to load in capacitors and indeed these two methods yield different structures. However, one can apply the standard theory of Howitt [5] to generally find all physical resistive coupling circuits, the ones containing operational amplifiers usually being included in the result.

The theory has been given for time-invariant systems. The primary reason for excluding time-variable ones at this point is that one can not generally expect the decomposition of the state into the components $s = [s^{cn}, s^{co}, s^{un}, s^{uo}]$ to hold for all time unless there is some restriction placed upon the system. Of course time-invariance is a sufficient restriction in which case a constant transformation exists to bring the realization into canonical form. Nevertheless much can be said about the time-variable case where the use of proper transformations, which may be time-variable even in the time-invariant case, yields a different canonical form [6]. Perhaps the flow pattern of Fig. V-4 is of interest in depicting the structure of the actual decomposition.

The somewhat complete nature of the equivalence results, which have not been obtained by other means, should give sufficient justification for the existence and study of state variable theory. Nevertheless the concepts of controllability and observability can be expressed in terms of cancellations in $[p1_{mk}-A]^{-1}B$ and $C[p1_k-A]^{-1}$, respectively [1, pp. 389, 408]. Likewise, if internal variables are considered in an $\alpha v = \beta i$ description the concepts can be expressed in terms of the α and β matrices [7].



Flow Pattern for Canonical Realization

Figure V-4

In summary, using the dual concepts of controllability and observability we have been able to obtain a feeling for the internal structure of time-invariant systems through the form of canonical realizations. Using the results we have also been able to obtain all canonical state variable equations, thus allowing a designer maximum freedom of choice to obtain a desired circuit configuration.

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G. Exercises

1. Complete two syntheses of the canonical equations of Eq. (V-10). Compare the results and discuss relationships between them.
2. Find all canonical equations using two state parameters for equivalents to the circuit of Fig. V-2. What changes if an arbitrary number of capacitors are allowed?
3. Suppose that it is possible to find a time-variable transformation $\underset{\sim}{T}_c(t)$ to bring the state to the canonical form of Eq. (V-6). Discuss the changes in Eqs. (V-8) and Fig. V-1.
4. Discuss why the basis change $\underset{\sim}{T}$ for Fig. V-1 could actually be omitted from the figure.
5. Show how $\underset{\sim}{T}_c$ can be created, at least to a great extent, directly from $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ [4, p. 374].
6. Find all equivalents for the integrator of Fig. III-4a) and discuss factors influencing the choice of one over another.

Rein n'est visible et cependant nous voyons tout. Ils ont peur de nous, parce que nous les avertissons sans cesse et malgré nous; et à peine les avons-nous abordés qu'ils sentent que nous réagissons contre leur avenir.

M. Maeterlinck
"Les Avertis" du "Tresor des Humbles"

CHAPTER VI

SENSITIVITY AND TRANSITION MATRICES

A. Summary

Using the canonical equations transfer function sensitivity can be conveniently expressed, this being done here for scalar transfer functions. Time domain calculations can also be made in which case convenient methods of computation for fundamental matrices are also presented.

B. Scalar Transfer Function Sensitivity

In terms of precent changes it is of interest to know how much a transfer function changes with a given change in some parameter. Thus, for transistor circuits it is of interest often to know the effects of replacing one transistor by another one having the same characteristics except for a different current gain, β. Or alternatively with integrated circuits one would desire to know how the overall performance is affected by a change in temperature. To study such, the sensitivity of a (scalar) transfer function $T(p)$ to a parameter x has been defined as [1]

$$S_x^{T(p)} = \frac{x}{T(p)} \frac{\partial T(p)}{\partial x} \qquad \text{(VI-1)}$$

Note that in this definition the sensitivity is a complex valued function of a complex variable p. In most cases of interest one really desires to know the behavior of the magnitude of the transfer function for sinusoidal signals, that is the actually desired quantity is $S_x^{|T(j\omega)|}$.

However this latter is analytically difficult to work with and one does have the relationships

$$S_x^{T(p)} = S_x^{|T(p)|} + jx\frac{\partial \measuredangle T(p)}{\partial x} \tag{VI-2a}$$

and

$$|S_x^{T(p)}| \geq |S_x^{|T(p)|}| \tag{VI-2b}$$

both of which are relatively easy to check.

The sensitivity can be evaluated in terms of a state space realization through differentiation of

$$T(p) = D + \underset{\sim}{C}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B} \tag{VI-3a}$$

If for any matrix $\underset{\sim}{G}$ we realize that

$$\frac{\partial \underset{\sim}{G}^{-1}}{\partial x} = -\underset{\sim}{G}^{-1}\frac{\partial \underset{\sim}{G}}{\partial x}\underset{\sim}{G}^{-1} \tag{VI-3b}$$

then we obtain

$$\frac{\partial T}{\partial x} = \frac{\partial D}{\partial x} + \frac{\partial \underset{\sim}{C}}{\partial x}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B} + \underset{\sim}{C}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\frac{\partial \underset{\sim}{A}}{\partial x}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B}$$
$$+ \underset{\sim}{C}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\frac{\partial \underset{\sim}{B}}{\partial x} \tag{VI-4}$$

We observe that, except for the derivations, the only operations involved are those already used in forming the transfer function from the realization. Consequently, this method of determining the sensitivity is quite applicable to computer analysis of circuits where we have previously seen that there are convenient methods of obtaining the realization $R = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, D\}$ from the circuit diagram. We observe, for example, that if the realization is set up in the special form of Eq. (IV-3) where $\underset{\sim}{C} = [1, 0, \ldots, 0]$, then $\partial\underset{\sim}{C}/\partial x = \underset{\sim}{0}$ while $\partial\underset{\sim}{A}/\partial x$ also takes a simple form (having only nonzero entries in the last row).

As an example let us consider the sensitivity to the damping factor ζ of

$$T(p) = \frac{1}{p^2 + 2\zeta\omega_n p + \omega_n^2} \tag{VI-5a}$$

From Eq. (IV-15f) we have

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad \underset{\sim}{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underset{\sim}{C} = [1, 0], \quad D = 0 \tag{VI-5b}$$

Then we have

$$(p\underset{\sim}{1}_2 - \underset{\sim}{A})^{-1} = T(p)\begin{bmatrix} p+2\zeta\omega_n & 1 \\ -\omega_n^2 & p \end{bmatrix}, \quad \frac{\partial \underset{\sim}{A}}{\partial \zeta} = \begin{bmatrix} 0 & 0 \\ 0 & -2\omega_n \end{bmatrix} \tag{VI-5c}$$

in which case Eq. (VI-4) gives

$$S_\zeta^T = \frac{\zeta}{T}[\underset{\sim}{C}(p\underset{\sim}{1}_2 - \underset{\sim}{A})^{-1} \frac{\partial \underset{\sim}{A}}{\partial \zeta}(p\underset{\sim}{1}_2 - \underset{\sim}{A})^{-1}\underset{\sim}{B}] = -2\zeta\omega_n pT(p) \tag{VI-5d}$$

If the sensitivity is desired at $p = j\omega_n$ we find $|S_\zeta^{T(j\omega_n)}| = 1 \geq |S_\zeta^{|T(j\omega_n)|}|$ in which case a 1% change in ζ causes no more than a 1% change in $|T(j\omega_n)|$. Note also that the sensitivity is zero at both zero and infinity frequencies. Of course we could have obtained the same results by differentiating $T(p)$ with respect to ζ directly. But if $T(p)$ is available in terms of the canonical equations and calculated in terms of a digital computer, this direct method of calculating the sensitivity generally calls for added routines over that using Eq. (VI-4).

C. Pole Position Sensitivities

A useful set of design parameters is the set of pole position sensitivities defined through

$$S_x^{p_k} = \frac{\partial p_k}{\partial x} \tag{VI-6}$$

where p_k is a pole of the transfer function $T(p)$. In general the poles of $T(p)$ are eigenvalues of $\underset{\sim}{A}$ or, what is the same, zeros of the determinant $\Delta(p)$ of $p\underset{\sim}{1}_k-\underset{\sim}{A}$. If we assume that p_k is a simple eigenvalue of $\underset{\sim}{A}$ then we can evaluate the pole sensitivity $S_x^{p_k}$ for p_k with respect to x as follows. We have, which serves to define the polynomial $K(p)$,

$$\Delta(p) = (p-p_k)K(p), \qquad K(p_k) \neq 0 \tag{VI-7a}$$

on differentiation

$$\frac{\partial\Delta(p)}{\partial x} = (p-p_k)\frac{\partial K(p)}{\partial x} - \frac{\partial p_k}{\partial x}K(p) \tag{VI-7b}$$

Solving for $\partial p_k/\partial x$ on letting $p = p_k$ gives, on noting that $K(p_k) = \partial\Delta(p)/\partial p$ evaluated at $p = p_k$,

$$S_x^{p_k} = \frac{\partial p_k}{\partial x} = - \frac{\partial\Delta(p)/\partial x}{\partial\Delta(p)/\partial p}\Bigg|_{p = p_k} \tag{VI-7c}$$

where

$$\Delta(p) = \det(p\underset{\sim}{1}_k-\underset{\sim}{A}) \tag{VI-7d}$$

As a consequence the pole position sensitivity is relatively easily evaluated in terms of the $\underset{\sim}{A}$ matrix and with the use of a computer [2].

To illustrate the situation let us again consider the transfer function of Eq. (VI-5a), we have

$$\Delta(p) = \det(p\underset{\sim}{1}_2-\underset{\sim}{A}) = p^2 + 2\zeta\omega_n p + \omega_n^2 \tag{VI-8a}$$

and thus

$$\partial\Delta(p)/\partial\zeta = 2\omega_n p, \qquad \partial\Delta(p)/\partial p = 2p + 2\zeta\omega_n \tag{VI-8b}$$

There are two poles of $T(p)$, let us consider

$$p_1 = -\omega_n[\zeta + \sqrt{\zeta^2 - 1}] \qquad \text{(VI-8c)}$$

Then Eq. (VI-7c) gives

$$s_\zeta^{p_1} = \omega_n\left[\frac{\zeta + \sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}\right] \qquad \text{(VI-8d)}$$

D. Time-Domain Variations

In many situations the quantity of most importance is the actual output change as a function of time due to a parameter change. In such situations the canonical state variable equations

$$\dot{\underset{\sim}{s}} = \underset{\sim}{A}\underset{\sim}{s} + \underset{\sim}{B}u \qquad \text{(VI-9a)}$$

$$y = \underset{\sim}{C}\underset{\sim}{s} + Du \qquad \text{(VI-9b)}$$

can advantageously be used.

Again let us consider a parameter x, as well as constant (in time) realization matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, D, the last one being a scalar by virtue of our treatment of single input single output systems. Then we find on differentiation with respect to x

$$\frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{s}}{\partial x}\right) = \underset{\sim}{A}\left(\frac{\partial \underset{\sim}{s}}{\partial x}\right) + \left[\frac{\partial \underset{\sim}{A}}{\partial x}\underset{\sim}{s} + \frac{\partial \underset{\sim}{B}}{\partial x}u\right] \qquad \text{(VI-10a)}$$

$$\frac{\partial y}{\partial x} = \underset{\sim}{C}\left(\frac{\partial \underset{\sim}{s}}{\partial x}\right) + \frac{\partial \underset{\sim}{C}}{\partial x}\underset{\sim}{s} + \frac{\partial D}{\partial x}u \qquad \text{(VI-10b)}$$

To determine $\partial y/\partial x$ we can first solve Eq. (VI-9a) for $\underset{\sim}{s}$ and then Eq. (VI-10a) for $\partial \underset{\sim}{s}/\partial x$. The important thing to observe is that the same matrix $\underset{\sim}{A}$ occurs in the two situations, only the forcing functions differ being $\underset{\sim}{B}u$ in the first case and $(\partial \underset{\sim}{A}/\partial x)\underset{\sim}{s} + (\partial \underset{\sim}{B}/\partial x)u$ in the second.

The problem in this case is one of solving the differential equation $\dot{\underset{\sim}{z}} = \underset{\sim}{A}\underset{\sim}{z} + \underset{\sim}{f}$ with $\underset{\sim}{f}$ known. Such solutions are obtained in a straightforward manner, and are in fact conveniently obtained on a digital computer, as discussed in the next section. Consequently, the variations in the output, $\partial y/\partial x$, as a function of time are conveniently obtained. Of course they can also be normalized, as for the transfer function, to give percent changes if so desired.

E. Transition Matrix Evaluation

Theoretically it is a relatively simple matter to solve the differential equation

$$\dot{z} = Az + f \qquad \text{(VI-11a)}$$

where f is a known forcing function independent of z and A is a square $k \times k$ matrix, also independent of z but perhaps not of time. To solve Eq. (VI-11a), which is the type of equation appearing in Eqs. (VI-9a, 10a), we first solve the equation

$$\dot{Z} = AZ, \qquad Z(t_o) = 1_k \qquad \text{(VI-11b)}$$

which is the original one with the k-vector z replaced by the $k \times k$ matrix Z, without the forcing function and with the identity matrix for initial conditions. The solution to the latter equation can be denoted by $\Phi(t,t_o)$ and is called the transition matrix for the system. In the case where A is constant in time this transition matrix can be explicitly evaluated as

$$\Phi(t,t_o) = e^{A(t-t_o)}, \quad \text{constant } A \qquad \text{(VI-12a)}$$

where the exponential of a matrix is defined precisely by

$$e^{At} = 1_k + At + A^2 \frac{t^2}{2!} + \ldots + A^i \frac{t^i}{i!} + \ldots \qquad \text{(VI-12b)}$$

In fact one can directly check that the exponential transition matrix of Eq. (VI-12a) does solve the unforced differential equation of (VI-11b).

As an example, if as in Eq. (V-3b) we have

$$\underset{\sim}{A} = a\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{(VI-13a)}$$

then

$$\underset{\sim}{A}^2 = a^2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \underset{\sim}{A}^3 = a^3\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \underset{\sim}{A}^4 = a^4\underset{\sim}{1}_2, \quad \underset{\sim}{A}^5 = a^5\underset{\sim}{A} \qquad \text{(VI-13b)}$$

in which case Eq. (VI-12b) gives

$$e^{\underset{\sim}{A}t} = \begin{bmatrix} 1-a^2t^2/2!+a^4t^4/4!+\ldots & -at+a^3t^3/3!-a^5t^5/5!+\ldots \\ at-a^3t^3/3!+a^5t^5/5!+\ldots & 1-a^2t^2/2!+a^4t^4/4!+\ldots \end{bmatrix}$$

$$= \begin{bmatrix} \cos jat & -j \sin jat \\ j \sin jat & \cos jat \end{bmatrix} \qquad \text{(VI-13c)}$$

In the case of the zero input situation with k-vector $\underset{\sim}{z}$ we simply multiply $\underset{\sim}{Z}(t_o)$ by $\underset{\sim}{z}(t_o)$ to get

$$\underset{\sim}{z}(t) = \underset{\sim}{\Phi}(t,t_o)\underset{\sim}{z}(t_o) \qquad \text{(VI-14)}$$

which yields the zero input response. If $\underset{\sim}{f} \neq \underset{\sim}{0}$, then by treating $\underset{\sim}{z}$ as the output we can apply the fundamental decomposition of Eq. (I-4). In the time-invariant case we then wish to convolute the impulse response $\underset{\sim}{\Phi}(t,0)1(t)$ with $\underset{\sim}{f}(t)$, where $1(t)$ is the unit-step function. Thus the general solution of interest to Eq. (V-11a) is

$$\underset{\sim}{z}(t) = e^{\underset{\sim}{A}(t-t_o)}\underset{\sim}{z}(t_o) + \int_{t_o}^{t} e^{\underset{\sim}{A}(t-\tau)}\underset{\sim}{f}(\tau)d\tau, \quad t > t_o \qquad \text{(VI-15)}$$

One can check that this latter is a solution by direct substitution in the original differential equation.

