UNIVERSITE CATHOLIQUE DE LOUVAIN

LABORATOIRE D'ELECTRONIQUE

NETWORK THEORY

THE STATE - SPACE APPROACH

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LOUVAIN

Network Theory: The State Space Approach

For my English speaking friends I have decided to give the following in translation, for it would be a shame to have the philosophical content wasted and this is perhaps the most significant part of the work. May it show that theories are truly more than theories, that knowledge stems from the heart as much as the head, and that feelings exist. Of course the technical content forms the core of the work but I would hope that my choice of a Flemish author writing in French would show a unity of spirit in a country I love but which would sometimes get overly burdened with differences. Let there remain also a wish for all my friends, those for which words can do no justice, those met and known since, and that which has silently touched, a song. R.W.N. 10/13/69

Preface: Here reigned also that love which no longer expresses itself because it does not participate in the life of the world. It would support, perhaps, no test; it seems at each instant betrayingly revealed and the least ordinary friendships has the air of vanishing it; however, that life is more profound than our own and perhaps it only seems to us indifferent because it knows itself reserved for further and surer times.

End of Preface:

Because the boy in business all
Kings and persons wish to make
Just like the clerks, a wonder 'tis,
So have I decided thus
That I will now bring forth
What every poet ought to do,
He who shall rightly versify,
For sure to verse is not a pity
One should well think before
How he should measure line begin
The middle write and place an end.

Acknowledgments: One feels that it is finally the hour to affirm a thing more grave, more human, more real and more profound than friendships, pity or love; a thing which mortally strikes from the wing to the foot of the bosom, and which one has never spoken, and which it is never possible to speak, for so many lives pass in silence! ... And time presses.

Chapter I: Often we have not the time to perceive them; they go away with nothing to say and they remain to us forever unknown.

Chapter II: But others linger awhile, watching us in smiling attentively, they seem on the point of avowing that they understand all.

Chapter III: In haste, sagely and minutely, they prepare to live.
And then, toward the twentieth year they move off in haste,
stifling their steps as if they have come to discover that they were
mistaken in remaining and that they must go pass their life
among men whom they know not.

Chapter IV: They are strange. They seem closer than others and question
nothing; nevertheless their eyes have a certainty so profound that they
must know all and have had more than the time of one evening
to tell their secret.

Chapter V: There pass, between two beings who meet for the first time,
strange secrets of life and death; and indeed other secrets which
still have no name, but which take possession immediately of
our attitude, our gaze, and our face.

Chapter VI: Nothing is visible and nevertheless we see all. They have fear of
us because we warn them without cease and in spite of ourselves;
and scarcely have we approached them before they feel that we
are reacting against their future.

Chapter VII: It is possible that there are no secret thoughts between two
men, but there are some things more imperative and more profound
than thought. Several times I have been witness of these things,
and one day I have seen them so near that I no longer knew if
it was a question of another or of myself.

Chapter VIII: It seems that at moments we look out from the height of a
tower. It is true that nothing is hidden; and all of you who meet
me, you know what I have done and what I will do, you know
that which I think and that which I have thought.

Chapter IX: There is thus one part of life — and it is the best, the purest
and the grandest — which does not mix in the ordinary life, and the
eyes of lovers themselves never penetrate that slice of silence and love.

Conclusion: We live at the side of our veritable life and we feel that our
most intimate and most profound thoughts even do not look upon
us, for we are another thing than our thoughts and our dreams.
And it is only at certain moments and even by distraction that
we live ourselves.

- - - - - - - -

What day will we become that which we are?
We put ourselves aside without saying anything and we
comprehend all without knowing anything.

NETWORK THEORY: THE STATE-SPACE APPROACH

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Ici régnait aussi cet amour qui ne
s'exprime plus parce qu'il ne participe
pas a la vie de ce monde. Il ne
supporterait peut-être aucune épreuve,
il semble a chaque instant trahi, et
la moindre amitié ordinaire a l'air
de la vaincre, et cependant sa vie est
plus profonde que nous-mêmes et
peut-être ne nous semble-t-il
indifférent que parce qu'il se sait
réservé pour des temps plus longs
et plus sûrs.

M. Maeterlinck
"Les Avertis" du "Trésor des Humbles"

PREFACE

The nine chapters which follow represent the set of lectures given as a final year one semester course at L'Université Catholique de Louvain for the first semester of the 1967-68 school year. Because of the presence of two national languages with the lectures given in a third it was decided to record the material as covered for student assistance and availability for future studies. Also the material often records in a consistent whole unavailable research results, and puts on further record the nature of joint cooperation between our associated research groups at Stanford and Louvain.

In the field of electrical engineering the theory of state-variables has raised some rather paradoxical situations. On the one hand it is often claimed that nothing can be achieved with state-variables that can not be done with more classical methods. This point is most frequently raised by those who wish to construct working circuits. On the other hand the mathematically inclined have a tendency to develop rather minute points or to get involved in the elegence of the theory with an attendant sacrifice of the practically important aspects. As a consequence the two natures of theory and practice tend to become further separated when state-variables are involved. Here we would at least make an attempt to resolve this paradoxical situation; that is, we would try to bring theory closer to practice and vice versa. This is done by presenting a coherent whole with emphasis upon those aspects of the theory for which use can almost immediately be seen or which have proven themselves in practice. Actually the subject was suggested by the Université; as we felt that some value could result in the intended types of treatment we have enjoyed the challenge and hope that the venture has proven profitable for all concerned.

It should be remembered that the material represents lectures and not a polished book, even though it has somewhat the form of a book for convenience of the user. As a consequence of its lecture form as well as the circumstances of its construction, there is much omitted which could profitably be contained. For example, there are points of derivations which could profitably be put into notes for completeness but which have been omitted in order to cover the material desired in the allotted time. Of equal importance is the scarcity of references; generally only a single reference available to the author's students at the time is given while multiple referencing would be much preferable. Likewise there are some topics, as topological and nonlinear synthesis, which have been almost entirely omitted but which should properly not be for completeness. Among works which we would have liked to add, perhaps to be saved for a revised edition, are those of J. Hiller (active theory), P. Wang (infinite dimensional theory), H. Watanabe (nonlinear theory), R. Yarlagada (topological synthesis), and D. Youla (lumped-distributed synthesis). A list of symbols and an index is appended for convenience.

In conjunction with our belief that life should be constructive and associated with a masculine spirit of verse which enhances its poetry, we incorporate some nontrivial concepts of the Flemish writer in French, M. Maeterlinck.

R. Newcomb
Louvain, January 1968

Om dat die leeke van allen zaken
Rime ende dichte willen maken
Gheijc clerken, dat wonder es,
So hebbic mi bewonden des
Dat ic nu wil bringhen voort
Wat enen dichter toe behoort,
Die te rechte sal dichten wel;
Want dichten en is gheen spel.
Men sal ooc voren versinnen,
Hoemen dat dicht zal beghinnen,
Middelen ende daer toe enden.

Jan Boendale
"De Leke Spieghel, III"

ACKNOWLEDGMENTS

It is with the greatest pleasure that the author takes this opportunity to acknowledge, and publicly thank, Professor N. Rouche whose efforts, immediate and through past cooperative researches, made our stay at Louvain possible. Perhaps this work can be considered as a tribute to the program carried out by Professor Rouche. Likewise we owe an equal debt of gratitude to Professor V. Belevitch who first proposed such a visit to us. Another special debt is owed to Colonel B. R. Agins and Captain A. Dayton of the US AFOSR who supported much of the research presented. Among many others who have been helpful during our stay we would acknowledge the following who have been of particular assistance: B. Anderson, M. Bhushan, M. Białko, G. Biorci, R. Boite, B. Cayphas, S. Chiappone, M. Davio, H. P. Debruyn, C. Desoer, J. Deutsch, P. Dewilde (especially), V. and S. Doležal, T. Duson, G. Francois, L. Fritz, A. Friziani, A. González-Domínguez, E. Gödör, W. Heinlein, W. Holmes, P. Jespers, Y. Kamp, J. Linvill, L. Lloyd, M. Martens, J. Neirynck, M. Novák, R. G. de Oliveira, L. Pope, E. and C. Sautter, L. Silverman, R. Spence, F. Stumpers, B. Tellegen, M. E. Terry, P. Van Bastelaer, A. Vander Vorst, E. Van Lantschoot, R. Van Overstraeten, J. Winkler.

In the words of M. Maeterlinck ["Les Avertis" du "Tresor des Humbles"]

> L'on sent que c'est l'heure enfin
> d'affirmer une chose plus grave, plus humaine,
> plus réelle et plus profonde que l'amitié,
> la pitié ou l'amour; une chose qui bat
> mortellement de l'aile tout au fond de la
> gorge, et qu'on ignore, et qu'on n'a
> jamais dite, et qu'il n'est plus possible
> de dire, car tant de vies se passent
> à se taire! ... Et le temps presse.

for

M. A. Gillett

Souvent, nous n'avons pas le temps de les apercevoir; ils s'en vont sans rien dire et ceux-là nous demeurent à jamais inconnus.