Several points of observation are worth observing. We see that in the time-invariant case the transition matrix is found by summing an infinite series. Since the series is always uniformly convergent one can use the series summation as a method for finding the transition matrix on a digital computer. Such a method involves only summation and matrix multiplication and the error after a finite number of terms are considered is relatively easily determined [3]. Alternate methods result from noting that $\exp \underset{\sim}{A}t$ is the inverse Laplace transform of $(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}$ as Eq. (VI-11b) shows. Consequently, all entries in $\exp \underset{\sim}{A}t$ are exponentials or time multiplied exponentials; these can be determined from a partial fraction expansion of $(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}$ where in fact iterative methods can be used to replace evaluation of this inverse by simple matrix multiplications [4] [5]. If also $\underset{\sim}{f}$ has a rational Laplace transform the final $\underset{\sim}{z}(t)$ for Eq. (VI-15) can be relatively simply found by inversion of Laplace transforms. Alternatively the needed convolution can be carried out directly, though less conveniently, on the computer.

F. Discussion

In terms of the realization matrices several types of sensitivity have been discussed and evaluated, all for scalar transfer functions of time-invariant networks. Both transfer function and pole position sensitivity are relatively easily evaluated while time domain variations require a solution of the canonical equations to find the transition matrix $\exp \underset{\sim}{A}t$.

Actually to determine the variations in the output $y(t)$ due to x parameter changes, $\partial y/\partial x$, requires two solutions of the equations $\dot{\underset{\sim}{z}} = \underset{\sim}{A}\underset{\sim}{z} + \underset{\sim}{f}$, first with $\underset{\sim}{f} = \underset{\sim}{B}u$, with $\underset{\sim}{z}(t_o) = \underset{\sim}{s}(t_o)$, and then with $\underset{\sim}{f} = (\partial \underset{\sim}{A}/\partial x)\underset{\sim}{s} + (\partial \underset{\sim}{B}/\partial x)u$ subject to $\partial \underset{\sim}{s}(t_o)/\partial x = \underset{\sim}{z}(t_o)$, this latter often being taken as zero. Typical results in the somewhat unrealistic situations where $x = a_{11}$ are plotted in [2, p. 341].

Because changes in responses due to circuit element variations can be disturbing it is often desirable to try to find circuitry which minimizes such variations. One can see from the formula $T(p) = D +$

$C(p1_k - A)^{-1}B$ that if the entire transfer function is obtained by a single realization then the feedback supplied by the configuration will generally mean that each circuit element can possibly strongly interact with all other components resulting in relatively high sensitivity. On the other hand if the transfer function is broken into degree one or two factors as $T(p) = \prod_i [d_i + \frac{c_i b_i}{p+a_i}] \prod_j [D_j + C_j (p1_2 - A_j)^{-1} B_j]$ then those circuit elements occurring in a given portion only relatively strongly interact with those components associated with the appropriate degree one or two realization. Consequently there is practical value in designs based upon the factorization of transfer functions into small degree portions.

Finally we mention that, as with most other state variable techniques, the theory of sensitivity is made practical for the use of digital computers through the techniques discussed.

G. References

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3. Liou, M. L., "A Novel Method of Evaluating Transient Response," Proceedings of the IEEE, Vol. 54, No. 1, January 1966, pp. 20-23.
4. Liou, M. L., "Evaluation of the Transition Matrix," Proceedings of the IEEE, Vol. 55, No. 2, February 1967, pp. [illegible].
5. Thomson, W. E., "Evaluation of Transient Response," Proceedings of the IEEE, Vol. 54, No. 11, November 1966, p. [illegible].

H. Exercises

1. Exhibit a formula for $\partial y/\partial x$ in terms of the realization matrices and the initial state and input.
2. Show [4] that

$$(p1_k - A)^{-1} = \sum_{i=0}^{k-1} \frac{p^{k-i-1}}{d(p)} B_i$$

where

$$d(p) = p^k + d_1 p^{k-1} + \dots + d_{k-1} p + d_k$$

and

$$\underset{\sim}{B}_o = \underset{\sim}{1}_k, \quad \underset{\sim}{B}_1 = \underset{\sim}{B}_o \underset{\sim}{A} + d_1 \underset{\sim}{1}_k \quad \dots$$

$$\underset{\sim}{B}_{k-1} = \underset{\sim}{B}_{k-2} \underset{\sim}{A} + d_{k-1} \underset{\sim}{1}_k, \quad \underset{\sim}{0} = \underset{\sim}{B}_{k-1} \underset{\sim}{A} + d_k \underset{\sim}{1}_k$$

3. Find the sensitivity of the Brune section, Fig. I-5, to variations in the two gyrators. From this determine which gyrator should be most stably constructed.
4. Discuss the actual programming involved in setting up Eq. (VI-15). Give a flow chart for a program to determine $\partial y/\partial x$ on a digital computer.

Il se peut qu'il n'y ait aucune arrière - pensée entre deux hommes, mais il y a des choses plus impérieuses et plus profondes que la pensée. J'ai été plusieurs fois témoin de ces choses, et un jour je les ai vues de si près que je ne savais plus s'il s'agissait d'un autre ou de moi-même ...

M. Maeterlinck
"Les Avertis" du "Tresor des Humbles"

CHAPTER VII

POSITIVE-REAL ADMITTANCE SYNTHESIS

A. Summary

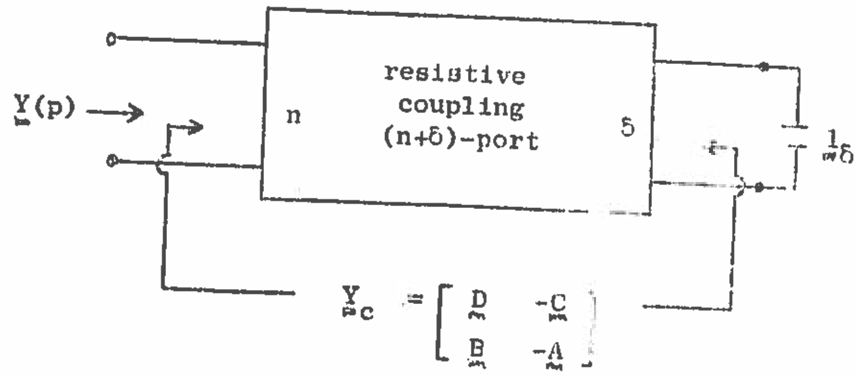
The results of the Positive-Real Lemma, whose proof is merely outlined, are applied to obtain a transformation which yields a positive-real coupling admittance to load in capacitors such that a passive circuit synthesises a positive-real admittance $\underset{\sim}{Y}(p)$.

B. Introductory Remarks

Previously, Section III D), we saw that if an admittance matrix $\underset{\sim}{Y}(p)$ has a state-variable realization $R = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$ then a physical structure yielding $\underset{\sim}{Y}(p)$ as the input n-port admittance results from loading a resistive coupling (n+k)-port structure described by the admittance

$$\underset{\sim}{Y}_c = \begin{bmatrix} \underset{\sim}{D} & -\underset{\sim}{C} \\ \underset{\sim}{B} & -\underset{\sim}{A} \end{bmatrix} \qquad \text{(III-7c)}$$

in k unit capacitors. Here $\underset{\sim}{Y}(p)$ is an $n \times n$ matrix while k is the size of the state; conveniently k is taken as the minimal value δ, this being the degree of $\underset{\sim}{Y}(p)$. The structure is as in Fig. VII-1 which is Fig. III-7 repeated for convenience.



Realization Structure

δ = δ[Y(p)] for Minimal

Figure VII-1

However, even when $Y(p)$ can be obtained through the use of only passive circuit elements, this method may require other than passive elements since Y_c may not be obtainable without the use of active elements. Consequently we recall that all minimal equivalents can be obtained by transformations performed upon Y_c; thus all minimal capacitor structures result by allowing T to vary in

$$\hat{Y}_c = \begin{bmatrix} D & -CT \\ T^{-1}B & -T^{-1}AT \end{bmatrix} \qquad \text{(VII-1)}$$

Our interest here is to search for a proper choice of the transformation T such that the new coupling admittance matrix $\hat{Y}_c$ can be realized by passive resistors (and gyrators, recall Fig. III-8).

We recall that the condition for a given rational n×n matrix $Y(p)$ to be the admittance matrix of a passive n-port constructed of only passive circuit elements is that $Y(p)$ is positive-real [1, p. 240]. By definition a matrix $Y(p)$ is positive-real if

a) $Y(p)$ is holomorphic in $\operatorname{Re} p > 0$

b) $Y(p^*) = Y^*(p)$ in $\operatorname{Re} p > 0$

c) The Hermitian part $Y_H(p)$, $2Y_H(p) = Y(p) + \tilde{Y}^*(p)$, is nonnegative definite in $\operatorname{Re} p > 0$,

where the superscript asterisk denotes complex conjugation. If $Y(p)$ is positive-real and rational we will call it PR for convenience.

Since it is known that any rational positive-real matrix has a passive synthesis in the form of Fig. VII-1, it is then a matter of searching for a suitable transformation T to make $\hat{Y}_c$ positive-real when $Y(p)$ is. The purpose of the next sections is to obtain the desired T.

C. The PR Lemma

First we recall that any PR matrix $Y(p)$ can be decomposed into the sum of two matrices

$$Y(p) = Y_L(p) + Y_o(p) \tag{VII-2}$$

where Y_L, the lossless part has all its poles on the $j\omega$ axis [and satisfies $Y_L(p) = -\tilde{Y}_L(-p)$] while $Y_o(p)$ has poles only in the open left half plane; both Y_L and Y_o are PR while the decomposition can be obtained through a partial fraction expansion. Since the poles of Y_L and Y_o can not coincide, a minimal realization for Y_L can be "added" to a minimal realization for Y_o to obtain one for Y. As a consequence we will first obtain properties of these separate realizations and then show how to combine them to give the proper meaning to the word "added." For convenience we assume $Y(\infty) = Y_o(\infty)$.

The basic result in the theory is as follows [2].

The PR Lemma: Let $Y(p)$ be an $n \times n$ rational matrix with real coefficients and with no poles in $\operatorname{Re} p \geq 0$, and let $R = \{A, B, C, D\}$ be a minimal realization. Then $Y(p)$ is PR if and only if there exist matrices W_∞, L, and a (unique) positive definite (symmetric) P satisfying

$$PA + \tilde{A}P = -\tilde{L}L \tag{VII-3a}$$

$$PB = \tilde{C} - \tilde{L}W_\infty \tag{VII-3b}$$

$$\tilde{W}_{\infty} W_{\infty} = \tilde{D} + D \qquad \text{(VII-3c)}$$

Outline of Demonstration: As the steps in the proof are detailed and involved [2] we merely outline the main ideas with emphasis upon those points of interest for actual calculations.

To see that if Eqs. (VII-3) hold then $Y(p)$ is PR is straightforward from the following calculations, since $Y(p)$ is assumed holomorphic in $\operatorname{Re} p \geq 0$ and has real coefficients.

$$
\begin{aligned}
2Y_H(p) &= \tilde{Y}(p^*) + Y(p) = \tilde{D}+D + \tilde{B}(p^* 1_\nu - \tilde{A})^{-1}\tilde{C} + C(p 1_\nu - A)^{-1}B \\
&= \tilde{W}_\infty W_\infty + \tilde{B}[(p^* 1_\nu - \tilde{A})^{-1}P + P(p 1_\nu - A)^{-1}]B \\
&\quad + \tilde{B}(p^* 1_\nu - \tilde{A})^{-1} L W_\infty + \tilde{W}_\infty \tilde{L}(p 1_\nu - A)^{-1}B \\
&= \tilde{W}_\infty W_\infty + \tilde{B}(p^* 1_\nu - \tilde{A})^{-1}[P(p+p^*) + L\tilde{L}](p 1_\nu - A)^{-1}B \\
&\quad + \tilde{B}(p^* 1_\nu - \tilde{A})^{-1} L W_\infty + \tilde{W}_\infty \tilde{L}(p 1_\nu - A)^{-1}B \\
&= [\tilde{W}_\infty + \tilde{B}(p^* 1_\nu - \tilde{A})^{-1} L][W_\infty + \tilde{L}(p 1_\nu - A)^{-1}B] \\
&\quad + \tilde{B}(p^* 1_\nu - \tilde{A})^{-1}[(p+p^*)P](p 1_\nu - A)^{-1}B
\end{aligned}
\qquad \text{(VII-4)}
$$

This last shows that $Y_H(p)$ is positive semidefinite for all p with $p+p^* \geq 0$, that is in the right half plane, since P can also be factored into $P = P^{1/2}P^{1/2}$ with the square roots also symmetric.

To show that $Y(p)$ is PR only if Eqs. (VII-3) hold is more difficult. We first find a $W(p)$ satisfying

$$Y(p) + \tilde{Y}(-p) = \tilde{W}(-p)W(p) \qquad \text{(VII-5a)}$$

where further $W(p)$ is holomorphic, together with its right inverse W_r^{-1}, in the right half plane. Such $W(p)$ can be found conveniently, but the calculations can become involved [?]. The use of this particular $W(p)$ is used to guarantee the minimality of $W(p)$, often simpler factorizations as the one of Gauss [1, p. 168] can be used to advantage.