M. Maeterlinck
"Les Avertis" du "Trésor des Humbles"

CHAPTER I

INTRODUCTION - THE STATE

A. Summary

Here we briefly review the philosophical nature of the state giving a more or less precise mathematical formulation in terms of system transformations and network relationships. An example concerning the Brune structure is given to illustrate various points of the theory to be considered.

B. The State - Intuitively

Intuitively an object can be described at a given instant by a certain set of conditions which in fact are specified by the object being described; these conditions are often referred to as the state of the object. However, in scientific discussions the state is usually taken to mean that set of conditions which when specified at a given instant of initiation of an excitation lead to a predicted response over the period of excitation. Thus the concept is generally applied to causal (that is, nonanticipatory or equivalently antecedal) systems where it is possible to predict the output to a given input. A specification of the necessary conditions to allow determination of the output, that is an assignment of initial conditions, is essentially a specification of the state. The state then is that entity, described through a set of parameters (perhaps uncountably infinite in number), which when prescribed initially allows a unique motion of the entity under determinate excitations. We shall soon make the concept precise mathematically at which time we will see that a slight modification is of interest for treating networks.

C. The State - Uses

Although of the most recent development, our primary interest will be the use of the state for design or synthesis. For synthesis we need to develop a formulation which is convenient for decomposition and construction. In obtaining a suitable development we shall investigate analysis methods from which we will see that by isolating a set of state variables a convenient analysis method is obtained. The method is especially convenient for digital computer formulation, and thus, we will obtain several methods for digital computer analysis for circuits. The results are further useful for investigation of the transient and frequency responses of networks as well as for the determination of natural frequencies. Similarly a useful technique for investigating sensitivity is obtained. Of particular importance is also the means of determining "all" possible equivalents. By reversing the analysis process one is led to several design formulations. For example, given a transfer function one can algebraically set up a canonical set of state variable equations, by a means suitable for digital computer programming. From the canonical equations one can revert to an analog computer realization, the result being of considerable use for integrated circuit design using operational amplifiers. By another interpretation of the canonical equations one can obtain an alternate minimal capacitor synthesis by loading a gyrator-resistor network. By proper generalization of multivariable functions we can also develop a synthesis for lumped-distributed circuits.

Although it can be claimed that the state variables are nothing more than an appropriate choice of variables for initial conditions, such an outlook is rather narrow. In fact previous results obtained from an "initial condition" outlook are rather weak and shallow when compared to what has been achieved by the state variable outlook. From the previous paragraph we can summarize the results of state variable theory to be discussed in the sequel by the following topics:

1. Digital Computer Analysis
    a. Formulation of canonical equations
        (1) Topological means
        (2) Reactive extractions
    b. Transient analysis
    c. Frequency response
2. Analog Simulation
    a. Integrated circuits
    b. Filter design
3. Equivalence
    a. Minimal realization transformations
    b. Nonminimal (encirclements)
4. Sensitivity
5. Finite Synthesis
    a. Minimal realizations
    b. Loaded n-port theory
    c. Lossless synthesis (hybrid)
6. Multivariable Realizations
    a. Minimal realizations, etc.
    b. Lumped-distributed synthesis
    c. Noncommensurate line synthesis
7. Distributional Generalizations
    a. Representations
    b. Time-variable circuits
8. Inifinite-Dimensional Extensions

D. The State - Mathematical

Let us consider as given a system designed to map inputs $\underset{\sim}{u}$ into outputs $\underset{\sim}{y}$. If we know all inputs applied to the system from its time of construction to the time of observation, t, then $\underset{\sim}{y}(t)$ is "uniquely" known and is determined through a knowledge of the system transformation. However, it is more frequent that we have on hand a given system which we will begin to use at time t_o, generally without a knowledge of the inputs applied before t_o. We will assume that there is a set of parameters

$s(t_o)$ which we can measure, or somehow determine, such that if the input $u(t)$ is known for $t \geq t_o$ then also for $t \geq t_o$ the output $y(t)$ is uniquely determined [upon a specification of the state $s(t_o)$]. Since the output is uniquely determined, there exists a transformation $T[\cdot,\cdot]$ such that

$$y = T[u,s(t_o)], \qquad t \geq t_o \qquad \text{(I-1)}$$

Since t_o can vary, the state s is also a "function" of time as is of course reasonable on intuitive grounds. We point out that in general y, u, and particularly s are multidimensional quantities; we will take u as an m-vector, y as an n-vector, and s as a k-vector [for example, k will often be the number of capacitors and inductors in a circuit]. Pictorially Eq. (I-1) is represented as in Fig. I-1.

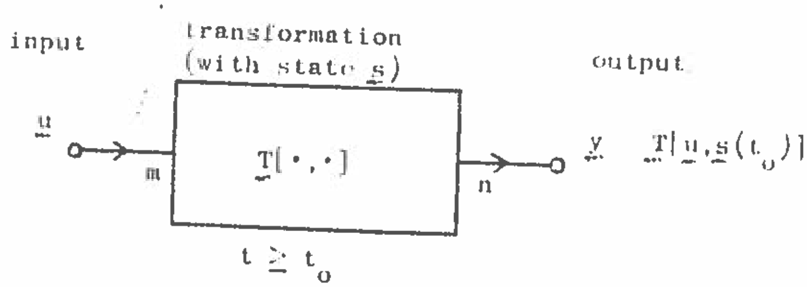


Fig. I-1. SYSTEMS REPRESENTATION.

A system which can be represented by a transformation of the form of Eq. (I-1) is conveniently called a state determined system. One can in fact make a detailed study of the general types of state determined systems [1, p. 67] but it seems more important for our purposes to proceed to other studies. However, we define a few useful concepts. First is that of the zero state θ, defined through

$$0 = T[0,\theta(t_o)], \qquad t \geq t_o \qquad \text{(I-2)}$$

In other words a zero state is any state which gives a zero output for a zero input. As an example of a nonzero zero state consider the balanced bridge circuit of Fig. I-2 where the capacitor voltage serves as the state, $\underset{\sim}{s}(t) = [v_c(t)]$, and we take the applied voltage as input with the source current as output. When the applied voltage is zero no input current flows as is seen by the redrawing shown in the (b) portion of the figure; thus, $\underset{\sim}{\theta} = [\underset{\sim}{v}_c(t)]$. We observe that in this system all

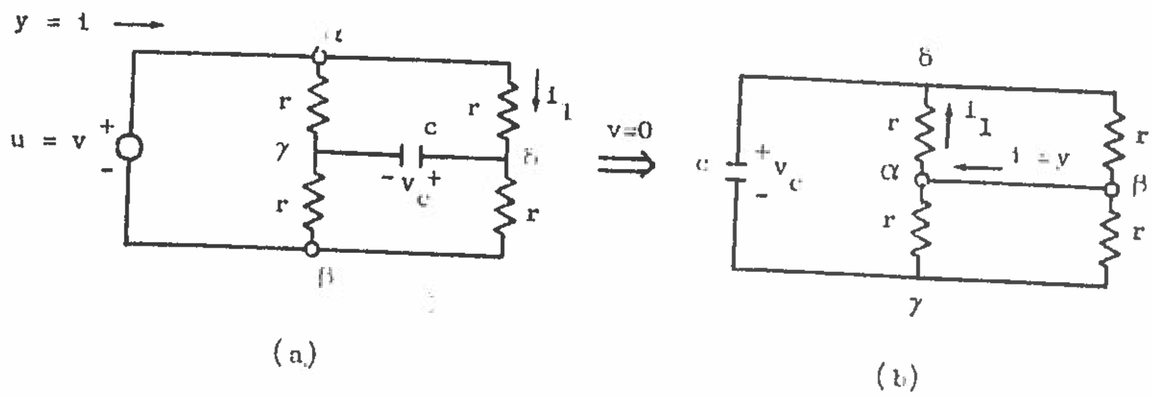


Fig. I-2. NONZERO ZERO-STATE EXAMPLE.

states are the zero state, but in general such will not be the case. For example if we had taken i_1 as the output, the output would only have been zero if $v_c = 0$, that is for this new system, with $u = v$, $y = i_1$ the state $\underset{\sim}{s} = [v_c(t)]$ is only the zero state when it is zero; $\underset{\sim}{\theta} = [0]$.

With the concept of the zero state on hand we can consider the definition of a linear system. A system is called linear (with respect to inputs) if for all constants k, all initial states $\underset{\sim}{s}(t_o)$, all zero states $\underset{\sim}{\theta}(t_o)$, and all inputs $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$.

$$\underset{\sim}{T}[k(\underset{\sim}{u}_1-\underset{\sim}{u}_2),\underset{\sim}{\theta}(t_o)] = k\underset{\sim}{T}[\underset{\sim}{u}_1,\underset{\sim}{s}(t_o)]-k\underset{\sim}{T}[\underset{\sim}{u}_2,\underset{\sim}{s}(t_o)] \qquad \text{(I-3)}$$

We observe that because of the need to consider the state there is a difference between a linear system (in its mathematical representation) and a linear transformation. An immediate consequence of this definition of linearity is the fundamental decomposition obtained by taking $k = 1$, $\underset{\sim}{u}_1 = u$, $\underset{\sim}{u}_2 = 0$.

$$\underset{\sim}{T}[\underset{\sim}{u},\underset{\sim}{s}(t_0)] = \underset{\sim}{T}[\underset{\sim}{0},\underset{\sim}{s}(t_0)] + \underset{\sim}{T}[\underset{\sim}{u},\underset{\sim}{\theta}(t_0)] \qquad \text{(I-4)}$$

That is, for a linear system the total response can be broken into the sum of two parts, one of which is the zero input response and the other of which is the zero state response. Thus, superposition not only holds with respect to inputs, as Eq. (I-3) shows, but also with respect to the response from initial conditions.

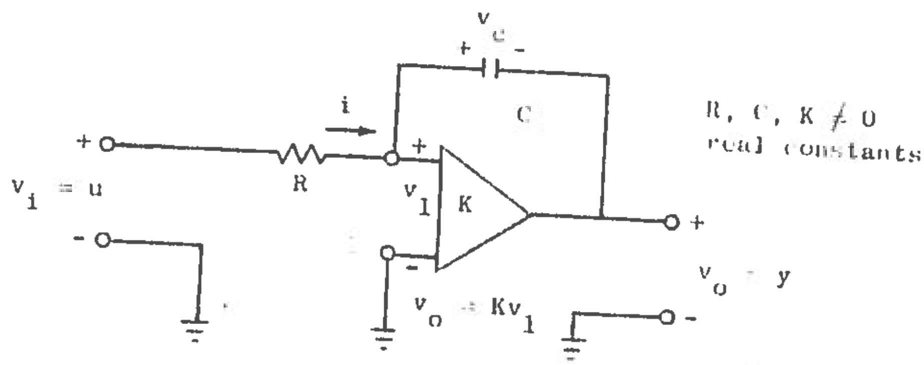


Fig. I-3. INTEGRATOR.

As an example of the decomposition let us consider the integrator of Fig. I-3. The describing equations can be taken as

$$Ri = v_i - \frac{v_o}{K}, \qquad i = \frac{Cd[v_o/K - v_o]}{dt}$$

which upon simple substitution of the first into the second yields the following differential equation completely in terms of input and output variables.

$$\frac{RC(1-K)}{K}\frac{dv_o}{dt} + \frac{v_o}{K} = v_i \qquad \text{(I-5a)}$$

To obtain the transformation mapping the input into the output this differential equation must be solved. We find by any of several means (Laplace transforms, for example)

$$v_o(t) = \underbrace{v_o(t_o)\exp\left[-\frac{1}{RC(1-K)}(t-t_o)\right]}_{T[\underset{\sim}{0},\underset{\sim}{s}(t_o)]} + \underbrace{\int_{t_o}^{t}\left\{\exp\left[-\frac{1}{RC(1-K)}(t-\tau)\right]\right\}\left[\frac{K}{RC(1-K)}v_i(\tau)\right]d\tau}_{T[\underset{\sim}{u},\underset{\sim}{0}(t_o)]} \quad \text{(I-5b)}$$

We see that Eq. (I-4) is satisfied and that $\underset{\sim}{s}(t) = \{v_o(t)\} = \{y(t)\}$ is a suitable choice for the state. Since $v_c = \frac{1-K}{K} v_o$ we also see that an appropriate (alternate) choice for the state is $\underset{\sim}{s}(t) = \{v_c(t)\}$.

Perhaps much more should be said about the domains of definition of the various quantities but such discussions can also get lengthy. We merely mention that for a given system there is usually some restriction on the type of inputs allowed as well as the range of outputs for which the mathematical transformation $\underset{\sim}{T}[\cdots]$ is valid. In our study we will most often assume that the input and output are zero before $t = t_o$ and that they, along with the state, are real valued.

For linear systems it will often be possible to find a description in the form

$$\frac{d\underset{\sim}{s}(t)}{dt} = \underset{\sim}{A}(t)\underset{\sim}{s}(t) + \underset{\sim}{B}(t)\underset{\sim}{u}(t) \quad \text{(I-6a)}$$

$$\underset{\sim}{y}(t) = \underset{\sim}{C}(t)\underset{\sim}{s}(t) + \underset{\sim}{D}(t)\underset{\sim}{u}(t) + \underset{\sim}{E}(t)\frac{d\underset{\sim}{u}(t)}{dt} \quad \text{(I-6b)}$$

If such can be found, these equations are called a <u>canonical representation</u> and the set

$$R = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D},\underset{\sim}{E}\}$$

is called a <u>realization</u>. For such a system having the dimension, k, of the state finite, we ascribe the name <u>finite</u> or <u>differential system</u>. Likewise, if the coefficient matrices, $\underset{\sim}{A}(t)$, ..., are constant then the system is called <u>time-invariant</u> (actually this time-invariance is a special case of a more general definition applicable to any state determined system [2, p. 1]. In most situations of interest the useful information about the system is contained in the matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$, so we will often assume that either $\underset{\sim}{E} = \underset{\sim}{0}$ or $\underset{\sim}{D} = \underset{\sim}{E} = \underset{\sim}{0}$. Thus, most of our concern will be with the canonical set of equations

$$\frac{d\underset{\sim}{s}}{dt} = \underset{\sim}{A}\,\underset{\sim}{s} + \underset{\sim}{B}\,\underset{\sim}{u} \tag{I-7a}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{s} + \underset{\sim}{D}\,\underset{\sim}{u} \tag{I-7b}$$

and the realization

$$R = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}\} \tag{I-7c}$$

It is possible to interrelate the canonical equations with the zero state response, $\underset{\sim}{T}[\underset{\sim}{u},\underset{\sim}{\theta}(t_o)]$, in the time-invariant case (a similar development holds for time-varying systems). When the realization R is constant, Eqs. (I-6) yield a continuous transformation, in the sense of distribution theory, mapping inputs into outputs (in the zero state). Consequently, there exists a matrix $\underset{\sim}{h}(t)$ such that [3, p. 223]

$$\underset{\sim}{T}[\underset{\sim}{u},\underset{\sim}{\theta}(t_o)] = \underset{\sim}{h} * \underset{\sim}{u}$$

$$= \int_{-\infty}^{\infty} \underset{\sim}{h}(t-\tau)\,\underset{\sim}{u}(\tau)\,d\tau \tag{I-8}$$

where $*$ denotes convolution, that is, the integration exhibited (recall that $\underset{\sim}{u}(\tau)$ is zero for $\tau < t_o$). The $n \times m$ matrix $\underset{\sim}{h}$ consists of distributions (functions, impulses, etc.) and is called a <u>distributional kernel</u>; physically it represents a matrix of impulse responses. For Fig. I-2 we have, for example,

$$h(t) = \frac{1}{r}\,\delta(t) \tag{I-9a}$$

while for Fig. I-3 we have

$$h(t) = \frac{K}{RC(1-K)}\left\{\exp\left[-\frac{t}{RC(1-K)}\right]\right\}1(t) \qquad \text{(I-9b)}$$

where $1(t)$ is the unit step function and $\delta(t) = d1(t)/dt$ is the unit impulse. By taking Laplace transforms, denoted by $\mathcal{L}[\ ]$, we have from Eq. (I-8)

$$\mathcal{L}[\underset{\sim}{T}[\underset{\sim}{u},\underset{\sim}{\theta}]] = \underset{\sim}{H}(p)\ \mathcal{L}[\underset{\sim}{u}] \qquad \text{(I-10a)}$$

$$\underset{\sim}{H}(p) = \mathcal{L}[\underset{\sim}{h}] \qquad \text{(I-10b)}$$

where $\underset{\sim}{H}(p)$, $p = \sigma + j\omega$, is called the <u>transfer function matrix</u> (it is $n \times m$ also). By taking Laplace transforms in Eq. (I-6) we can obtain, by straightforward substitution, an alternate expression for the transfer function matrix

$$\underset{\sim}{H}(p) = p\underset{\sim}{E} + \underset{\sim}{D} + \underset{\sim}{C}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B} \qquad \text{(I-10c)}$$

where $\underset{\sim}{1}_k$ is the $k \times k$ identity matrix. One of the problems of the theory is then to find a realization $R = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D},\underset{\sim}{E}\}$ given a transfer function $\underset{\sim}{H}(p)$ since then the canonical equations are on hand. A similar problem is to obtain the canonical equations from a given physical structure. We comment that Eq. (I-10c) shows that the transfer functions resulting from the canonical equations are always rational, when k is finite, and possess at most a simple pole at infinity; in the more commonly treated case where $\underset{\sim}{E} = \underset{\sim}{0}$, $\underset{\sim}{H}(p)$ has no pole at infinity.