One can then show that $\underset{\sim}{W}(p)$ has the minimal realization $\mathcal{R} = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{L}, \underset{\sim}{W}_\infty\}$ which serves to define $\underset{\sim}{L}$, note that the matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are identical for $\underset{\sim}{Y}(p)$ and $\underset{\sim}{W}(p)$. We then transform the minimal realization

$$\mathcal{R}_1 = \left\{ \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \tilde{\underset{\sim}{L}}\underset{\sim}{L} & -\tilde{\underset{\sim}{A}} \end{bmatrix}, \begin{bmatrix} \underset{\sim}{B} \\ \tilde{\underset{\sim}{L}}\underset{\sim}{W}_\infty \end{bmatrix}, [\tilde{\underset{\sim}{W}}_\infty \underset{\sim}{L}, -\tilde{\underset{\sim}{B}}], \tilde{\underset{\sim}{W}}_\infty \underset{\sim}{W}_\infty \right\} \quad \text{(VII-5b)}$$

of $\tilde{\underset{\sim}{W}}(-p)\underset{\sim}{W}(p)$ through Eq. (V-2a) using

$$\underset{\sim}{T} = \begin{bmatrix} \underset{\sim}{1}_\delta & \underset{\sim}{0} \\ -\underset{\sim}{P} & \underset{\sim}{1}_\delta \end{bmatrix} \quad \text{(VII-5c)}$$

to get the equivalent realization for $\tilde{\underset{\sim}{W}}(-p)\underset{\sim}{W}(p)$

$$\mathcal{R}_2 = \left\{ \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\tilde{\underset{\sim}{A}} \end{bmatrix}, \begin{bmatrix} \underset{\sim}{B} \\ \underset{\sim}{P}\underset{\sim}{B} + \tilde{\underset{\sim}{L}}\underset{\sim}{W}_\infty \end{bmatrix}, [\tilde{\underset{\sim}{W}}_\infty \underset{\sim}{L} + \tilde{\underset{\sim}{B}}\underset{\sim}{P}, -\tilde{\underset{\sim}{B}}], \tilde{\underset{\sim}{W}}_\infty \underset{\sim}{W}_\infty \right\} \quad \text{(VII-5d)}$$

Here $\underset{\sim}{P}$ is the unique positive definite solution of the equation

$$\underset{\sim}{P}\underset{\sim}{A} + \tilde{\underset{\sim}{A}}\underset{\sim}{P} = -\tilde{\underset{\sim}{L}}\underset{\sim}{L} \quad \text{(VII-4a)}$$

Next we note that a realization for $\underset{\sim}{Y}(p)+\tilde{\underset{\sim}{Y}}(-p)$ is

$$\mathcal{R}_3 = \left\{ \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\tilde{\underset{\sim}{A}} \end{bmatrix}, \begin{bmatrix} \underset{\sim}{B} \\ \tilde{\underset{\sim}{C}} \end{bmatrix}, [\underset{\sim}{C}, -\tilde{\underset{\sim}{B}}], \tilde{\underset{\sim}{D}} + \underset{\sim}{D} \right\} \quad \text{(VII-5e)}$$

On noting the conditions for equivalence and identification of realizations we obtain $\mathcal{R}_2 = \mathcal{R}_3$ and the PR Lemma follows, Q.E.D.

On noting that almost all of the previous holds except that $\underset{\sim}{W} = \underset{\sim}{0}$, and hence $\underset{\sim}{L} = \underset{\sim}{0}$, when $\underset{\sim}{Y}$ is lossless and zero at infinity, we conclude that in the lossless case there exists a positive definite (symmetric) $\underset{\sim}{P}_I$ such that

$$\underset{\sim}{P}_L \underset{\sim}{A}_L + \tilde{\underset{\sim}{A}}_L \underset{\sim}{P}_L = \underset{\sim}{0} \qquad \text{(VII-6a)}$$

$$\underset{\sim}{P}_L \underset{\sim}{B}_L = \tilde{\underset{\sim}{C}}_L \qquad \text{(VII-6b)}$$

where $\{\underset{\sim}{A}_L, \underset{\sim}{B}_L, \underset{\sim}{C}_L, \underset{\sim}{0}\}$ is a minimal realization of the lossless PR admittance which is zero at infinity. As a consequence we can replace the conditions of the PR Lemma to allow simple poles on the $j\omega$ axis, none though at infinity, if we use

$$\underset{\sim}{P} = \underset{\sim}{P}_L \dot{+} \underset{\sim}{P}_o, \qquad \underset{\sim}{A} = \underset{\sim}{A}_L \dot{+} \underset{\sim}{A}_o$$

$$\underset{\sim}{B} = \begin{bmatrix} \underset{\sim}{B}_L \\ \underset{\sim}{B}_o \end{bmatrix}, \qquad \underset{\sim}{C} = [\underset{\sim}{C}_L, \underset{\sim}{C}_o], \qquad \underset{\sim}{L} = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{L}_o \end{bmatrix} \qquad \text{(VII-7)}$$

where the subscript zeros refer to the realization of $\underset{\sim}{Y}_o$, that portion of $\underset{\sim}{Y}(p)$ with only open left half plane poles. Note, however, that now $\underset{\sim}{P}$ is no longer unique by virtue of the presence of $\underset{\sim}{P}_L$.

In conclusion, if $\underset{\sim}{Y}(p)$ is PR with no pole at infinity then Eqs. (VII-3) hold with the various matrices obtained using Eq. (VII-7) upon decomposing $\underset{\sim}{Y}(p)$ into the sum of a lossless part $\underset{\sim}{Y}_L(p)$ and a nonlossless part $\underset{\sim}{Y}_o(p)$. The calculations are theoretically very straightforward but the computation for $\underset{\sim}{W}(p)$ with the proper holomorphic inverse gives considerable difficulty in practice. However once such a $\underset{\sim}{W}(p)$ is found Eqs. (VII-3a) can be solved for $\underset{\sim}{P}_o$ in a very straightforward manner as a set of linear algebraic equations subject to the positive definite constraints. As it stands the method does not allow the direct treatment of poles at infinity and these must therefore be extracted separately as an added term $p\underset{\sim}{C}_\infty$, for the right side of Eq. (VII-2), to be independently considered for synthesis purposes.

D. PR Admittance Synthesis

We assume as given an $n \times n$ PR admittance matrix which we can, as a consequence, decompose into

$$\underset{\sim}{Y}(p) = \underset{\sim}{Y}_o(p) + \underset{\sim}{Y}_L(p) + p\underset{\sim}{C}_\infty \qquad \text{(VII-8)}$$

where $\underset{\sim}{Y}_o(p)$ is holomorphic in $\text{Re } p \geq 0$, all the poles of $\underset{\sim}{Y}_L(p)$ are on the $j\omega$ axis and simple with none at infinity, and all three terms on the right of Eq. (VI-8) are separately PR. The term $p\underset{\sim}{C}_\infty$ is separately synthesized, using for example only capacitors loading transformers [1, p. 204]; the resulting network for $p\underset{\sim}{C}_\infty$ is connected in parallel with one of $\underset{\sim}{Y}_o + \underset{\sim}{Y}_L$.

To synthesize $\underset{\sim}{Y}_o + \underset{\sim}{Y}_L$ we find any minimal realization $R = \left([\underset{\sim}{A}_L \dot{+} \underset{\sim}{A}_o], \begin{bmatrix} \underset{\sim}{B}_L \\ \underset{\sim}{B}_o \end{bmatrix}, [\underset{\sim}{C}_L, \underset{\sim}{C}_o], \underset{\sim}{D}\right)$ and then determine a desired $\underset{\sim}{P} = \underset{\sim}{P}_L \dot{+} \underset{\sim}{P}_o$ as for Eq. (VII-7). Since $\underset{\sim}{P}$ is positive definite we find its (unique) positive definite square root $\underset{\sim}{P}^{1/2}$. Thus

$$\underset{\sim}{P} = \underset{\sim}{P}^{1/2}\underset{\sim}{P}^{1/2} \qquad \text{(VII-9a)}$$

In actual fact, since $\underset{\sim}{P}$ is in direct sum form we can also write $\underset{\sim}{P}^{1/2}$ in direct sum form as

$$\underset{\sim}{P}^{1/2} = \underset{\sim}{P}_L^{1/2} \dot{+} \underset{\sim}{P}_o^{1/2} \qquad \text{(VII-9b)}$$

Next we apply the theory of equivalence of Chapter V, choosing

$$\underset{\sim}{T} = \underset{\sim}{P}^{-1/2} \qquad \text{(VII-9c)}$$

where $\underset{\sim}{P}^{-1/2}$ is the inverse of $\underset{\sim}{P}^{1/2}$ [note that the $\underset{\sim}{P}$ of Eq. (V-2b) has a different meaning than the $\underset{\sim}{P}$ of Eq. (VII-9c) whereas the $\underset{\sim}{T}$'s are the same]. We then have a realization $\hat{R} = (\underset{\sim}{P}^{1/2}\underset{\sim}{A}\underset{\sim}{P}^{-1/2}, \underset{\sim}{P}^{1/2}\underset{\sim}{B}, \underset{\sim}{C}\underset{\sim}{P}^{-1/2}, \underset{\sim}{D})$ derived from the original R having its entries as given by Eqs. (VII-7). As a consequence, by our introductory comments and Eq. (VII-1) we can form

$$\hat{\underset{\sim}{Y}}_c = \begin{bmatrix} \underset{\sim}{D} & -\underset{\sim}{C}\underset{\sim}{P}^{1/2} \\ \underset{\sim}{P}^{1/2}\underset{\sim}{B} & -\underset{\sim}{P}^{1/2}\underset{\sim}{A}\underset{\sim}{P}^{-1/2} \end{bmatrix} \qquad \text{(VII-10a)}$$

$$= \begin{bmatrix} Y_L(\infty) + Y_o(\infty) & -[C_L C_o](P_L^{-1/2} \dot{+} P_o^{-1/2}) \\ (P_L^{1/2} \dot{+} P_o^{1/2})\begin{bmatrix} B_L \\ B_o \end{bmatrix} & -(P_L^{1/2} \dot{+} P_o^{1/2})[A_L \dot{+} A_o](P_L^{-1/2} \dot{+} P_o^{-1/2}) \end{bmatrix} \quad \text{(VII-10b)}$$

By our previous reasoning $Y(p)-pC_\infty$ results from loading the resistive coupling network having the admittance matrix $\hat{Y}_c$ in δ unit capacitors, where δ is the degree of $Y(p)-pC_\infty$. Our claim is now that $\hat{Y}_c$ is PR if $Y(p)$ is, such that a circuit structure from $\hat{Y}_c$ need use only gyrators and positive resistors. That is, the choice $T = P^{-1/2}$ has allowed a completely passive synthesis of a PR admittance matrix.

To see that $\hat{Y}_c$ is PR we merely need to check to see if it has a positive semidefinite Hermitian part. Thus we form

$$\hat{Y}_c + \tilde{\hat{Y}}_c = \begin{bmatrix} D + \tilde{D} & -CP^{-1/2} + \tilde{B}P^{1/2} \\ P^{1/2}B - \tilde{P}^{-1/2}\tilde{C} & -P^{1/2}AP^{-1/2} - \tilde{P}^{-1/2}\tilde{A}\tilde{P}^{1/2} \end{bmatrix} \quad \text{(VII-11a)}$$

$$= (1_n \dot{+} P^{-1/2}) \begin{bmatrix} \tilde{W}_\infty W_\infty & -C + \tilde{B}P \\ PB - \tilde{C} & -PA - \tilde{A}P \end{bmatrix} (1_n \dot{+} P^{-1/2}) \quad \text{(VII-11b)}$$

$$= (1_n \dot{+} P_L^{-1/2} \dot{+} P_o^{-1/2}) \begin{bmatrix} \tilde{W}_\infty W_\infty & 0 & -\tilde{W}_\infty L_o \\ 0 & 0 & 0 \\ -\tilde{L}_o W_\infty & 0 & \tilde{L}_o L_o \end{bmatrix} (1_n \dot{+} P_L^{-1/2} \dot{+} P_o^{-1/2}) \quad \text{(VII-11c)}$$

where we have used the fact that $P^{1/2}$ is symmetric, $\tilde{P}^{1/2} = P^{1/2}$, as well as Eqs. (VII-3) in their extended form valid for the inclusion of lossless parts, Eq. (VII-7). That is, W_∞ is that $W(\infty)$ which corresponds to $Y_o(p)$ while $L = [0, L_o]$. If A_∞ has rank r, that is if r is the rank of $Y_o(p) + \tilde{Y}_o(p)$, then we can rewrite Eq. (VII-11c) as

$$\hat{Y}_c + \tilde{\hat{Y}}_c = \begin{bmatrix} \tilde{W}_\infty \\ 0 \\ -\tilde{P}_o^{-1/2}\tilde{L}_o \end{bmatrix} 1_r [W_\infty, 0, -L_o P_o^{-1/2}] \quad \text{(VII-11d)}$$

As shown by Section III D), $\underset{\sim}{Y}_c$ can now be synthesized by gyrators and r positive resistors. For instance Fig. III-8 applies to synthesize the symmetric part, which is one-half of Eq. (VII-11d), with r unit resistors and a gyrator coupling network described by the gyrator conductance matrix

$$\underset{\sim}{G} = \frac{1}{\sqrt{2}}\begin{bmatrix} \tilde{\underset{\sim}{W}}_{co} \\ \underset{\sim}{0} \\ -\underset{\sim}{P}_o^{-1/2}\underset{\sim}{L}_o \end{bmatrix} \qquad \text{(VII-11e)}$$

We comment that zeros in $\underset{\sim}{G}$ which designate rows and columns of zeros in the symmetric part of $\hat{\underset{\sim}{Y}}_c$ are as expected since they are associated with the lossless part $\underset{\sim}{Y}_L(p)$ for which no resistors are necessary. In fact since

$$r = \operatorname{rank}[\underset{\sim}{Y}(p)+\underset{\sim}{Y}(-p)] \qquad \text{(VII-11f)}$$

and since this rank corresponds to the minimum number of resistors possible in a synthesis, we see that besides using a minimum number of capacitors this method uses the minimum number of resistors. In fact in the case where the original $\underset{\sim}{Y}(p)$ is lossless, $\underset{\sim}{G}$ of Eq. (VII-11e) reduces completely to zero. Of course the vanishing of the $\underset{\sim}{P}_L$ portions of Eq. (VII-11d) does not mean that $\underset{\sim}{P}_L$ never enters into consideration; for example $\underset{\sim}{P}_L$ occurs in the skew-symmetric portion which acts through gyrators to couple the capacitors to the input ports in a lossless manner.

E. Example

Let us apply the method to the PR scalar

$$y(p) = \frac{4p^3 + 2p^2 + 18p}{p^3 + 2p^2 + 4p + 8} \qquad \text{(VII-12a)}$$

$$= \frac{2p}{p^2+4} + \frac{4p}{p+2} \qquad \text{(VII-12b)}$$

The latter split gives the decomposition into lossless and nonlossless parts; thus $y_L(p) = 2p/(p^2+4)$, $y_o(p) = 4p/(p+2)$.