We will illustrate some of the above points, while exhibiting a set of canonical equations, in the following example of a Brune section. However, first we comment that we have considered a given construct as a system by "orienting" its variables, that is, by specifying inputs and outputs. Thus, as we already saw in Fig. I-2, a given construct can yield several different systems by having different inputs and outputs assigned. Nevertheless, the state will generally remain invariant; that is, given a construct, there is an associated state which in fact can be used with all systems obtained from the construct. Further, a network has been defined by the set of all pairs $[\underset{\sim}{v}, \underset{\sim}{i}]$ of voltages $\underset{\sim}{v}$ and currents $\underset{\sim}{i}$ allowed at its ports [4, p. 7]. We could proceed from this definition of a network to introduce the state as a set of parameters needed at time t_o to specify allowed pairs $[\underset{\sim}{v}, \underset{\sim}{i}]$ for $t > t_o$. But for our purposes it is sufficient to orient variables at the network ports and work with inputs and outputs, as for example through the admittance or scattering matrices. We note, though, that in any characterization there is a minimum value for the size, k, of the state. This minimum size is often referred to as the <u>degree</u> δ of the system; through Eq. (I-10c) we see that δ is characterized through $\underset{\sim}{H}$; thus we can write $\delta[\underset{\sim}{H}(p)]$ or (precisely only when $\underset{\sim}{E} = \underset{\sim}{0}$)

$$\delta = \min k = \delta[\underset{\sim}{H}(p)] = \text{system degree}$$

We will later see how to calculate δ directly from $\underset{\sim}{H}(p)$ but for now we merely comment that δ physically represents the minimum number of integrators necessary for an analog simulation of the system described by the canonical equations (I-7). We do mention that it is sometimes of interest to have more than the minimum number of components of the state present, especially for the determination of equivalent realizations to satisfy some specified constraints (as for example the desire to incorporate only a certain type of transistor in a design). Figure I-2 has already illustrated an example of a nonminimal realization, where we define a <u>minimal realization</u> as one where the $\underset{\sim}{A}$ matrix is $\delta > \delta$, that is, has its order equal to the degree of $\underset{\sim}{H}(p)$. In this case $\underset{\sim}{H}(p) = 1/r$, $\delta[\underset{\sim}{H}] = 0$, and we see that the system of Fig. I-2 is equivalent to a resistor, the situation being as shown in Fig. I-4, where Fig. I-2a has been redrawn in the (b) portion.

E. <u>The State - Brune Section Example</u>

At this point let us set up the canonical equations for the nonreciprocal Brune section of Fig. I-5 [5, p.], where we make the

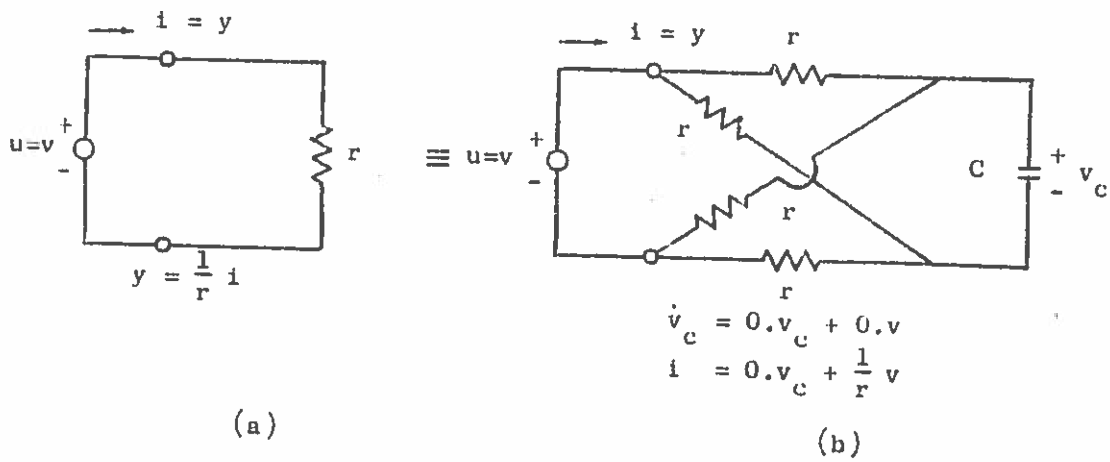


Fig. I-4. ZERO AND ONE-DIMENSION REALIZATIONS OF $H(p) = 1/r$.

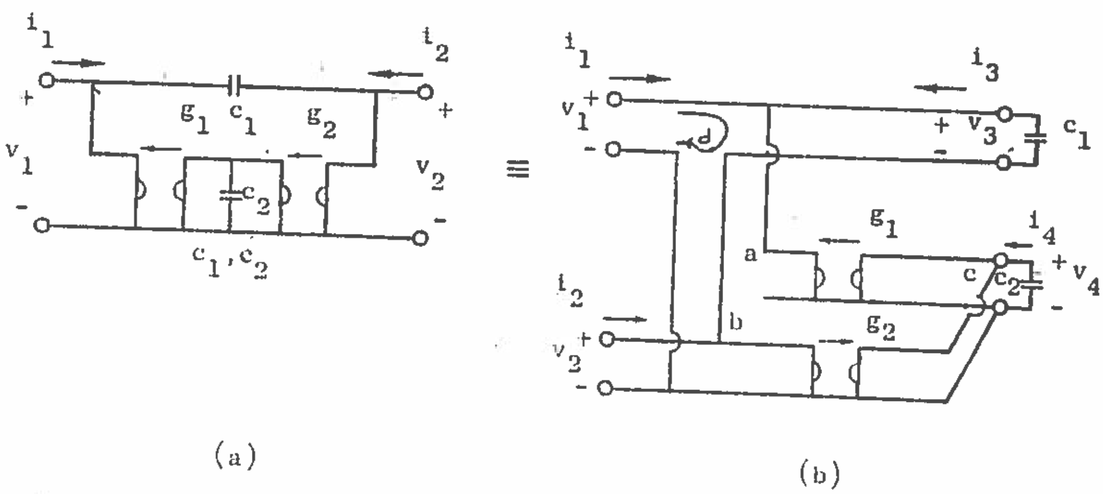


Fig. I-5. NONRECIPROCAL BRUNE SECTION (a) WITH CAPACITOR EXTRACTION (b).

particular choice of input and output (of later use for modeling of filters for integrated circuit realization).

$$\underset{\sim}{u} = \begin{bmatrix} v_1 \\ -i_2 \end{bmatrix}, \qquad \underset{\sim}{y} = \begin{bmatrix} v_2 \\ i_1 \end{bmatrix} \tag{I-9a}$$

In order to analyze the Brune section to obtain the canonical state variable equations we first separate the dynamical elements by removing the capacitors as a load on a purely resistive 4-port, as shown in

Fig. I-5b. We also take as a convention for the gyrators the symbolism of Fig. I-6.

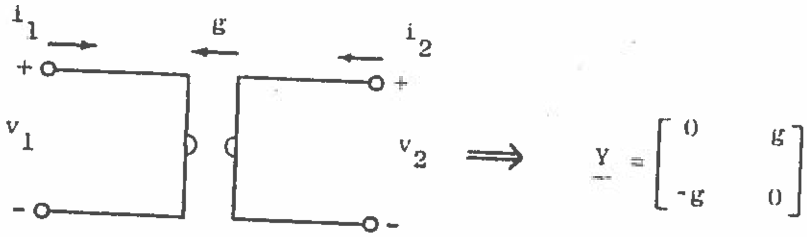


Fig. I-6. GYRATOR CONVENTIONS.

By summing currents at the nodes marked a, b, c (in Fig. I-5) and summing voltages around the loop d, respectively, we obtain

$$\begin{bmatrix} 0 & 0 & 0 & g_1 \\ 0 & 0 & 0 & -g_2 \\ -g_1 & g_2 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} \qquad \text{(I-9b)}$$

A suitable choice for the state is generally the set of capacitor voltages or charges and inductor currents or flux, thus we let

$$\underset{\sim}{s} = \begin{bmatrix} c_1 v_3 \\ c_2 v_4 \end{bmatrix} \qquad \text{(I-9c)}$$

for which it follows, from Fig. I-5b, that

$$\dot{\underset{\sim}{s}} = -\begin{bmatrix} i_3 \\ i_4 \end{bmatrix} \qquad \text{(I-9d)}$$

We can therefore rewrite Eq. (I-9b) to specifically exhibit the quantities of interest by rearranging the columns.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -g_1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ -i_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ g_2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} 0 & g_1/c_2 \\ 0 & -g_2/c_2 \\ 0 & 0 \\ 1/c_1 & 0 \end{bmatrix} \begin{bmatrix} c_1 v_3 \\ c_2 v_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -i_3 \\ -i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(I-9e)}$$

If we add the second row to the first and $-g_2$ times the last row to the third, we can isolate $\mathbf{y}$ from $\dot{\mathbf{s}}$ to get

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ g_2-g_1 & 0 \\ -1 & 0 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 & (g_1-g_2)/c_2 \\ 0 & -g_2/c_2 \\ -g_2/c_1 & 0 \\ 1/c_1 & 0 \end{bmatrix} \mathbf{s} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{\mathbf{s}} = \mathbf{0}$$

(I-9f)

Using the third row multiplied by -1 and the second row gives

$$\dot{\mathbf{s}} = \begin{bmatrix} 0 & -g_2/c_2 \\ g_2/c_1 & 0 \end{bmatrix} \mathbf{s} + \begin{bmatrix} 0 & 1 \\ g_1-g_2 & 0 \end{bmatrix} \mathbf{u} \qquad \text{(I-9g)}$$

while the fourth (by -1) row and then the first give the desired output equation

$$\mathbf{y} = \begin{bmatrix} -1/c_1 & 0 \\ 0 & (g_1-g_2)/c_2 \end{bmatrix} \mathbf{s} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \qquad \text{(I-9h)}$$

These last two equations are the canonical equations for the Brune section.