For y_L and y_o appropriate realizations R_L and R_o are obtained from Eq. (IV-3) as

$$R_L = \left\{ \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, [1,\ 0],\ [0] \right\} \qquad \text{(VII-12c)}$$

$$R_o = \{[-2],\ [-8],\ [1],\ [4]\} \qquad \text{(VII-12d)}$$

For y_o we have

$$y_o(p) + y_o(-p) = \frac{-8p^2}{-p^2+4} = \left(\frac{-2\sqrt{2}p}{-p+2}\right)\left(\frac{2\sqrt{2}p}{p+2}\right) \qquad \text{(VII-12e)}$$

We observe that $W(p)$ is unique to within a minus sign; we choose

$$W(p) = \frac{2\sqrt{2}p}{p+2} \qquad \text{(VII-12f)}$$

for which a realization following Eq. (IV-3) is $\{[-2], [-4\sqrt{2}], [1], [2\sqrt{2}]\}$. We thus desire to choose a transformation $T = 1/\sqrt{2}$ to bring this B of $-4\sqrt{2}$ to $T^{-1}B = -8$. Thus we have as the appropriate realization R_w for W

$$R_w = \{[-2],\ [-8],\ [1/\sqrt{2}],\ [2\sqrt{2}]\} \qquad \text{(VII-12g)}$$

We have at this point $L_o = 1/\sqrt{2}$ and $W_\infty = 2\sqrt{2}$. The transformation P_o is found from

$$P_oA_o + A_oP_o = -4P_o = -L_oL_o' = -1/2 \qquad \text{(VII-12h)}$$

or

$$P_o = 1/8, \quad P_o^{1/2} = 1/2\sqrt{2} \qquad \text{(VII-12i)}$$

To find P_L we observe that $y_L(p) + y_L(-p) = 0$ in which case $L_L = 0$ and we simply solve for a positive definite P_L satisfying $P_L A_L + \tilde{A}_L P_L = 0_2$, that is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(VII-12j)}$$

The (1,1) and (2,2) entries require $p_{12} = 0$ while the (1,2) entry gives $p_{11} = 4p_{22}$ as does the (2,1) term. Positive definiteness merely requires $p_{22} > 0$ while $P_L B_L = \tilde{C}_L$ requires $8p_{22} = 1$. Thus

$$P_L = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/8 \end{bmatrix}, \qquad P_L^{1/2} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/2\sqrt{2} \end{bmatrix} \qquad \text{(VII-12k)}$$

Now the original coupling admittance, before the application of $P^{1/2}$ is

$$Y_c = \left[\begin{array}{c|cc|c} 4 & -1 & 0 & -1 \\ \hline 2 & 0 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ \hline -8 & 0 & 0 & 2 \end{array}\right] \qquad \text{(VII-12l)}$$

which is not PR as can be seen by the principal middle submatrix A_L. We then form

$$\hat{Y}_c = [1 \dotplus P^{1/2}] Y_c [1 \dotplus P^{-1/2}] \qquad \text{(VII-12m)}$$

where $P^{1/2} = P_L^{1/2} \dotplus P_o$ or

$$\hat{\mathbf{Y}}_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ -8 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -\sqrt{2} & 0 & -2\sqrt{2} \\ \sqrt{2} & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 2 \end{bmatrix}$$

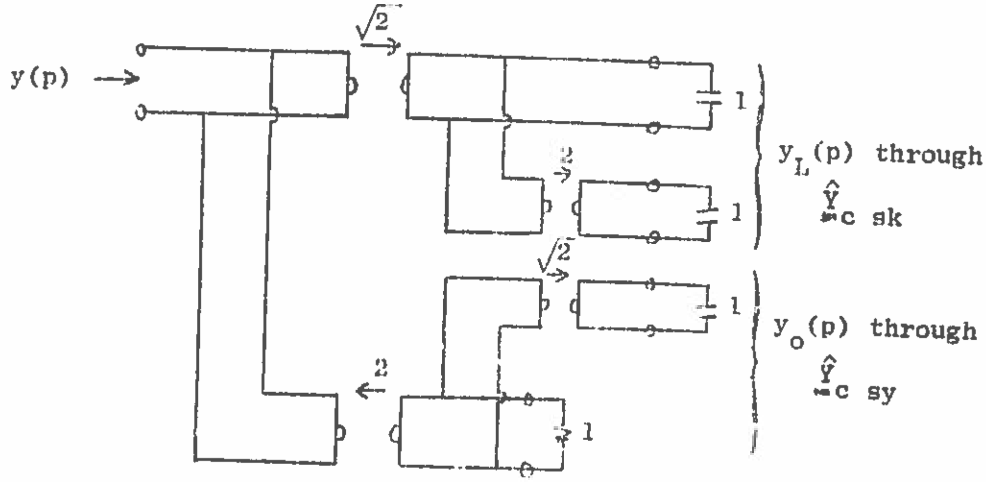
Finally we have for $\hat{\mathbf{Y}}_c = \hat{\mathbf{Y}}_{c\ sy} + \hat{\mathbf{Y}}_{c\ sk}$

$$\hat{\mathbf{Y}}_{c\ sy} = \begin{bmatrix} 4 & 0 & 0 & -2\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -\sqrt{2} \end{bmatrix} [2 \;\; 0 \;\; 0 \;\; -\sqrt{2}]$$

$$\hat{\mathbf{Y}}_{c\ sk} = \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $\hat{\mathbf{Y}}_{c\ sy}$ takes the form predicted by Eq. (VII-11d). The final circuit diagram is shown in Fig. VII-2.

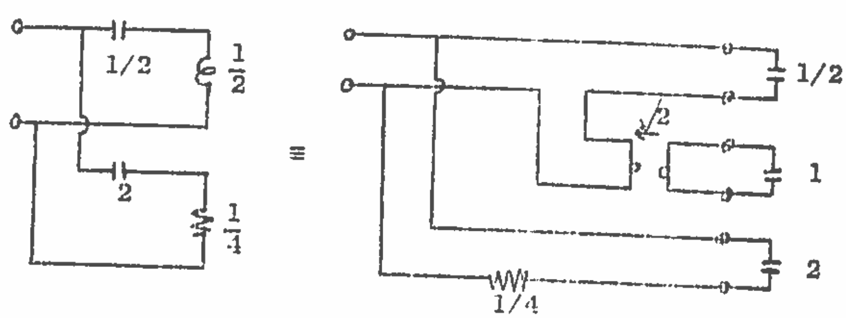
In the figure we observe that y_L and y_o are separately realized and then connected in parallel. In all situations $\hat{\mathbf{Y}}_{c\ sy}$ will be associated only with $y_o(p)$ but in this case the skew symmetric part has only occurred while synthesizing $y_L(p)$. Note that even though a minimum number of capacitors and resistors have been used an excess number of gyrators occurs. By shifting elements through the gyrators



Final Example Configuration

Figure VII-2

we can easily obtain Fig. VII-3a) from Fig. VII-2, or by direct synthesis. Decomposing this latter circuit yields the resistive circuit of Fig. VII-3b), loaded in capacitors. We observe however that this latter configuration possesses no admittance coupling matrix. Our conclusion is that always our synthesis of PR admittances will work but that in some instances more than the minimum number of gyrators will be used, though never more than the minimum number of capacitors and resistors is needed.



a) b)

Minimal Gyrator Realization

Figure VII-3

F. Discussion

In this chapter we have presented a method of synthesis of positive-real rational admittance matrices, and by duality impedance matrices. The method is based upon only algebraic operations and thus is readily programmed on a computer. The key point of the theory is the proper application of the PR Lemma to obtain the appropriate transformation. However it is in the application of this Lemma where the greatest difficulty occurs since a rather complicated factorization of the para-Hermitian part of $\underset{\sim}{Y}(p)$ sometimes must be undertaken in order to obtain $\underset{\sim}{W}(p)$. For nonpositive-real matrices or positive-real matrices of infinite dimension similar steps appear to be possible but these have not been extensively studied.

The ideas of the method can be applied to a hybrid coupling matrix in such a manner that some promise holds for minimal gyrator synthesis [4]. That is, $\underset{\sim}{Y}_c$ can be interpreted as a hybrid matrix if some ports are loaded in inductors in place of capacitors; in such a case one still desires $\underset{\sim}{Y}_c$ PR when $\underset{\sim}{Y}(p)$ is. Alternately, by using the hybrid interpretation one can give a synthesis in terms of the cut set and tie set matrices previously studied, at least in the lossless (and gyratorless) case [5]. However, as with the minimal gyrator situation improved conditions are still needed to complete the theory. Nevertheless the nonlinear theory has been interestingly investigated [6].

Because of the situation illustrated in Fig. III-3, where no coupling admittance matrix exists, it seems important to extend the method to scattering matrices where partial results of the PR Lemma type are available [7]. The work of Youla and Tissi represents a step in this direction [8].

Since it was possible to find one transformation $\underset{\sim}{T}$ taking any minimal realization into a passive one it is of interest to find all such $\underset{\sim}{T}$. As yet little solid theory is available in this direction but the theory of continuous transformation groups seems applicable.

G. References

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H. Exercises

1. Synthesize the PR impedance matrix

$$Z(p) = \frac{1}{p^2+2}\begin{bmatrix} p & 1 \\ 1 & 4p \end{bmatrix} + \frac{1}{p+1}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

by converting to the admittance matrix and applying the methods of this chapter.

2. Fill in the steps of the PR Lemma proof.

*3. Discuss a positive-real lemma for nonrational matrices and how this might be used for synthesis.

4. Investigate possible methods of determining $W(p)$, [3], [1, p. 168] and discuss the simplest for machine calculation.

5. Show that the synthesis of the text uses both the minimum number of resistors and capacitors.

*6. For the example of the text:

   a) find all minimal realizations and isolate those for which $\hat{Y}_c$ is PR.

   b) investigate possible ways of accounting for Fig. VII-3.

   c) find all minimal realizations on a scattering matrix basis.

7. Discuss the various methods of calculating the matrix P [9].

*8. Investigate methods of synthesizing bounded-real rational matrices by the techniques of the text [7].

*9. Show how the same techniques can be extended to cover nonminimal synthesis of PR matrices and discuss how such may be of importance for minimal gyrator synthesis.

10. Apply the PR Lemma to show how to synthesize through the equations [10]

$$\dot{s} = \frac{1}{2}(A-\tilde{A})s + Bv - \frac{1}{\sqrt{2}}\tilde{L}v^*$$

$$i = \tilde{B}s$$

$$i^* = \frac{1}{\sqrt{2}} Ls$$

subject to $i = -v^*$. In particular show that a network realization occurs by terminating the gyrator network

$$Y_c = \begin{bmatrix} 0 & 0 & -\tilde{B} \\ 0 & 0 & \frac{1}{\sqrt{2}}L \\ B & \frac{-L}{\sqrt{2}} & \frac{1}{2}(A-\tilde{A}) \end{bmatrix}$$

in unit resistors and unit capacitors. Show that the minimum number of resistors and capacitors are used.

Ils semblaient par moments nous regarder du haut d'une tour. Il est vrai que rien n'est caché; et vous tous qui me rencontrez, vous savez ce que j'ai fait et ce que je ferai, vous savez ce que je pense et ce que j'ai pensé.

M. Maeterlinck
"Les Avertis" du "Tresor les Humbles"

CHAPTER VIII

LUMPED-DISTRIBUTED LOSSLESS SYNTHESIS

A. Summary

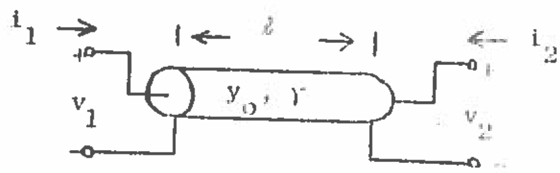
Here we briefly summarize the application of the previously discussed techniques to the synthesis of networks constructed of lossless lumped circuit elements and LC transmission lines. The theory is based upon the use of frequency transformations to obtain lossless but rational 2-variable matrices.

B. Introductory Material

We first review some properties of LC transmission lines as well as a method of treating circuits constructed from lumped circuit elements in conjunction with the LC lines. This will lead us to positive-real and rational 2-variable matrices and their synthesis. As we will see the admittance description, which we adhere to, is not rational in the true frequency variable, and as a consequence we introduce a second frequency variable to obtain rationality.

Let us first consider a lossless transmission line of length ℓ and inductance L and capacitance C per unit length. As shown in Fig. VIII-1 this line can be treated as a 2-port having the admittance matrix [1, p. 66]

$$\underset{\sim}{Y}(p) = y_o \begin{bmatrix} \operatorname{ctnh} \gamma p & -\operatorname{csch} \gamma p \\ -\operatorname{csch} \gamma p & \operatorname{ctnh} \gamma p \end{bmatrix} \qquad \begin{matrix} \gamma = \sqrt{LC}\,\ell \\ y_o = \sqrt{C/L} \end{matrix} \tag{VIII-1a}$$



Lossless Transmission Line

Figure VIII-1

We observe that the admittance matrix $\underset{\sim}{Y}(p)$ is not rational in p but that the positive-real frequency transformation

$$\lambda = \operatorname{ctnh}(\gamma p/2) \qquad \text{(VIII-1b)}$$

yields a rational positive-real admittance description

$$\textcircled{\underset{\sim}{Y}}(\lambda) = \underset{\sim}{Y}(p) = y_o \begin{bmatrix} \frac{1+\lambda^2}{2\lambda} & \frac{1-\lambda^2}{2\lambda} \\ \frac{1-\lambda^2}{2\lambda} & \frac{1+\lambda^2}{2\lambda} \end{bmatrix} \qquad \text{(VIII-1c)}$$

In fact we observe that any transmission line which has its γ an integer multiple of this basic line also has an admittance matrix which is rational in λ. Since given a set of transmission lines for which the γ's are rationally related there always exists a smallest γ for which the admittance description is Eq. (VIII-1c), we will assume that all lines under consideration are rationally related, that is have rationally related γ's.

If next we assume the presence of lumped capacitors, inductors and gyrators, as well as the rationally related LC lines considered in the λ plane, a node analysis yields branch admittances of the form

$$y_{ij} = C_{ij}p + (1/L_{ij}p) + c_{ij}\lambda + (1/\ell_{ij}\lambda) + g_{ij} \qquad \text{(VIII-2a)}$$

and for $i \neq j$

$$y_{ji} = C_{ij}p + (1/L_{ij}p) + c_{ij}\lambda + (1/\ell_{ij}\lambda) - g_{ij} \qquad \text{(VIII-2b)}$$

Such a network we will call lumped distributed. We note that for passive elements a lumped distributed network has an admittance matrix $\underset{\sim}{y}(p,\lambda)$ at any ports which is positive-real in both variables and satisfies the lossless constraint

$$\underset{\sim}{y}(p,\lambda) = -\underset{\sim}{\tilde{y}}(-p,-\lambda) \qquad \text{(VIII-2c)}$$

In actual fact $\underset{\sim}{y}(p,\lambda)$ satisfies the 2-variable positive-real constraints. That is, by definition a matrix is 2-variable positive-real if [2, p. 252]

a) $\underset{\sim}{y}(p,\lambda)$ is holomorphic in $\operatorname{Re} p > 0$, $\operatorname{Re} \lambda > 0$,

b) $\underset{\sim}{y}(p,\lambda)$ is real for p and λ real in $\operatorname{Re} p > 0$, $\operatorname{Re} \lambda > 0$,

c) the Hermitian part of $\underset{\sim}{y}(p,\lambda)$ is positive semi-definite in $\operatorname{Re} p > 0$, $\operatorname{Re} \lambda > 0$.

A rational 2-variable positive-real matrix will also be called PR.

A property of interest for synthesis is that the poles on the imaginary axes can be separately extracted to yield [3, p. 34]

$$\underset{\sim}{y}(p,\lambda) = \underset{\sim}{y}_1(p) + \underset{\sim}{y}_2(\lambda) + \underset{\sim}{y}_o(p,\lambda) \qquad \text{(VIII-3)}$$

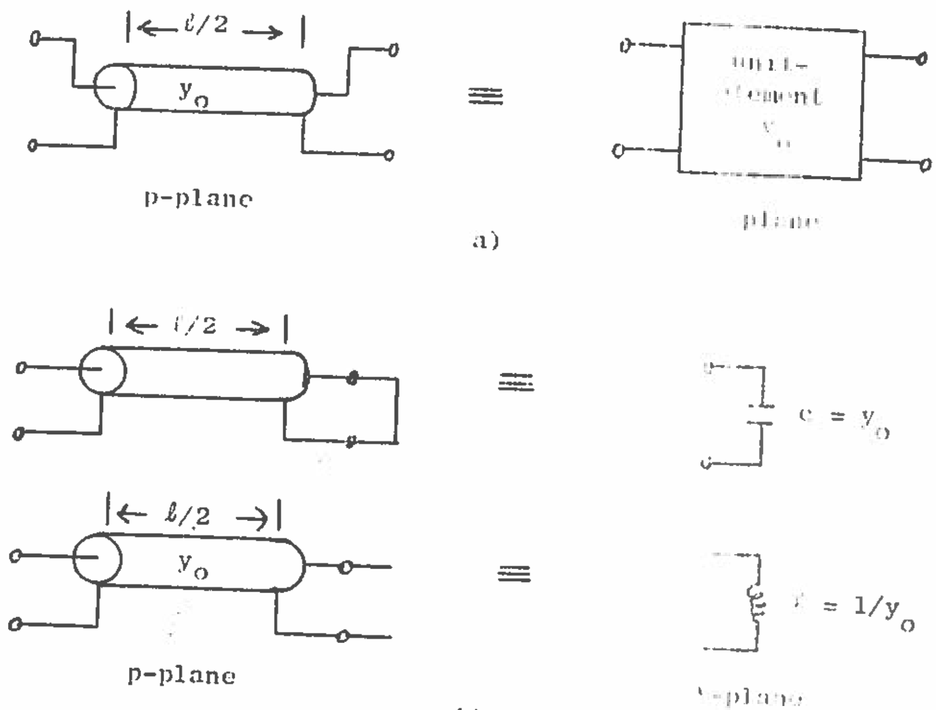
where $\underset{\sim}{y}_o$, $\underset{\sim}{y}_1$, and $\underset{\sim}{y}_2$ are all separately positive-real and rational when $\underset{\sim}{y}$ is rational; here $\underset{\sim}{y}_o$ has only poles which explicitly depend upon both p and λ. Of primary interest is the fact that $\underset{\sim}{y}_o(p,\lambda)$ has no poles at infinity in either variable.

To head toward synthesis it is of importance to note that those lines which have lengths one-half the basic length, called unit-elements, are described by

$$\textcircled{\underset{\sim}{Y}}_{ue}(\lambda) = \underset{\sim}{Y}_{ue}(p) = y_o \begin{bmatrix} \lambda & -\sqrt{\lambda^2-1} \\ -\sqrt{\lambda^2-1} & \lambda \end{bmatrix} \qquad \text{(VIII-4)}$$

Although such a description is not rational we observe that when loaded in a short circuit the unit-element appears as a capacitor of capacitance

y_o in the λ-plane when observed at the input. Similarly loading in an open circuit yields a λ-plane inductor at the input. These relationships can be depicted as shown in Fig. VIII-2.



p vs. λ-Plane Elements

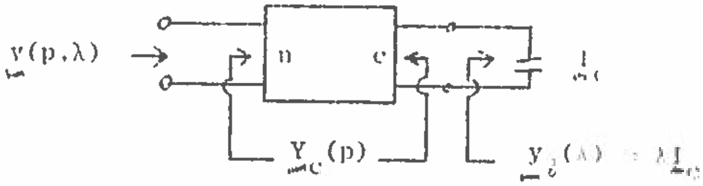
Figure VIII-2

With this last observation we see that a synthesis method could possibly arise by loading a p-plane (n+c)-port described by

$$\underset{\sim}{Y}_{nc}(p) = \begin{bmatrix} \underset{\sim}{y}_{11}(p) & \underset{\sim}{y}_{12}(p) \\ \underset{\sim}{y}_{21}(p) & \underset{\sim}{y}_{22}(p) \end{bmatrix} \qquad \text{(VIII-5a)}$$

by a set of c unit λ-plane capacitors (which are p-plane shorted unit-elements), as shown in Fig. VIII-3. If such occurs then one obtains

$$\underset{\sim}{y}(p,\lambda) = \underset{\sim}{y}_{11}(p) - \underset{\sim}{y}_{12}(p)[\lambda \underset{\sim}{1}_c + \underset{\sim}{y}_{22}(p)]^{-1}\underset{\sim}{y}_{21}(p) \qquad \text{(VIII-5b)}$$



Possible Configuration

Figure VIII-3

We observe the following. In the general expansion of a 2-variable PR matrix, Eq. (VIII-3), the matrix $\underset{\sim}{y}_2(\lambda)$ can not be absorbed in Eq. (VIII-5b) while $\underset{\sim}{y}_1(p)$ can. However, both $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2$ can be synthesized by standard methods with the resulting networks being placed in parallel with that for $\underset{\sim}{y}_o$. Hence we really need only consider Eq. (VIII-5b) for $\underset{\sim}{y}_o(p,\lambda)$. Now Eq. (VIII-5b) is in the form of previous results except that the realization matrices vary with p. Thus we are after a realization $R(p) = \{\underset{\sim}{A}(p), \underset{\sim}{B}(p), \underset{\sim}{C}(p), \underset{\sim}{D}(p)\} = \{-\underset{\sim}{y}_{22}, \underset{\sim}{y}_{12}, -\underset{\sim}{y}_{21}, \underset{\sim}{y}_{11}\}$ in which case the previous theory should hold. In fact we can use the methods of Chapter IV to create a minimal realization $R(p)$. However, the transformation to bring $\underset{\sim}{Y}_c(p)$ to be PR though obtainable in theory is not known in explicite form. Thus we proceed by directly finding a PR coupling admittance, this being possible because of the lossless nature imposed.

C. Minimal Realization Creation

To obtain a realization $R(p)$ for an $n\times n$ PR $\underset{\sim}{y}_o(p,\lambda)$, for which $\underset{\sim}{Y}_c(p)$ is also PR we will simply modify the previous realization theory, presenting the method of Rao [4], in some places omitting the details of proof which can be rather lengthy for their content.

As before we write

$$\underset{\sim}{y}_o(p,\lambda) = \frac{\lambda^r \underset{\sim}{B}_{r+1}(p) + \lambda^{r-1}\underset{\sim}{B}_r(p) + \ldots + \lambda\underset{\sim}{B}_2(p) + B_1(p)}{\lambda^r + a_r(p)\lambda^{r-1} + \ldots + a_1(p)}$$

$$= \underset{\sim}{A}_{-1}(p) + \sum_{i=0}^{\infty} \frac{\underset{\sim}{A}_i(p)}{[illegible]^{i+1}} \qquad \text{(VIII-6a)}$$

where the latter is the expansion about [illegible]. The [illegible] companion matrix is defined as

$$\underset{\sim}{\Omega}_n(p) = \begin{bmatrix} \underset{\sim}{0}_n & \underset{\sim}{1}_n & [illegible] & \cdots & \vdots \\ \underset{\sim}{0}_n & \underset{\sim}{0}_n & [illegible] & & \underset{\sim}{0} \\ \vdots & \vdots & & & \vdots \\ -a_1(p)\underset{\sim}{1}_n & -a_2(p)\underset{\sim}{1}_n & \cdots & & -a_r(p)\underset{\sim}{1}_n \end{bmatrix} \qquad \text{(VIII-6b)}$$

and a modified Hankel matrix defined by

$$\underset{\sim}{T}_r(p) = \begin{bmatrix} \underset{\sim}{A}_0(p) & \underset{\sim}{A}_1(p) & \cdots & \underset{\sim}{A}_{r-1}(p) \\ -\underset{\sim}{A}_1(p) & -\underset{\sim}{A}_2(p) & \cdots & -\underset{\sim}{A}_r(p) \\ \underset{\sim}{A}_2(p) & \underset{\sim}{A}_3(p) & \cdots & \underset{\sim}{A}_{r+1}(p) \\ \vdots & & & \vdots \\ (-1)^{r-1}\underset{\sim}{A}_{r-1}(p) & & \cdots & (-1)^{r-1}\underset{\sim}{A}_{2r-2}(p) \end{bmatrix} \qquad \text{(VIII-6c)}$$

Because of the lossless nature of $\underset{\sim}{Y}_0$, $\underset{\sim}{T}_r(p)$ [illegible] equal to $T_r(-p)$ [the para-Hermitian property] and it is non[illegible] semi-definite for $p = j\omega$. Consequently $\underset{\sim}{T}_r(p)$ can be factored, [illegible] the method used at Eq. (VIII-5a), to obtain

$$\underset{\sim}{T}_r(p) = \underset{\sim}{U}(p)\underset{\sim}{U}(-p) \qquad \text{(VIII-6d)}$$

where $\underset{\sim}{U}(p)$ as well as its <u>left inverse</u> $\underset{\sim}{U}^{-1}(p)$ are holomorphic in $\mathrm{Re}\ p > 0$; this factorization preserves the [illegible] nature of $\underset{\sim}{T}_r$, that is, $\underset{\sim}{U}(p)$ is also rational with real coefficients. Further the matrix $\underset{\sim}{U}$ can be taken of size $nr \times \delta_\lambda$ where δ_λ is the rank of $\underset{\sim}{T}_r(p)$ and then partitioned into $n \times \delta_\lambda$ blocks to define the entries in

$$\tilde{\underset{\sim}{U}}(p) = [\underset{\sim}{U}_o(p), \underset{\sim}{U}_1(p), \ldots, \underset{\sim}{U}_{r-1}(p)] \qquad \text{(VIII-6e)}$$

Noting that $\underset{\sim}{A}_o(p) = -\underset{\sim}{y}_{12}(p)\underset{\sim}{y}_{21}(p) = \underset{\sim}{U}_o(p)\tilde{\underset{\sim}{U}}_o(p)(-p)$ we see that as we desire $\underset{\sim}{y}_{21}(p) = -\tilde{\underset{\sim}{y}}_{12}(-p)$ because of the lossless constraint, we are led to define

$$\underset{\sim}{y}_{12}(p) = -\tilde{\underset{\sim}{y}}_{21}(-p) = \underset{\sim}{U}_o(p) \qquad \text{(VIII-7a)}$$

Noting further the previous method of defining $\underset{\sim}{A}$ by Eq. (IV-9) somewhat justifies the definition

$$\underset{\sim}{y}_{22}(p) = \underset{\sim}{U}_\ell^{-1}(p)\underset{\sim}{\Omega}_n(p)\underset{\sim}{U}_o(p) \qquad \text{(VIII-7b)}$$

Of course we also define

$$\underset{\sim}{y}_{11}(p) = \underset{\sim}{y}_o(p,\infty) \qquad \text{(VIII-7c)}$$

With these the coupling admittance matrix of Eq. (VIII-5a) is completely specified. In fact $\underset{\sim}{Y}_c(p)$ is PR and satisfies the lossless condition $\underset{\sim}{Y}_c(p) = -\tilde{\underset{\sim}{Y}}_c(-p)$ though both these properties, especially the PR one, are rather delicate to prove; the interested reader is referred to [4]. Further, the degree of $\underset{\sim}{Y}_c(p)$ is the minimum possible and equal to the p degree of $\underset{\sim}{y}_o(p,\lambda)$ defined as $\delta_p = \max_{\lambda_o} \delta[\underset{\sim}{y}_o(p,\lambda_o)]$. The number of λ-plane capacitors is equal to δ_λ where in fact $\delta_\lambda = \max_{p_o} \delta[\underset{\sim}{y}_o(p_o,\lambda)] = \text{rank}\, \underset{\sim}{T}_r(p)$. We comment that the whole process could have been undertaken by making p-plane capacitor extractions from which we conclude that δ_p represents the minimum possible number of p-plane reactive elements, while δ_λ gives the minimum number of λ-plane reactive elements.

In summary, loading the PR $(n+\delta_\lambda)\times(n+\delta_\lambda)$ matrix

$$\underset{\sim}{Y}_c(p) = \begin{bmatrix} \underset{\sim}{y}_o(p,\infty) & \underset{\sim}{U}_o(p) \\ -\tilde{\underset{\sim}{U}}_o(-p) & \underset{\sim}{U}_\ell^{-1}(p)\underset{\sim}{\Omega}_n(-p)\underset{\sim}{U}_o(p) \end{bmatrix} \qquad \text{(VIII-7d)}$$

realization in ξ_λ unit λ-plane capacitors (which are shorted unit-elements) yields $y_o(p,\lambda) = y_o(p, \operatorname{ctnh}[\gamma p/2])$ at the n input ports. A synthesis of the lossless coupling admittance $Y_c(p)$ by a minimum number of reactive p-plane reactive elements, which is readily possible [1, Chap 8], yields a network possessing a minimum number of lumped reactive elements as well as (p-plane) transmission lines.

D. Examples

Let us synthesize the function

$$y(p,\lambda) = \frac{\lambda(p/2) + 1}{\lambda + ([2+p^2]/2p)} \qquad \text{(VIII-8a)}$$

We have

$$T_r = \frac{1}{2} - \frac{p^2}{4} = [-(\frac{1}{\sqrt{2}} + \frac{p}{2})][-(\frac{1}{\sqrt{2}} - \frac{p}{2})] \qquad \text{(VIII-8b)}$$

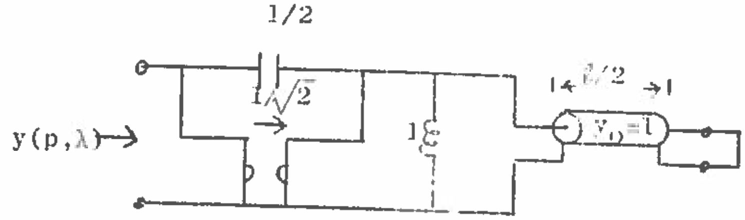
with

$$U_o(p) = -\frac{1}{2}(p+\sqrt{2}), \qquad U_\ell^{-1} = \frac{-2}{p+\sqrt{2}}, \qquad \eta = -\frac{2+p^2}{2p} \qquad \text{(VIII-8c)}$$

for which

$$Y_c(p) = \frac{1}{2}\begin{bmatrix} p & p+\sqrt{2} \\ -p+\sqrt{2} & p+\frac{2}{p} \end{bmatrix} \qquad \text{(VIII-8d)}$$

Using a short circuited transmission line for the λ-plane capacitor yields the circuit of Fig. VIII-4.