Using $\underset{\sim}{H}(p) = \underset{\sim}{D} + \underset{\sim}{C}\,(p\underset{\sim}{1}_2 - \underset{\sim}{A})^{-1}\underset{\sim}{B}$ we can find the transfer function.

$$\underset{\sim}{H}(p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1/c_1 & 0 \\ 0 & (g_1-g_2)/c_2 \end{bmatrix} \begin{bmatrix} p & g_2/c_2 \\ -g_2/c_1 & p \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ g_1-g_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{p^2 + g_2^2/c_1c_2} \begin{bmatrix} \frac{-1}{c_1} & p \\ 0 & \frac{(g_1-g_2)}{c_2} \end{bmatrix} \begin{bmatrix} p & -g_2/c_2 \\ g_2/c_1 & p \end{bmatrix} \begin{bmatrix} 0 & 1 \\ g_1-g_2 & 0 \end{bmatrix}$$

$$= \frac{1}{p^2 + g_2^2/c_1c_2} \begin{bmatrix} p^2 + \frac{g_1g_2}{c_1c_2} & \frac{-p}{c_1} \\ p\,\frac{(g_1-g_2)^2}{c_2} & p^2 + \frac{g_1g_2}{c_1c_2} \end{bmatrix} \qquad \text{(I-9i)}$$

We comment that one of the alternate choices available for the state is

$$\hat{\underset{\sim}{s}} = \begin{bmatrix} v_3 \\ v_1 \end{bmatrix}$$

and that for this, or any other choice for the state, we obtain the same transfer function. In fact we observe that there is a nonsingular transformation mapping one choice for the state into another, that is,

$$\underset{\sim}{s} = \underset{\sim}{T}\,\hat{\underset{\sim}{s}}\,, \quad \underset{\sim}{T} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \qquad \text{(I-10)}$$

As we will later see, any minimal realization is related to any other through a nonsingular transformation on the state as in Eq. (I-10). In this case $\delta[\underset{\sim}{H}] = 2$, and thus the realization

$$R = \left\{ \begin{bmatrix} 0 & -\frac{g_2}{c_2} \\ \frac{g_2}{c_1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ g_1 - g_2 & 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{c_1} & 0 \\ 0 & \frac{(g_1-g_2)}{c_2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is minimal.

F. Discussion

By way of introduction (or review, depending upon previous background), we have considered the meaning of the state and given the primary equations related to our further studies. For differential systems the equations of most interest are the canonical ones.

$$\dot{\underset{\sim}{s}} = \underset{\sim}{A}\,\underset{\sim}{s} + \underset{\sim}{B}\,\underset{\sim}{u} \qquad \text{(I-11a)}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{s} + \underset{\sim}{D}\,\underset{\sim}{u} \qquad \text{(I-11b)}$$

with the associated transfer function yielding the output $\underset{\sim}{y}$ in terms of the input $\underset{\sim}{u}$, when initially in the zero state $\underset{\sim}{s}(t_o) = \underset{\sim}{0}(t_o)$, through

$$\underset{\sim}{y} = \underset{\sim}{h} * \underset{\sim}{u} \qquad \text{(I-11c)}$$

given by

$$\mathcal{L}[\underset{\sim}{h}] = \underset{\sim}{H}(p) = \underset{\sim}{D} + \underset{\sim}{C}\,(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B} \qquad \text{(I-11d)}$$

We observe that in this differential system case the state is that set of parameters for which a matrix set of first order differential equations can be set up in terms of the transfer function and its realization. The matrix $\underset{\sim}{h}$ is the impulse response matrix with its Laplace transform $\mathcal{L}[\underset{\sim}{h}]$ being the transfer function. From the expression for $\underset{\sim}{H}(p)$ in terms of the realization $\underset{\sim}{R} = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}\}$ matrices, it is clear that the poles of $\underset{\sim}{H}(p)$ are zeros of the determinant of $p\underset{\sim}{1}_k - \underset{\sim}{A}$, that is, the natural frequencies of the system are eigenvalues of the matrix $\underset{\sim}{A}$.

We also observe that if we have two realizations $R = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}\}$ and $\hat{R} = \{\hat{\underset{\sim}{A}},\hat{\underset{\sim}{B}},\hat{\underset{\sim}{C}}\,\hat{\underset{\sim}{D}}\}$ related through

$$\hat{\underset{\sim}{A}} = \underset{\sim}{T}^{-1}\underset{\sim}{A}\,\underset{\sim}{T}, \quad \hat{\underset{\sim}{B}} = \underset{\sim}{T}^{-1}\underset{\sim}{B}, \quad \hat{\underset{\sim}{C}} = \underset{\sim}{C}\,\underset{\sim}{T}, \quad \hat{\underset{\sim}{D}} = \underset{\sim}{D} \tag{I-11e}$$

with $\underset{\sim}{T}$ nonsingular, then the two transfer functions are identical. Thus we have

$$\hat{\underset{\sim}{H}}(p) = \hat{\underset{\sim}{D}} + \hat{\underset{\sim}{C}}\,(p\underset{\sim}{1}_k - \hat{\underset{\sim}{A}})^{-1}\hat{\underset{\sim}{B}} = \underset{\sim}{D} + \underset{\sim}{C}\,\underset{\sim}{T}\,(p\underset{\sim}{T}^{-1}\underset{\sim}{T} - \underset{\sim}{T}^{-1}\underset{\sim}{A}\,\underset{\sim}{T})^{-1}\underset{\sim}{T}^{-1}\underset{\sim}{B}$$

$$= \underset{\sim}{D} + \underset{\sim}{C}\,(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B} = \underset{\sim}{H}(p)$$

consequently we can investigate equivalent systems by manipulating the state variable equations through methods associated with the transformation of Eq. (I-11e), which in fact can be interpreted in terms of the state as a basis change in the state space through $\underset{\sim}{s} = \underset{\sim}{T}\,\hat{\underset{\sim}{s}}$. We are then led to observe that there is a k-dimensional space, the state space, in which we have introduced (Cartesian) coordinates against which the components of $\underset{\sim}{s}$ for the canonical equations are measured. The actual state, for a given input $\underset{\sim}{u}(t)$ and an initial state $\underset{\sim}{s}(t_o)$, traverses the state space on a trajectory $\underset{\sim}{s}(t)$, this trajectory giving the "motion" or behavior of the system, as verified by Eq. (I-11a,b).

Our primary interest will be with linear networks considered as systems through the transformation formulation so far discussed. One could consider the more general nonlinear case described by the matrix differential equations

$$\dot{\underset{\sim}{s}} = \underset{\sim}{f}\,(\underset{\sim}{s},\ \underset{\sim}{u},\ \dot{\underset{\sim}{u}}) \tag{I-12a}$$

$$\underset{\sim}{y} = \underset{\sim}{g}\,(\underset{\sim}{s},\ \underset{\sim}{u},\ \dot{\underset{\sim}{u}}) \tag{I-12b}$$

However, very little is available in the way of synthesis for such equations, so we have chosen to concentrate on the linear case. We also choose to devote efforts primarily to the continuous-time case since it is of most interest for network studies. But because our treatment will generally be of an algebraic nature, the results are almost all valid for discrete-time systems, which in fact have considerable practical importance, for example, through the theory of automata.

In our treatment we have not proceeded in the most rigorous manner possible since we wish to bring out only the basic and most important points for our later use. Once the concepts we have treated are grasped in principle, the more detailed works are available to those interested [1], [6]. However we have not wished to sacrifice completely the rigor of the theory so have proceeded in a rather precise manner for the detail given. Although most of our emphasis will be upon networks, we have given a somewhat general systems formulation in order not to overly limit the treatment. As a consequence we will most frequently work with a network in an input-output situation, as for example through the admittance matrix where the input $\underset{\sim}{u}$ is the set of port voltages $\underset{\sim}{v}$, and the output $\underset{\sim}{y}$ the port currents $\underset{\sim}{i}$ (in which case $m = n$). Since such a (port) description tells very little about the internal structure we will use the state to discuss internal operation and construction of the network. A <u>network</u> is a system with electrical inputs and outputs.

It is of interest to know means of obtaining the canonical equations so we next turn to a discussion of the setting up of state variable equations.

G. References

1. Zadeh, L. A., and C. A. Desoer, "Linear System Theory," McGraw-Hill, New York, 1963.
2. Newcomb, R. W., "Concepts of Linear Systems and Controls," Brooks/Cole, Belmont, Calif., 1968.
3. Schwartz, L., "Theórie des noyaux," <u>Proc. Intern. Congr. of Mathematicians</u>, Cambridge, Mass., 1950, pp. 220-230.
4. Newcomb, R. W., "Linear Multiport Synthesis," McGraw-Hill, New York, 1966.
5. Newcomb, R. W., "Active Integrated Circuit Synthesis," Prentice-Hall, Englewood Cliffs, N. J. 1968.
6. Kalman, R. E., "Algebraic Structure of Linear Dynamical Systems," Proceedings of the National Academy of Science, USA, to appear.

H. Exercises

1. Set up the canonical equations for the Hazony section of Fig. EI-1. Do this for the input-output variables of Eq. (I-9a) as well as for the admittance and impedance matrices as transfer functions.