Example Synthesis

Figure VIII-4

To illustrate the difficulties of the more general situations consider the lossless PR

$$y(p,\lambda) = \frac{\lambda^2 p+2\lambda}{\lambda^2+p\lambda+2} \qquad \text{(VIII-9a)}$$

The expansion about $\lambda = \infty$ gives

$$y(p,\lambda) = p + \frac{(2-p^2)}{\lambda} + \frac{(p^3-4p)}{\lambda^2} + \frac{(-p^4+6p^2-4)}{\lambda^3} + \dots \qquad \text{(VIII-9b)}$$

Thus

$$\underset{\sim}{T}_2 = \begin{bmatrix} 2-p^2 & p^3-4p \\ -p^3+4p & -p^4+6p^2-4 \end{bmatrix} \qquad \text{(VIII-9c)}$$

One then needs to factor this as discussed at Eq. (VIII-6d), which is no simple task. Hence we drop this example at this point with the comment that a simple factorization to produce the holomorphic factor would be most welcome.

E. Symmetrization

As we saw in the last figure the method may use gyrators where actually none are apparently required. Here we show how these gyrators can be avoided by the procedure of Koga [3, p. 44].

Given the PR admittance $\underset{\sim}{Y}_c(p)$, of Eq. (VIII-7d) for example, if it is not already symmetric we form the following coupling admittance matrix

$$\underset{\sim}{Y}_S(p) = \begin{bmatrix} \underset{\sim}{y}_{11} & \underset{\sim}{y}_{12S} & -\underset{\sim}{y}_{12A} \\ \tilde{\underset{\sim}{y}}_{12S} & \underset{\sim}{y}_{22S} & -\underset{\sim}{y}_{22A} \\ \tilde{\underset{\sim}{y}}_{12A} & \tilde{\underset{\sim}{y}}_{22A} & \underset{\sim}{y}_{22S} \end{bmatrix} \qquad \text{(VIII-10a)}$$

where

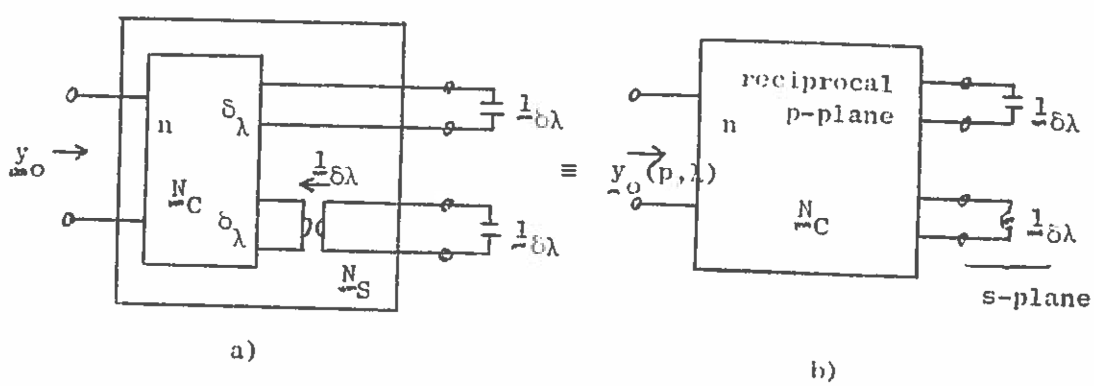
$$2\underset{\sim}{y}_{12S} = \underset{\sim}{y}_{12} + \underset{\sim}{\tilde{y}}_{21} \qquad 2\underset{\sim}{y}_{12A} = \underset{\sim}{y}_{12} - \underset{\sim}{\tilde{y}}_{21} \tag{VIII-10b}$$

$$2\underset{\sim}{y}_{22S} = \underset{\sim}{y}_{22} + \underset{\sim}{\tilde{y}}_{22} \qquad 2\underset{\sim}{y}_{22A} = \underset{\sim}{y}_{22} - \underset{\sim}{\tilde{y}}_{22} \tag{VIII-10c}$$

with the subscript S and A standing for the symmetric and (skew) antisymmetric parts. The matrix $\underset{\sim}{Y}_S(p)$ is PR and lossless with $\underset{\sim}{Y}_c(p)$, and $\underset{\sim}{y}_o(p,\lambda)$ results at the first n ports of a circuit realization by loading the final $2\delta_\lambda$ ports in unit λ-plane capacitors, as we will discuss below. If next we extract a (cascade) gyrator from each of the final δ_λ ports, as shown in Fig. VIII-5a), we obtain a symmetric coupling admittance matrix $\underset{\sim}{Y}_C$; for example, when (as is the normal situation) $\underset{\sim}{y}_{22S}$ is nonsingular

$$\underset{\sim}{Y}_C(p) = \begin{bmatrix} \underset{\sim}{y}_{11}+\underset{\sim}{y}_{12A}\underset{\sim}{y}_{22S}^{-1}\underset{\sim}{\tilde{y}}_{12A} & \underset{\sim}{y}_{12S}+\underset{\sim}{y}_{12A}\underset{\sim}{y}_{22S}^{-1}\underset{\sim}{\tilde{y}}_{22A} & \underset{\sim}{y}_{12A}\underset{\sim}{y}_{22S}^{-1} \\ \underset{\sim}{\tilde{y}}_{12S}+\underset{\sim}{y}_{22A}\underset{\sim}{y}_{22S}^{-1}\underset{\sim}{\tilde{y}}_{12A} & \underset{\sim}{y}_{22S}+\underset{\sim}{y}_{22A}\underset{\sim}{y}_{22S}^{-1}\underset{\sim}{\tilde{y}}_{22A} & \underset{\sim}{y}_{22A}\underset{\sim}{y}_{22S}^{-1} \\ \underset{\sim}{y}_{22S}^{-1}\underset{\sim}{\tilde{y}}_{12A} & \underset{\sim}{y}_{22S}^{-1}\underset{\sim}{\tilde{y}}_{22A} & \underset{\sim}{y}_{22S}^{-1} \end{bmatrix} \tag{VIII-10d}$$

The extracted gyrators can be combined with the loading capacitors to yield s-plane inductors while $\underset{\sim}{Y}_C(p)$ can be synthesized by a reciprocal, passive, lossless p-plane configuration. The overall structure is then reciprocal with $\underset{\sim}{y}_o(p,\lambda)$ and as shown in Fig. VIII-5b).



Procedure for Reciprocal Synthesis
of a Symmetric $\underset{\sim}{y}_o(p,\lambda)$

Figure VIII-5

To see why the method works let us reason as follows. Since $\underset{\sim}{y}_o(p,\lambda)$ is assumed symmetric we can write

$$\underset{\sim}{y}_o = \frac{1}{2}(\underset{\sim}{y}_o + \tilde{\underset{\sim}{y}}_o) = \underset{\sim}{y}_{11}(p) - \frac{1}{2}\{\underset{\sim}{y}_{12}(p)[\underset{\sim}{y}_{22}(p) + \lambda\underset{\sim}{1}_{\varepsilon\lambda}]^{-1}\underset{\sim}{y}_{21}(p) \qquad \text{(VIII-11a)}$$

$$+ \tilde{\underset{\sim}{y}}_{21}(p)[\tilde{\underset{\sim}{y}}_{22}(p) + \lambda\underset{\sim}{1}_{\varepsilon\lambda}]^{-1}\tilde{\underset{\sim}{y}}_{12}(p)\}$$

for which a realization is seen to come from the coupling admittance matrix

$$\underset{\sim}{Y}_1(p) = \begin{bmatrix} \underset{\sim}{y}_{11} & \frac{1}{\sqrt{2}}\underset{\sim}{y}_{12} & \frac{1}{\sqrt{2}}\tilde{\underset{\sim}{y}}_{21} \\ \frac{1}{\sqrt{2}}\underset{\sim}{y}_{21} & \underset{\sim}{y}_{22} & \underset{\sim}{0} \\ \frac{1}{\sqrt{2}}\tilde{\underset{\sim}{y}}_{12} & \underset{\sim}{0} & \tilde{\underset{\sim}{y}}_{22} \end{bmatrix} \qquad \text{(VIII-11b)}$$

That is, a circuit realization for $\underset{\sim}{Y}_1$ yields $\underset{\sim}{y}_o$ at the input when terminated by $2\varepsilon_\lambda$ unit λ-plane capacitors. Next we find an equivalent realization using Eq. (V-2a) with the orthogonal transformation

$$\underset{\sim}{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} \underset{\sim}{1}_{\varepsilon\lambda} & -\underset{\sim}{1}_{\varepsilon\lambda} \\ \underset{\sim}{1}_{\varepsilon\lambda} & \underset{\sim}{1}_{\varepsilon\lambda} \end{bmatrix} \qquad \text{(VIII-11c)}$$

Thus we obtain

$$\underset{\sim}{Y}_s(p) = [\underset{\sim}{1}_n \dotplus \tilde{\underset{\sim}{T}}]\underset{\sim}{Y}_1(p)[\underset{\sim}{1}_n \dotplus \underset{\sim}{T}] \qquad \text{(VIII-11d)}$$

which gives Eq. (VIII-11a). The PR property as well as losslessness is preserved through these operations. Finally we comment that if $\underset{\sim}{y}_{22s}$ is not nonsingular for Eq. (VIII-10d) it can be made so by an orthogonal transformation to yield [illegible] $= \underset{\sim}{T}_1 \underset{\sim}{y}_{22s} \tilde{\underset{\sim}{T}}_1$ with [illegible] nonsingular (i.e. [illegible]).

The previous example of Eq. (VIII-8d) illustrates the procedure. We have

$$\underset{\sim}{Y}_S(p) = \frac{1}{2}\begin{bmatrix} p & -p & \sqrt{2} \\ -p & p + \frac{2}{p} & 0 \\ -\sqrt{2} & 0 & p + \frac{2}{p} \end{bmatrix} \qquad \text{(VIII-12a)}$$

Extraction of the gyrator at port three yields

$$\underset{\sim}{Y}_C(p) = \frac{1}{2}\begin{bmatrix} p + \frac{2p}{p^2+2} & -p & \frac{-\sqrt{2}p}{p^2+2} \\ -p & p + \frac{2}{p} & 0 \\ \frac{-\sqrt{2}p}{p^2+2} & 0 & \frac{p}{p^2+2} \end{bmatrix} \qquad \text{(VIII-12b)}$$

Synthesis of $\underset{\sim}{Y}_C(p)$, which is symmetric, yields $y_0(p,\lambda)$ at the input when the second port is loaded in a unit capacitor and the third port in a unit inductor, the latter two being p-plane short and open circuited LC transmission lines. Note however that four p-plane (lumped) reactive elements must be used to synthesize $\underset{\sim}{Y}_C(p)$, in contrast to the two used at Fig. VIII-4.

F. Discussion

Given a nonrational admittance matrix in p, $\underset{\sim}{Y}(p)$, if there exists a γ such that $\underset{\sim}{Y}(p) = \underset{\sim}{y}(p,\lambda)$ is rational, PR, and lossless in the two variables p and $\lambda = \text{ctanh}(\gamma p/2)$, then the procedures of this chapter can be used to obtain a synthesis. In particular the synthesis uses both lumped and distributed LC components, a minimum number of all types when gyrators are also allowed. If the original matrix is symmetric then also a series of operations can be used to eliminate the gyrators, but an excess number of reactive elements is needed for the given procedure, though it seems that other methods should be available to reduce this number.

In the treatment given we have extracted λ-plane elements as the load to obtain realization matrices which depend upon the other variable p, $R(p) = \{\underset{\sim}{A}(p), \underset{\sim}{B}(p), \underset{\sim}{C}(p), \underset{\sim}{D}(p)\}$. Of course we could have reversed the role of λ and p since in $y(p,\lambda)$ there is no real preference. The only difference occurs in the synthesis where the extraction of the lumped p-plane elements means that the λ-plane coupling network needs to be synthesized in terms of distributed elements. This latter though can be conveniently carried out in terms of cascade synthesis methods using the unit-elements [1, Chap. 7] and is, thus, in some ways superior.

The same methods can be used for the synthesis of lumped-distributed RC networks of considerable interest to the theory of integrated circuits. For such one introduces a different variable $s = \sqrt{p}$. Then a given admittance $Y(p)$ can be synthesized by a synthesis of the lossless admittance [5]

$$y(s,\lambda) = \frac{1}{\sqrt{p}} Y(p) \qquad \text{(VIII-13)}$$

Such a synthesis can follow that of the text with the s-plane elements replaced by resistors (for the inductors) and capacitors while the λ-plane elements are replaced by RC lines to obtain the original p-plane $Y(p)$.

In the case where there are nonrationally related lines the methods discussed can be extended by considering ν-variable matrices, with $\nu > 2$. Although minimal realizations can relatively easily be given, as yet it has not been possible to obtain a PR coupling admittance in terms of $\nu-1$ of the variables.

G. References

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H. Exercises

1. Synthesize the lossless PR

a) $y(p,\lambda) = \dfrac{2\lambda(2p^2+1)}{4\lambda p+2p^2+1}$

b) $y(p,\lambda) = \dfrac{4\lambda p+2p^2+1}{2\lambda(2p^2+1)}$

2. Prove that $\underset{\sim}{Y}_c(p)$ of Eq. (VIII-7d) is PR and lossless.
3. Carry out the steps for p-plane, instead of λ-plane, extractions.
4. Show that the gyrator extraction of Fig. VIII-5a) yields $\underset{\sim}{Y}_C(p)$ of Eq. (VIII-10d). Carry out the details when $\underset{\sim}{y}_{22S}$ is singular.
5. Obtain a realization for $\underset{\sim}{y}_o(p,\lambda)$ using the method of Chapter IV and show how to obtain the realization of this chapter from the other.
6. Analyze any 2-port lossless lumped-distributed circuit and from the resulting $\underset{\sim}{y}(p,\lambda)$ synthesize the network by the methods of this chapter. Compare the final circuit with the original and discuss the problems raised.

Il y a ainsi une part de la vie, -- et c'est la meilleure, la plus pure et la plus grande, -- qui ne se mêle pas à la vie ordinaire, et les yeux des amants eux-mêmes ne percent presque jamais cette digue de silence et d'amour.

M. Maeterlinck
"Les Avertis" du "Trésor des Humbles"

CHAPTER IX

TIME-VARIABLE SYNTHESIS

A. Summary

Using similar but generally somewhat different techniques than for time-invariant structures, realizations for impulse responses can be obtained and manipulated to yield synthesis results. Of primary interest is that discussed for transfer voltage functions and that for special types of admittances.