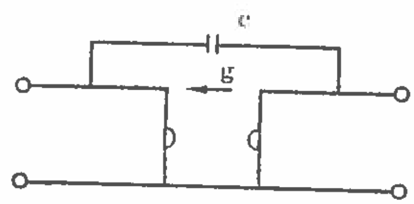


Fig. EI-1. HAZONY SECTION.

*2. Given the canonical equations for the admittance matrix (as the transfer function) and those for the impedance matrix, find the relations between the two realization set matrices. Repeat for the scattering matrix and the admittance matrix given.

3. A given network has the canonical equations

$$\dot{\underset{\sim}{s}} = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \underset{\sim}{s} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad \text{(EI-1a)}$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \underset{\sim}{s} + \begin{bmatrix} 2 \end{bmatrix} u \qquad \text{(EI-1b)}$$

a. Find the transfer function.

b. Find the zero input response for $\underset{\sim}{s}(t_o) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Plot the trajectory $\underset{\sim}{s}(t)$ in state space.

*4. Discuss a formulation for "transfer functions" in terms of the realization matrices for time-variable networks.

5. As we have mentioned, the state applies to much more than scientific or physical systems. Investigate the concept in terms of, for example, language formation or motion picture production.

6. Consider any network of interest and set up the appropriate state space equations. From these, investigate the minimality of the realization as well as other sets of canonical equations yielding the desired transfer function.

Mais d'autres s'attardent un peu, nous regardent en souriant attentivement, semblant sur le point d'avouer qu'ils ont tout compris.

M. Maeterlinck
"Les Avertis" du "Trésor des Humbles"

CHAPTER II

FORMULATION OF CANONICAL EQUATIONS

A. Summary

By the use of appropriate replacements and capacitor extractions a simple method of equation formulation suitable for digital computer use is presented; the method is described in terms of the admittance description but can be used in other situations. This method is followed by the outline of a topological one which exhibits a more general set of equations.

B. Capacitor Extractions

Let us consider as given a finite circuit, that is, a connection of a finite number of resistors, capacitors, inductors, transformers, gyrators, and devices, such as transistors, which can be modeled by the above elements. (We assume linear but perhaps time-variable and active elements at this point; that is, negative as well as positive element values which may vary with time are allowed.) To illustrate the method, we search for the canonical state variable equations for the admittance matrix as transfer function [1]. To concentrate on fundamental concepts, we replace all inductors by the capacitor-loaded gyrator equivalent shown in Fig. II-1.

After making such a replacement we extract all capacitors into a separate network which loads a multiport described completely by algebraic constraints. If the admittance matrix is $n \times n$ and if there are c capacitors extracted, the situation is shown in Fig. II-2, where the "resistive" $(n + c)$-port is loaded by a capacitive c-port.

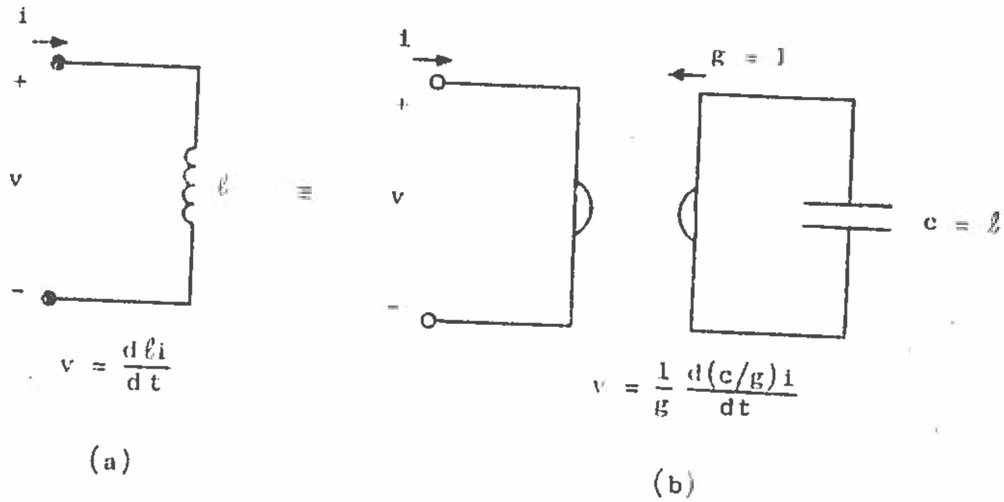


Fig. II-1. INDUCTOR EQUIVALENT.

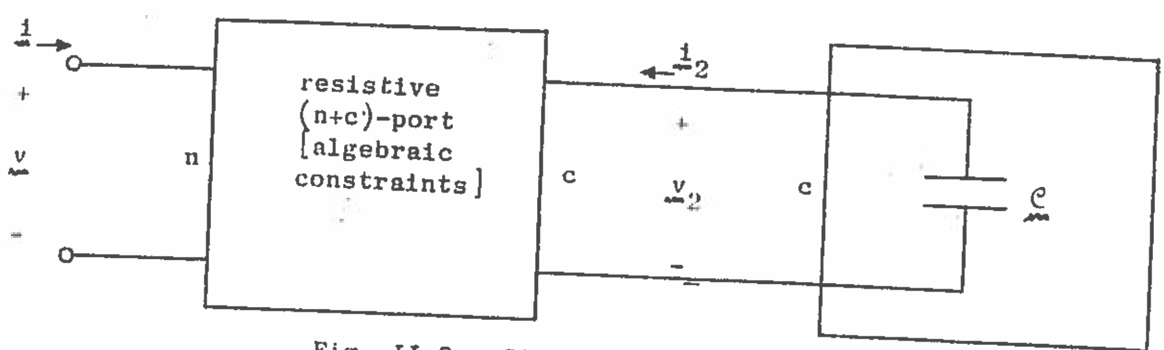


Fig. II-2. CAPACITOR EXTRACTION.

Our reason, of course, for isolating the capacitors is that their charges, or voltages, can serve as state variables. We can obtain a general description, that is, an $\mathcal{A}\, v = \mathcal{B}\, i$ characterization, for the resistive (n + c)-port, but let us assume that this (n + c)-port also possesses an admittance description Y_c, where since we are allowing the presence of time-variable circuit elements, we have that $Y_c = Y_c(t)$. In order to be able to apply the load constraints to obtain the state-variable description, we can partition Y_c according to the ports.

$$\begin{bmatrix} i \\ i_2 \end{bmatrix} = Y_c \begin{bmatrix} v \\ v_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v \\ v_2 \end{bmatrix} \qquad \text{(II-1)}$$

We point out that the existence of Y_c is an assumption of the theory, and one which places a restriction (which is often not too severe) on the class of circuits considered.

At this point it is convenient to rewrite the above equations in a partitioned form more useful for finding the canonical equations. Thus,

$$\begin{bmatrix} 1_n \\ 0 \end{bmatrix} \begin{bmatrix} i \end{bmatrix} + \begin{bmatrix} 0 \\ 1_c \end{bmatrix} \begin{bmatrix} i_2 \end{bmatrix} - \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} \begin{bmatrix} v \end{bmatrix} - \begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix} \begin{bmatrix} v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(II-2a)}$$

Next we observe that we should be able to choose the capacitor charge as the state, in which case we define

$$s = C\, v_2 \qquad \text{(II-2b)}$$

while from the load constraint we observe

$$i_2 = -\dot{s} = -\frac{d\, C\, v_2}{dt} \qquad \text{(II-2c)}$$

Here we have taken the matrix C as the $c \times c$ diagonal matrix of capacitance values; any capacitive coupling we assume to have been taken into account through transformers absorbed into the resistive $(n + c)$-port. We also assume $C(t)$ to be nonsingular. (Any singularity can actually be accounted for again by a change in the $(n + c)$-port, but we omit discussion of this rather tricky point in order to clearly proceed.) Substituting the (b) and (c) portions of Eq. (II-2) into the (a) one yields

$$\begin{bmatrix} 1_n \\ 0 \end{bmatrix} \begin{bmatrix} i \end{bmatrix} - \begin{bmatrix} 0 \\ 1_c \end{bmatrix} \begin{bmatrix} \dot{s} \end{bmatrix} - \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} \begin{bmatrix} v \end{bmatrix} - \begin{bmatrix} Y_{12}\, C^{-1} \\ Y_{22}\, C^{-1} \end{bmatrix} \begin{bmatrix} s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(II-3)}$$

The second set of (c) equations gives the derivative portion of the canonical equations, while the first set of n equations gives the output portion. Thus,

$$\dot{s} = -y_{22}\, C^{-1}\, s - y_{21}\, v \tag{II-4a}$$

$$i = y_{12}\, C^{-1}\, s + y_{11}\, v \tag{II-4b}$$

We have obtained the realization

$$R = \left\{ -y_{22}\, C^{-1},\ y_{21},\ y_{12}\, C^{-1},\ y_{11} \right\} \tag{II-4c}$$

in a simple manner. It is worth mentioning that if time-variable elements are present the realization matrices are functions of time, in which case we have succeeded in setting up the canonical state-variable equations for time-variable circuits. In the time-invariant situation we observe that the method proceeds only when there is no pole in the (n-port) admittance matrix at infinity; we will later (Sec. C) obtain a graph theory condition for no pole at infinity such that a test can be directly made on the circuit graph. In any case, time-variable or not, the method proceeds if and only if the coupling admittance matrix Y_c exists; the existence of Y_c is equivalent to the existence of the inverse of the $\mathcal{B}$ matrix in the general description, $\mathcal{A}\, v = \mathcal{B}\, i$, for the (n + c)-port coupling network.

As an example, let us consider the 2-port of Fig. II-3, which is a subportion of the nonreciprocal Brune section, useful for its own sake (since it is equivalent to a series inductor in cascade with a transformer).

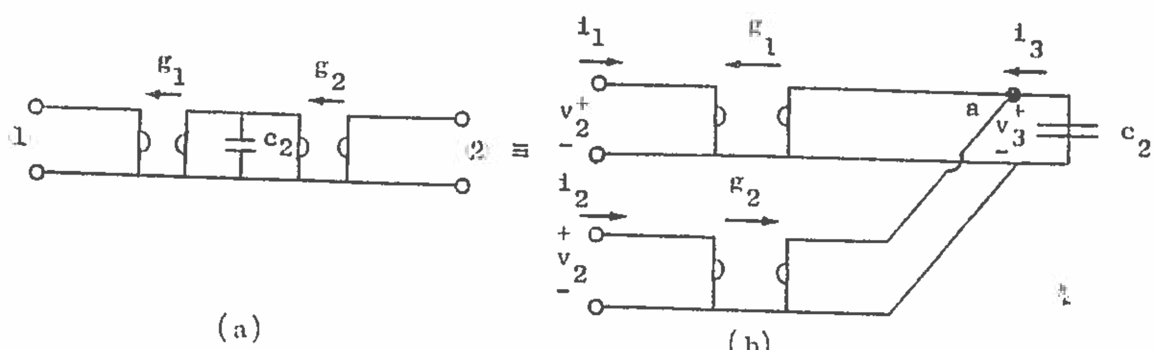


Fig. II-3. SHUNT-CAPACITOR LOADED-GYRATOR CASCADE.

By extracting the capacitor as shown in the (b) portion of the figure, we can obtain the appropriate equations. First we write the general description for the 3-port coupling structure (by respectively summing currents at node a and then writing i_1 and i_2 in terms of v_3 through the gyrator relationships).

$$\begin{bmatrix} -g_1 & g_2 & 0 \\ 0 & 0 & g_1 \\ 0 & 0 & -g_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \qquad \text{(II-5a)}$$

The coefficient matrix of the currents is nonsingular, being a permutation matrix, and thus on premultiplying Eq. (II-5a) by its inverse we find

$$\underset{\sim}{Y}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -g_1 & g_2 & 0 \\ 0 & 0 & g_1 \\ 0 & 0 & -g_2 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & g_1 \\ 0 & 0 & -g_2 \\ \hline -g_1 & g_2 & 0 \end{array}\right] \qquad \text{(II-5b)}$$

where we have made the partition appropriate to Eq. (II-1). Note that $\underset{\sim}{Y}_c$ is skew-symmetric, $\underset{\sim}{Y}_c = -\underset{\sim}{\tilde{Y}}_c$ (where ~ means transpose), as expected, since it is constructed solely from gyrators.

Equation (II-3) is directly

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} - \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right] [\dot{s}] - \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline -g_1 & g_2 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \left[\begin{array}{c} g_1/c_2 \\ -g_2/c_2 \\ \hline 0 \end{array}\right] [s] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \end{array}\right] \qquad \text{(II-5c)}$$

where we have partitioned the last $c = 1$ equations to be split off. Thus we have, by such a split, the canonical equations directly as

$$\dot{s} = 0 \cdot s + [g_1 \ \ -g_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{(II-5d)}$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_1/c_2 \\ g_2/c_2 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{(II-5e)}$$

We comment that, since the nonreciprocal Brune section itself has a pole at infinity, no $\underset{\sim}{Y}_c$ exists for it. However, on removal of the pole at infinity, Eqs. (II-5) result; hence the canonical equations for the admittance description of the Brune section are merely obtained from Eqs. (II-5d,e) by adding

$$\begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}$$

to the right of Eq. (II-5e). Note also that the canonical equations previously found for the Brune section were for a different set of input-output variables (that is, a different system). Still the same method was applied at that point.

We also comment that, upon adding suitable ports and ignoring variables of no interest, we can use the same method to find almost any input-output canonical set of state-variable equations, perhaps also after simple transformations on the variables. This result is directly seen by setting up equations in hybrid form.

Since the steps carried out are easily programmed, the procedure is a very convenient one for use in setting up canonical equations on a digital computer. For such purposes one needs a method for obtaining the coupling admittance $\underset{\sim}{Y}_c$ on the computer. Perhaps the most convenient method is to reduce the indefinite admittance matrix [2, p. 78] for the resistive coupling network to obtain $\underset{\sim}{Y}_c$; several programs are available for finding the indefinite admittance matrix, but a program is also very easily written from scratch. An alternate and almost equally useful method is to use the topological methods which we now discuss.

C. Topological Formulation

Let us again consider a finite circuit for which the equivalence of Fig. II-1 is used to replace inductors; again this replacement is not necessary but is convenient for simplification of already complicated expressions. Also we will assume that the admittance description is desired for which voltage sources have been placed at the ports.

By replacing each circuit element branch by a line segment, with an arbitrarily assigned orientation, as shown in Fig. II-4, we obtain an oriented graph to represent the circuit, the branches of which we can

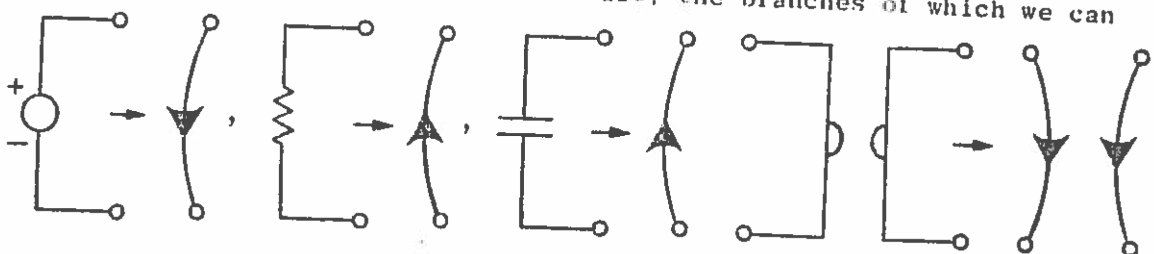


Fig. II-4. EXAMPLE GRAPH REPLACEMENTS.

number in some useful manner. A graph associated with a network or circuit structure will be called a network graph.

In order to proceed we introduce the following somewhat standard nomenclature associated with a network graph:

node=vertex	a dot on the graph (= a terminal of a circuit element branch)
branch	a line connecting two nodes (= a circuit element branch)
path	a sequence of branches and associated nodes
connected graph	a network graph in which every node is connected to every other node by a path
separate part	a maximally connected subgraph (that is, a subgraph for which all branches are connected to all other branches in the subgraph and to no others)
tree	a maximally connected subgraph of a separate part which contains no closed path
forest	a collection of trees of a graph, one for each separate part
cotree	the set of branches (in a separate part) which remain when a (fixed) tree is deleted
link	a branch of a (fixed) cotree

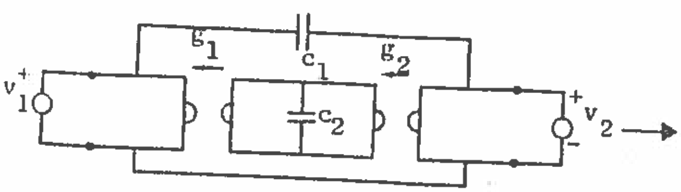
Although these definitions are not completely rigorous (for example "connected" and "closed path" are not made precise), they should be intuitively clear, perhaps after an example, and are sufficient for our purposes. To further proceed we introduce the following symbols:

- b = total number of branches
- ℓ = total number of links (cotree branches)
- s = number of separate parts
- t = total number of tree branches
- v = total number of nodes

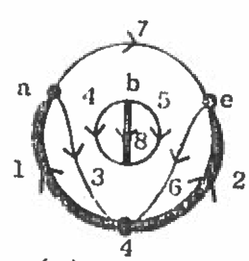
Here ℓ and t are formed by summing over all trees in a forest. For each separate part the number of tree branches is one less than the number of nodes while it is also clear that $b = \ell + t$. Thus we can directly predict the number of tree branches and links, without expressly exhibiting a tree, through

$$t = v - s, \qquad \ell = b - v + s \tag{II-6}$$

As an example, let us consider the 2-port of Fig. II-5 which has been closed, as mentioned above, on voltage sources (as will be appropriate to setting up the canonical equations; note that this network is identical in port behavior to the nonreciprocal Brune section of Fig. I-5). A possible network graph is shown in the (b) portion, with other graphs resulting by different choices of branch orientation and numbering. Note that by simple count $b = 8$, $s = 2$, $v = 5$, and thus, by Eq. (II-6), $t = 3$ and $\ell = 5$; these numbers are checked from the graph where a possible choice for a tree is shown in boldface (note that there are other choices for a tree, but that in a given analysis only one at a time is need).