B. Properties of Impulse Response Matrices

If we are given the state-variable equations with time variable coefficients

$$\dot{\underline{s}}(t) = \underline{A}(t)\underline{s}(t) + \underline{B}(t)\underline{u}(t) \qquad \text{(IX-1a)}$$

$$\underline{y}(t) = \underline{C}(t)\underline{s}(t) + \underline{D}(t)\underline{u}(t) \qquad \text{(IX-1b)}$$

we can find the zero state output through

$$\underline{y}(t) = \int_{-\infty}^{\infty}[\underline{D}(t)\delta(t-\tau)+\underline{C}(t)\underline{\Phi}(t,\tau)\underline{B}(\tau)]\underline{u}(\tau)d\tau \qquad \text{(IX-1c)}$$

where $\underline{\Phi}(t,\tau)$ is the state transition matrix satisfying

$$\frac{d\underline{\Phi}(t,\tau)}{dt} = \underline{A}(t)\underline{\Phi}(t,\tau), \quad t > \tau \qquad \text{(IX-1d)}$$

$$\underline{\Phi}(\tau,\tau) = \underline{1}_k, \quad \underline{\Phi}(t,\tau) = \underline{0}_k \quad t < \tau \qquad \text{(IX-1e)}$$

In actual fact since $\underline{\Phi}$ satisfies the differential equation of Eq. (IX-1d) it can be shown [1, p.530] to be the product of two matrices, one in t and one in τ

$$\underline{\Phi}(t,\tau) = \underline{\Xi}(t)\underline{\Lambda}(\tau)1(t-\tau) \qquad \text{(IX-1f)}$$

where $1(t-\tau)$ is the unit step function. Further the number of rows in $\Lambda(\tau)$ can be assumed reduced to its minimal size ℓ, this size being independent of τ for reasonably behaved $\underline{A}(t)$ [1, p.530]

As a consequence, we can associate with the state-variable equations an impulse response matrix [here $\delta(t)$ is the unit impulse]

$$\underline{T}(t,\tau) = \underline{D}(t)\delta(t-\tau)+\underline{C}(t)\underline{\Xi}(t)\underline{\Lambda}(\tau)\underline{B}(\tau)1(t-\tau) \qquad \text{(IX-2a)}$$

such that

$$\underline{y}(t) = \int_{-\infty}^{\infty} \underline{T}(t,\tau)\underline{u}(\tau)d\tau \qquad \text{(IX-2b)}$$

This latter can be conveniently denoted as

$$\underline{y} = \underline{T}\bullet\underline{u} \qquad \text{(IX-2c)}$$

Since $\underline{T}$ contains impulses it is often referred to as a distributional kernel defining the mapping of $\underline{u}$ into $\underline{y}$, $\underline{y} = \underline{T}\bullet\underline{u}$. If we have two such mappings defined by kernels $\underline{T}_1$ and $\underline{T}_2$ we can apply one after another, as might occur in a cascade of voltage transfer functions. This leads to the definition of the composition $\underline{T}_1 \circ \underline{T}_2$ through

$$\underline{y} = \underline{T}_1\bullet[\underline{T}_2\bullet\underline{u}] = [\underline{T}_1\circ\underline{T}_2]\bullet\underline{u} \qquad \text{(IX-2d)}$$

As an integral this composition takes the form

$$\underset{\sim}{T}_1 \circ \underset{\sim}{T}_2(t,\tau) = \int_{-\infty}^{\infty} \underset{\sim}{T}_1(t,\lambda)\underset{\sim}{T}_2(\lambda,\tau)\,d\lambda \qquad \text{(IX-2e)}$$

Through the concept of composition the inverse of a $k \times k$ kernel can be defined by

$$\underset{\sim}{T}^{-1} \circ \underset{\sim}{T} = \underset{\sim}{T} \circ \underset{\sim}{T}^{-1} = \delta(t-\tau)\underset{\sim}{1}_k \qquad \text{(IX-2f)}$$

Consequently $\underset{\sim}{T}$ can be given the representation alternate to Eq. (IX-2a) as

$$\underset{\sim}{T} = \underset{\sim}{D}\delta + (\underset{\sim}{C}\delta) \circ [\delta' \underset{\sim}{1}_k - \underset{\sim}{A}\delta]^{-1} \circ (\underset{\sim}{B}\delta) \qquad \text{(IX-3a)}$$

Since $[\delta'(t-\tau)]^{-1} = 1(t-\tau)$ we see by comparison with this last expression that if we are given

$$\underset{\sim}{T}(t,\tau) = \underset{\sim}{H}(t)\delta(t-\tau) + \underset{\sim}{\Psi}(t)\underset{\sim}{\theta}(\tau)1(t-\tau) \qquad \text{(IX-3b)}$$

then a possible realization is

$$\underset{\sim}{A} = \underset{\sim}{0}, \quad \underset{\sim}{B} = \underset{\sim}{\theta}, \quad \underset{\sim}{C} = \underset{\sim}{\Psi}, \quad \underset{\sim}{D} = \underset{\sim}{H} \qquad \text{(IX-3c)}$$

This is minimal if the number of rows in $\underset{\sim}{\theta}$ has been minimized.

If we make a transformation on the state

$$\hat{\underset{\sim}{s}}(t) = \underset{\sim}{\mathcal{T}}(t)\underset{\sim}{s}(t) \qquad \text{(IX-4a)}$$

then, since the transforming matrix must now be differentiated we have

$$\hat{\underset{\sim}{A}} = \underset{\sim}{\mathcal{T}}\underset{\sim}{A}\underset{\sim}{\mathcal{T}}^{-1} + \dot{\underset{\sim}{\mathcal{T}}}\underset{\sim}{\mathcal{T}}^{-1}, \quad \hat{\underset{\sim}{B}} = \underset{\sim}{\mathcal{T}}\underset{\sim}{B}, \quad \hat{\underset{\sim}{C}} = \underset{\sim}{C}\underset{\sim}{\mathcal{T}}^{-1} \qquad \text{(IX-4b)}$$

Consequently, the freedom of using time-variable transformations allows one to change the structure of the $\underset{\sim}{A}$ matrix, resulting in some rather interesting phenomena.

C. Passive Voltage Transfer Function Synthesis

Let us consider the problem of synthesis of kernels mapping voltages into voltages; the material follows to a large extent the ideas of Silverman [2].

As a preliminary, let us first observe that if we define, for a given $\underset{\sim}{A}(t)$ and a fixed t_o,

$$\underset{\sim}{V}(t) = \int_{t_o}^{t} \underset{\sim}{\Phi}(t,\tau)\tilde{\underset{\sim}{\Phi}}(t,\tau)d\tau, \quad t > t_o \tag{IX-5a}$$

(which is positive definite) then the choice

$$\underset{\sim}{\mathcal{T}} = (\underset{\sim}{V}^{-1})^{1/2} \tag{IX-5b}$$

yields on using Eqs. (IX-1d, 4b)

$$\hat{\underset{\sim}{A}} + \tilde{\hat{\underset{\sim}{A}}} = -\underset{\sim}{V}^{-1} \tag{IX-5c}$$

As a consequence, from what we previously learned at Eq. (VII-11) we should be able to use this transformation for a passive synthesis. We comment, however, that $\underset{\sim}{V}(t)$ varies with time even in the time-invariant case so that slightly different procedures are preferable when $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ are constant.

As the next preliminary let us synthesize a voltage to current transfer function (kernel) $\underset{\sim}{T}$, $\underset{\sim}{i}_2' = \underset{\sim}{T}\bullet\underset{\sim}{v}_1$, where $\underset{\sim}{i}_2'$ and $\underset{\sim}{v}_1$ are measured at different ports. Given any realization, say the one of Eq. (IX-3c) let us perform the transformation of Eq. (IX-5b) to obtain

$$\dot{\hat{\underset{\sim}{s}}} = \hat{\underset{\sim}{A}}\hat{\underset{\sim}{s}} + \hat{\underset{\sim}{B}}\underset{\sim}{v}_1 \tag{IX-6a}$$

$$\underset{\sim}{i}_2' = \hat{\underset{\sim}{C}}\hat{\underset{\sim}{s}} + \underset{\sim}{D}\underset{\sim}{v}_1 \tag{IX-6b}$$

Let us next introduce another set of variables, the current $\underset{\sim}{i}_1$ associated with the first ports and $\underset{\sim}{v}_2'$ the voltage associated with the final ports to write

$$\dot{\hat{s}} = \hat{A}\hat{s} + [\hat{B},\hat{C}]\begin{bmatrix} v_1 \\ v_2' \end{bmatrix} \qquad \text{(IX-6c)}$$

$$\begin{bmatrix} i_1 \\ i_2' \end{bmatrix} = \begin{bmatrix} \tilde{\hat{B}} \\ \tilde{\hat{C}} \end{bmatrix}\hat{s} + \begin{bmatrix} 0 & -\tilde{D} \\ D & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2' \end{bmatrix} \qquad \text{(IX-6d)}$$

Note that if we set $v_2' = 0$ and ignore the input port currents i_1 then the original description is returned. However, as in the time-invariant case, Eqs. (IX-6c,d) define a coupling (time-variable) resistive network through

$$Y_c(t,\tau) = \begin{bmatrix} 0 & -\tilde{D}(t) & -\tilde{\hat{B}}(t) \\ D(t) & 0 & -\tilde{\hat{C}}(t) \\ \hat{B}(t) & \hat{C}(t) & -\hat{A}(t) \end{bmatrix}\delta(t-\tau) \qquad \text{(IX-6e)}$$

Note that, by virtue of Eq. (IX-5c)

$$Y_c + \tilde{Y}_c = 0 \dotplus V^{-1}(t)\delta(t-\tau) \qquad \text{(IX-6f)}$$

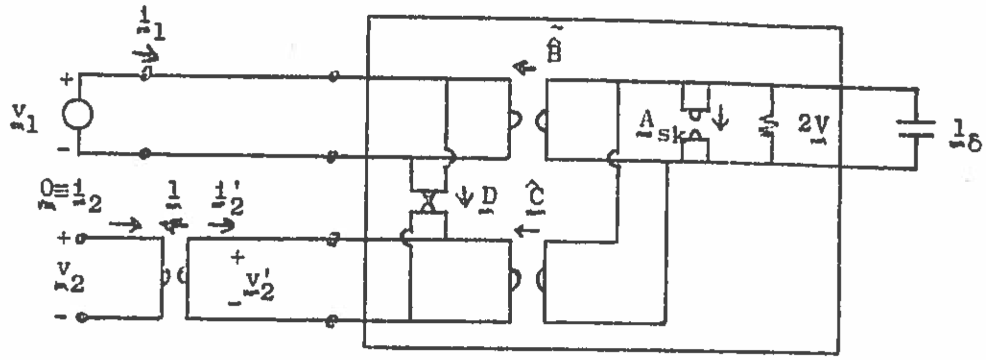
in which case Y_c can be synthesized by time-variable gyrators and resistors both of which are passive. Termination of the resultant network in unit capacitors yields Eq. (IX-6c,d). At the final ports we can next insert unit gyrators to obtain

$$v_2 = i_2', \qquad i_2 = v_2' \qquad \text{(IX-7a)}$$

Setting $v_2' = 0$ results in an open circuit load while $i_2' = v_2$ yields

$$v_2 = T \bullet v_1 \qquad \text{(IX-7b)}$$

As a consequence the procedure results, for $t > t_o$, in a passive realization of any $T(t,\tau)$ of the form of Eq. (IX-3b). Since practically such constructs are only used after a finite time, the $t > t_o$ restriction is of no practical restriction; but in some cases $t_o = -\infty$ can be used in which case the theory of Silverman results when $\underset{\sim}{D} = \underset{\sim}{0}$. The synthesis is summarized in Fig. IX-1.



Transfer Voltage Realization

Figure IX-1

As an example to illustrate the various points let us synthesize the time-invariant transfer function

$$\frac{V_2}{V_1} = \frac{3p}{p+2} = 3 + \frac{-6}{p+2} \qquad \text{(IX-8a)}$$

We have

$$T(t,\tau) = 3\delta(t-\tau) + (-6e^{-2t})(e^{2t})1(t-\tau) \qquad \text{(IX-8b)}$$

For a realization we can take

$$A = 0, \quad B = e^{2t}, \quad C = -6e^{-2t}, \quad D = 3 \qquad \text{(IX-8c)}$$

Then, for any fixed t_o,

$$V(t) = \int_{t_o}^{t} d\tau = t - t_o, \quad t > t_o \qquad \text{(IX-8d)}$$

which is positive definite for $t > t_o$ as expected; we have for Eq. (IX-5b)

$$\beta(t) = \frac{1}{\sqrt{t-t_o}} \qquad \text{(IX-8e)}$$

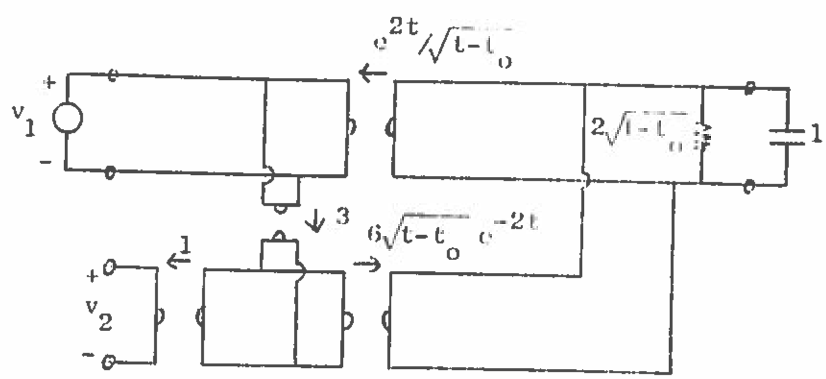
From the transformed realization equations we find

$$\hat{A} = \dot{\mathcal{T}}\mathcal{T}^{-1} = \frac{-1/2}{\sqrt{t-t_o}}, \quad \hat{B} = \frac{e^{2t}}{\sqrt{t-t_o}}, \quad \hat{C} = -6\sqrt{t-t_o}\, e^{-2t}, \quad D = 3 \qquad \text{(IX-8f)}$$

Thus $\underset{\sim}{Y}_c$ takes the form

$$\underset{\sim}{Y}_c(t,\tau) = \begin{bmatrix} 0 & -3 & -e^{2t}/\sqrt{t-t_o} \\ 3 & 0 & 6\sqrt{t-t_o}\, e^{-2t} \\ \frac{e^{2t}}{\sqrt{t-t_o}} & -6\sqrt{t-t_o}\, e^{-2t} & \frac{1/2}{\sqrt{t-t_o}} \end{bmatrix} \delta(t-\tau) \qquad \text{(IX-8g)}$$

The structure of the circuit realization is shown in Fig. IX-2. It should be observed that a) the elements are all passive, (b) the elements



Circuit for $V_2 = (3p/[p+2])V_1$

Figure IX-2

are time-variable even though the overall network is time-invariant, c) the elements become unbounded for t approaching t_o. If we would have chosen $A = -2$ and $t_o = -\infty$ this latter (unboundedness) could have been avoided while a slightly different approach (see the Exercises) would allow a time-invariant synthesis.

D. Passive Admittance Synthesis

Following the previous ideas we can form the coupling admittance matrix

$$Y_c(t,\tau) = \begin{bmatrix} D(t) & -C(t)\mathcal{T}^{-1}(t) \\ \mathcal{T}(t)B(t) & -\mathcal{T}(t)A(t)\mathcal{T}^{-1}(t)-\dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t) \end{bmatrix}\delta(t-\tau) \qquad \text{(IX-9a)}$$

We then wish for a passive synthesis to be able to choose $\mathcal{T}$ such that the symmetric part of Y_c is positive semidefinite. On evaluating this symmetric part we have, assuming a symmetric $\mathcal{T}$,

$$Y_c+\tilde{Y}_c = \begin{bmatrix} D+\tilde{D} & [\tilde{B}\mathcal{T}^2-C]\mathcal{T}^{-1} \\ \mathcal{T}^{-1}[\mathcal{T}^2B-\tilde{C}] & -\mathcal{T}^{-1}[\mathcal{T}^2A+\tilde{A}\mathcal{T}^2+[\dot{\mathcal{T}}^2]]\mathcal{T}^{-1} \end{bmatrix}\delta \qquad \text{(IX-9b)}$$

where we have also used $\dot{\mathcal{T}}^2 = \dot{\mathcal{T}}\mathcal{T}+\mathcal{T}\dot{\mathcal{T}}$. In the case where $D+\tilde{D}$, which is twice the symmetric part of D, is positive definite and A, B, C have bounded entries the (Riccati) equation

$$\mathcal{T}^2A+\tilde{A}\mathcal{T}^2+[\dot{\mathcal{T}}^2] = -[\mathcal{T}^2B-\tilde{C}][D+\tilde{D}]^{-1}[\tilde{B}\mathcal{T}^2-C] \qquad \text{(IX-9c)}$$

is known [3] to have a solution for a nonsingular <u>symmetric</u> $\mathcal{T}$. Consequently,

$$Y_c+\tilde{Y}_c = \begin{bmatrix} (D+\tilde{D})^{1/2} & 0 \\ 0 & \mathcal{T}^{-1}(\mathcal{T}^2B-\tilde{C})(D+\tilde{D})^{-1/2} \end{bmatrix}\begin{bmatrix} 1_n \\ 1_n \end{bmatrix}\begin{bmatrix} 1_n & 1_n \end{bmatrix}\begin{bmatrix} (D+\tilde{D})^{1/2} & 0 \\ 0 & (D+\tilde{D})^{-1/2}(B\mathcal{T}^2-C)\mathcal{T}^{-1} \end{bmatrix}$$

(IX-9d)

which shows that Y_c can be synthesized by n constant resistors loading time-variable gyrators (for the symmetric part) and time-variable gyrators (for the skew-symmetric part); here n is the number of terminal ports.

We conclude that if a given $n \times n$ admittance kernel

$$y(t,\tau) = D(t)\delta(t-\tau) + C(t)B(\tau)1(t-\tau) \qquad \text{(IX-10)}$$

has the symmetric part of D positive definite (as well as bounded minimal C and B matrices) then a passive synthesis can be given. Such results by solving the nonlinear Riccati equation, (IX-9c), for $\mathcal{T}(t)$, and forming Y_c which then yields $y(t,\tau)$ by loading of the passive coupling structure in unit capacitors. Several observations are worth noting. First $\mathcal{T}$ is very difficult to obtain, if not impossible practically, since a nonlinear variable coefficient differential equation must be solved. Second, Eq. (IX-9d) shows where difficulty arises if the symmetric part of D is singular; hence the method seems hard to extend to cover more general cases. Third, the presence of terms $E(t)\delta'(t-\tau)$ is handled by writing $E(t)\delta'(t-\tau) = \tilde{J}(t)J(\tau)\delta'(t-\tau) - J(t)\dot{\tilde{J}}(t)\delta(t-\tau)$; if y is known to come from a passive network this decomposition is always possible since E is then positive semidefinite. Fourth, although the passivity conditions on D (and E) are known, those on B and C are not, except in the lossless case where $B = \tilde{C}$ is possible and an alternate synthesis applies to cover all cases [4] [$\mathcal{T} = 1_n$ holds to yield a skew-symmetric Y_c].

An alternate and interesting method results from the following manipulation [5]. Let

$$\hat{A} = \mathcal{T}A\mathcal{T}^{-1} + \dot{\mathcal{T}}\mathcal{T}^{-1} \tag{IX-11a}$$

$$\hat{B} = \mathcal{T}B, \quad \hat{C} = C\mathcal{T}^{-1} \tag{IX-11b}$$

then from Eq. (IX-9c)

$$\begin{aligned} \hat{A}+\tilde{\hat{A}} &= -\mathcal{T}^{-1}[\mathcal{T}^2B-C][D+\tilde{D}]^{-1}[\tilde{B}\mathcal{T}^2-\tilde{C}]\mathcal{T}^{-1} \\ &= 2\tilde{\hat{L}}\hat{L} \end{aligned} \tag{IX-11c}$$

where $\hat{L}$ is defined as

$$\hat{L} = -\frac{1}{\sqrt{2}}(D+\tilde{D})^{-1/2}[\tilde{B}\mathcal{T}^2-\tilde{C}]\mathcal{T}^{-1} \tag{IX-11d}$$

If further we define

$$z = \left(\frac{D+\tilde{D}}{2}\right)^{1/2} = \tilde{z} \qquad \text{(IX-11e)}$$

where the positive definite symmetric square root is again meant, we obtain

$$\dot{s} = \frac{1}{2}(\hat{A}-\tilde{\hat{A}})s+(\hat{B}+\hat{L}z)v-\hat{L}v^* \qquad \text{(IX-12a)}$$

$$i = \widetilde{(\hat{B}+\hat{L}z)}s+zv^* \qquad \text{(IX-12b)}$$

$$i^* = -v^* = -\tilde{\hat{L}}s-zv \qquad \text{(IX-12c)}$$

Here direct substitution of the last constraint upon noticing that $2z\hat{L} = \hat{C}-\tilde{\hat{B}}$ yields the original set of equations

$$\dot{s} = \hat{A}s+\hat{B}v, \qquad i = \hat{C}s+\left[\frac{D+\tilde{D}}{2}\right]v \qquad \text{(IX-12d)}$$

The constraint $i^* = -v^*$ corresponds to resistive loads at the v^*, i^* ports. As a consequence we consider the coupling admittance matrix

$$\hat{Y}_c = \begin{bmatrix} 0 & z & -\tilde{\hat{B}}+z\hat{L} \\ -z & 0 & \tilde{\hat{L}} \\ \hat{B}+\hat{L}z & -\hat{L} & \frac{1}{2}(\hat{A}-\tilde{\hat{A}}) \end{bmatrix} \qquad \text{(IX-13)}$$

which is skew-symmetric and hence realizable by gyrators. When loaded at the final ports by unit resistors and at the next to final ports by unit capacitors, the input admittance $y(t,\tau)$ occurs at the input ports. In this manner an alternate synthesis results when $D+\tilde{D}$ is nonsingular, for a passive $y(t,\tau)$. It should be observed that this method requires that the skew-symmetric part of D must be extracted before Eqs. (IX-12) are considered, as seen from Eq. (IX-12d). Of course the skew-symmetric part of D is obtained by gyrators connected in parallel with the input ports. Note that this again shows that all time-variations for time-variable circuits can be placed in the gyrators.

E. Discussion

Because the state-variable equations are expressed in the time domain they are primarily suited for obtaining syntheses of time-variable networks. Here we have investigated two types of synthesis, one for voltage transfer and the other for n-port admittance impulse responses.

Emphasis has been placed upon passive structures but it is clear that the same ideas can be applied to synthesis using active elements, perhaps in an even simpler manner. The transfer function synthesis contains relatively simple calculations while the solution of a nonlinear differential equation makes the admittance syntheses extremely difficult to carry out. Consequently one would hope for a simpler admittance synthesis and in fact one which relaxes the unnecessary constraint of a nonsingular symmetric part for $\underset{\sim}{D}$.

In the time-invariant case the methods yield, in general, circuits with time-variable components. In some instances these can be combined to obtain time-invariant components but the result does show that perhaps some other synthesis methods exist which reduce to the known time-invariant techniques perviously discussed. It is worth observing though that many of the previous concepts discussed only for time-invariant structures do extend to the time-variable situation. For example it seems relatively simple to set up a theory of equivalence for time-variable structures from the discussions in Chapter V.

Although the n-port synthesis techniques have been given in terms of admittances the classical synthesis methods in terms of scattering matrices can also be extended [6] [7] [8] though as yet these latter time-variable methods have not really applied the concepts of state-variable theory for their success.

As a point of philosophical interest we point out that the passive synthesis of Section C can be applied to non-stable network functions, such as $T = 1/(p-1)$. Consequently one can relatively easily construct passive unstable networks, a rather paradoxical situation when it is realized that many intuitive deductions concerning passive networks have rested upon the "stability" of passive structures.

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G. Exercises

1. Synthesize by a passive structure the voltage transfer functions

   a) $$T(p) = \frac{p+2}{(p+3)(p+1)}$$

   b) $$T(t,\tau) = t\tau^2 e^{-(t-\tau)} 1(t-\tau)$$

2. Synthesize by the methods described the time-invariant admittance

$$y(p) = \frac{2p}{p+3}$$

From the result discuss various simplifications which can be made, or need to be made, in the theory.

3. Give a synthesis for time-invariant voltage transfer functions using ideas similar to those of Section C. For this one can choose any positive definite constant matrix $\underset{\sim}{V}$ and solve for $\underset{\sim}{\mathcal{J}}$ to yield Eq. (IX-5c).

4. Synthesize the voltage transfer function

$$T(p) = \frac{1}{p-1}$$

by the method of Section C. From the result discuss why a passive network need not be stable.

5. Discuss means of solving Eq. (IX-9c) for $\underset{\sim}{\mathcal{J}}$.

*6. Develop a state-variable synthesis of passive scattering matrices.

7. Extend the results of this chapter to nonpassive structures and discuss the meaning of your methods.

Nous vivons a côté de notre véritable vie
et nous sentons que nos pensées les
plus intimes et les plus profondes
même ne nous regardent pas, car nous
sommes autre chose que nos pensées
et que nos rêves. Et ce n'est qu'a certains
moments et presque par distraction que
nous vivons nous-mêmes.

M. Maeterlinck
"Les Avertis" du "Trésor des Humbles"

CONCLUSIONS

Paradoxically the simple expediant of introducing a set of first order differential equations to describe high order ones has led to the solution of previously unsolved problems, such as the determination of all equivalent active structures for a given network. As we have seen there are many areas where the ideas can be applied, perhaps with a possibility of gaining insight into the behavior of a system.

Thus, because most systems of practical significance possess an identifiable state, the state-variable equations give a general, or universal, means of observing systems. By keeping track of the solutions of the describing equations in state-variable form one can keep track of the behavior of the subparts of a system in orderly fashion. And because this tracking can be done orderly, the theory allows readily for the computer analysis of networks, this analysis having the possibility of proceeding in two ways, as we have seen in Chapter II in either the topological or capacitor extraction form. Once a computer analysis is set up in this manner the results can be used for other purposes than keeping a record of voltages and currents; for example Chapter VI has shown how a sensitivity analysis can proceed from a state-variable analysis program.

But the most striking uses of the theory occur when synthesis is considered. Here we have seen that minimal degree realizations, that is minimum reactive element circuits, result for general transfer functions by the theory of Chapter IV. Even though this latter is somewhat abstract its significance should not be overlooked. Because of its form it allows convenient integrated circuit constructions as well as analog modeling for simulation and preliminary testing of designs. Also because of its algebraic form the realization technique allows for the complete computer design of a system, though as yet such a program remains to be carried out. In the area of classical multiport synthesis, Chapter VII has shown that the introduction of state-variables can lead to a contribution since a minimal resistor and minimal capacitor circuit results by application of the given method.

Still it is by way of generalization of the positive-real admittance synthesis where the most significant contributions of state-variable theory seem to be made. We have illustrated this in two different ways. The first is through the introduction of a second variable to allow for design with both lumped and distributed elements, as covered in Chapter VIII. The second generalization is that of Chapter IX for the synthesis of time-variable circuits. Though this latter is as yet not completely finished, to us it represents a beautiful application of the theory which in almost all parts is carried out in the time domain.

Once a circuit has been designed the material of Chapter V on equivalence shows how many other circuits, in fact almost all, with the same terminal behavior can be found. To complete the picture any of these can be, in turn, analyzed by the methods of Chapters I and II to check its performance.

In summary, the theory of state-variables has allowed an almost complete picture of the theory of networks, in fact within the larger framework of scientific systems. It has, however, raised many fascinating problems, some of which we have tried to point out along the way. Thus, though the theory may offer little to some people it can offer an immense amount to those who would allow it -- so is it with almost all that we meet.

> Quel jour deviendrons-nous ce que nous sommes?
> Nous nous écartions sans rien dire et nous
> comprenions tout sans rien savior.
>
> M. Maeterlinck
> "Les Avertis" du "Trésor du Humbles"

LIST OF PRINCIPAL SYMBOLS

A, state coefficient matrix....7

b, number of branches.........26

B, input-state matrix..........7

C, state-output matrix.........7

$\mathcal{C}$, capacitance matrix.........21

cut set matrix.............27

D, canonical equation matrix...7

E, coefficient matrix..........7

g(p), least common denominator...........60

h, impulse response............8

H, transfer function...........9

i, port currents..............27

$\mathfrak{L}$[ ], Laplace transform........9

ℓ, number of links............26

m, input size..................4

n, output size..................4

$p = \sigma + j\omega$, frequency............9

P, observability matrix.......64

PR lemma matrix...........96

Q, quality factor.............48

Q, controllability matrix.....64

R, realization................7

resistance..................6

s, number of separate ports...26

$s_x^{p_k}$, pole position sensitivity............86

s, state......................4

$S_x^{T(p)}$, sensitivity............84

S_r, Hankel matrix.............62

t, number of tree branches....26

time.......................3

t_0, initial time.................3

T, state transition.........16

T[,], system transformation......4

T_r, modified Hankel matrix.....115

T, tie set matrix................28

state transformation........126

u, input.........................3

U, factor of T_r................115

U_ℓ^{-1}, left inverse of U.........115

v, number of vertices...........26

v, port voltages................27

V, transition energy form......127

$W(p)$, para-Hermitian factor.....97

W_r^{-1}, right inverse of W.........97

y, output.......................3

$y(t,\tau)$, admittance kernel......131

Y_c, coupling admittance....20, 128

Y_C, symmetric coupling admittance...............119

Y_g, gyrator coupling matrix.....51

δ, impulse, = δ[ ]...........9, 10

δ_λ, λ-degree = rank T_r....115, 116

λ, frequency variable..........111

θ, zero state...................4

ω_n, undamped natural frequency..48

Ω_n, companion matrix.......61, 115

0, zero matrix...................4

1(t), unit step...................9

1_k, k×k identity.................9

$1_{\rho,r}$, partial identity matrix...62

$\dot{+}$, direct sum....................51

$\tilde{}$, transpose..................23

•,o, compositions..............125

INDEX

No más, sino que Dios te guarde,
y á mí me dé paciencia para
llevar bien el mal que han de
decir de mí mas de cuatro
sotiles y almidonados-Vale.

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