(a)



(b)

Fig. II-5. EXAMPLE GRAPH FROM CIRCUIT STRUCTURE.

Since we will wish to sum currents at the nodes, we have also labeled them. We observe that for a node analysis we wish to choose tree branch voltages as independent variables while for a loop analysis we wish to choose link currents. In setting up the state-variable equations we actually will work with both types of variables.

Next we introduce the following (column) vector variables:

$\underset{\sim}{i}_b$ = vector of branch currents ($b \times 1$)

$\underset{\sim}{v}_b$ = vector of branch voltages ($b \times 1$)

$\underset{\sim}{i}_\ell$ = vector of link currents ($\ell \times 1$)

$\underset{\sim}{v}_t$ = vector of tree branch voltages ($t \times 1$)

$\underset{\sim}{v}$ = vector of port (source) voltages ($n \times 1$)

$\underset{\sim}{i}$ = vector of port currents ($n \times 1$)

Along with these variables we assume the polarity of a given branch's variables in conjunction with the given branch orientation as shown in Fig. II-6.

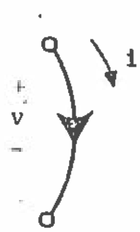


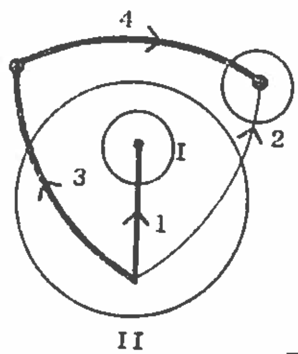
Fig. II-6. POLARITY OF VARIABLES.

Now we introduce the cut set and tie set matrices from which the analysis can truly begin. For a given circuit we pick a fixed forest. The cut set matrix $\underset{\sim}{C}$ is defined by considering the tree branches in numerical order; for each tree branch a circle (or similar curve) is drawn such that of all the tree branches only the prescribed one is cut by the circle. The (oriented) set of branches cut by any one circle is called a <u>cut set</u>. For any one cut set all the currents entering the circle on the cut set branches must sum to zero by Kirchhoff's current law; considering all t cut sets we obtain

$$\underset{\sim}{0} = \underset{\sim}{C}\,\underset{\sim}{i}_b \tag{II-7}$$

where $\underset{\sim}{C}$ is the $t \times b$ <u>cut set matrix</u> (consisting of 0 or $\pm$ 1's). As an example, Fig. II-7 shows the cuts for the particular graph. The resulting cut set matrix is given as the coefficient matrix in the equation

$$\begin{matrix}\text{cut I} \rightarrow \\ \text{cut II} \rightarrow \\ \text{cut III} \rightarrow\end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} \qquad \text{(II-8)}$$



cut set I = branch 1

cut set II = branch 2 (out) and branch 3 (out)

cut set III = branch 2 (in) and branch 4 (in)

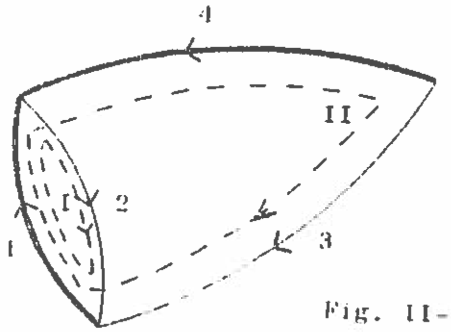
Fig. II-7. EXAMPLE CUTS FOR $\underset{\sim}{\mathcal{C}}$.

The tie set matrix $\underset{\sim}{\mathcal{T}}$ is defined in a somewhat dual manner. Again a forest is chosen. On removing all links a particular link is reinserted; the (oriented) branches forming a closed path under this reinsertion are the associated tie set. Ordering all tie sets according to the numerical order of the links defines, through Kirchhoff's voltage law (applied to each loop of tie set branches),

$$\underset{\sim}{0} = \underset{\sim}{\mathcal{T}}\, \underset{\sim}{v}_b \qquad \text{(II-9)}$$

where $\underset{\sim}{\mathcal{T}}$ is the $\ell \times b$ <u>tie set matrix</u> (again consisting of 0 or $\pm$ 1's). For example, Fig. II-8 has

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \qquad \text{(II-10)}$$



tie set I = branch 2 (+) and branch 1 (+)

tie set II = branch 3 (+), branch 1 (+) and branch 4 (-)

Fig. II-8. EXAMPLE TIES FOR $\underset{\sim}{J}$.

We also claim that it is possible to write (recall that $\tilde{}$ = transpose)

$$\underset{\sim}{v}_b = \tilde{\underset{\sim}{C}}\,\underset{\sim}{v}_t \qquad \text{(II-11a)}$$

$$\underset{\sim}{i}_b = \tilde{\underset{\sim}{J}}\,\underset{\sim}{i}_l \qquad \text{(II-11b)}$$

For the plausibility of Eq. (II-11a), say, let us argue as follows. By Kirchhoff's voltage law it should be clear that the tree branch voltages determine all link voltages; hence there is a linear transformation to give $\underset{\sim}{v}_b = \underset{\sim}{A}\,\underset{\sim}{v}_t$, where $\underset{\sim}{A}$ is some $b \times t$ matrix in fact consisting of zeros and (+ or -) ones. If we consider the graph as a closed system then the total input power is zero, $P_{in} = \tilde{\underset{\sim}{v}}_b\,\underset{\sim}{i}_b = 0$. Thus $\tilde{\underset{\sim}{v}}_b\,\underset{\sim}{i}_b = \tilde{\underset{\sim}{v}}_t\,\tilde{\underset{\sim}{A}}\,\underset{\sim}{i}_b = 0$. Since the tree branch voltages can be arbitrarily assigned (when the graph is considered as an abstract object), we must require $\tilde{\underset{\sim}{A}}\,\underset{\sim}{i}_b = \underset{\sim}{0}$. In other words if we choose Eq. (II-11a), then Eq. (II-7) follows as a possibility. [Of course, a proof requires that we argue in reverse, but this can be done by beginning with Eqs. (II-11) at first.]

For convenience of notation we next choose a numbering of branches such that all the branches occur first; thus

$$\underset{\sim}{v}_b = \begin{bmatrix} \underset{\sim}{v}_t \\ \underset{\sim}{v}_\ell \end{bmatrix}, \qquad \underset{\sim}{i}_b = \begin{bmatrix} \underset{\sim}{i}_t \\ \underset{\sim}{i}_l \end{bmatrix} \qquad \text{(II-11c)}$$

in which case Eqs. (II-11a,b) show that the cut set and tie set matrices can be partitioned as

$$\underset{\sim}{\mathcal{C}} = [\underset{\sim}{1}_t \,\vdots\, \underset{\sim}{C}] , \qquad \underset{\sim}{\mathcal{T}} = [\underset{\sim}{T} \,\vdots\, \underset{\sim}{1}_\ell] \tag{II-11d}$$

where $\underset{\sim}{C}$ and $\underset{\sim}{T}$ are, respectively $t \times \ell$ and $\ell \times t$ matrices; $\underset{\sim}{1}_t$ is, of course, the $t \times t$ identity. We observe that

$$\underset{\sim}{C} = - \underset{\sim}{\tilde{T}} \tag{II-11e}$$

since again

$$\underset{\sim}{\tilde{v}}_b \underset{\sim}{i}_b = 0 = \underset{\sim}{\tilde{v}}_t [\underset{\sim}{1}_t \;\; \underset{\sim}{C}] \begin{bmatrix} \underset{\sim}{\tilde{T}} \\ \\ \underset{\sim}{1}_\ell \end{bmatrix} \underset{\sim}{i}_\ell = \underset{\sim}{\tilde{v}}_t [\underset{\sim}{\tilde{T}} + \underset{\sim}{C}] \underset{\sim}{i}_\ell = 0$$

and $\underset{\sim}{v}_t$ and $\underset{\sim}{i}_\ell$ can be arbitrarily assigned.

Our next step is to place all voltage sources in tree branches. (We remark that we are only considering the presence of voltage sources; if current sources are present, only simple modifications are necessary, or one can use the equivalence of Fig. II-9.) Next we place as many

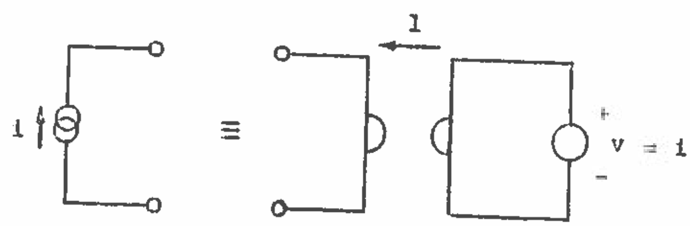


Fig. II-9. CURRENT SOURCE EQUIVALENT.

as possible of the capacitors in tree branches--any left over are somehow "excess"; but the need for considering these excess capacitors is in fact the reason for our treating this topological method. It follows that if there is a capacitor link then the path formed by the associated tie set branches consists entirely of voltage sources and capacitors--such gives rise to a pole at infinity, for example, in the admittance matrix. Let us now further fix our numbering of branches such that $\underset{\sim}{v}_t$ and $\underset{\sim}{v}_\ell$ take the form

$$\underset{\sim}{v}_t = \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{v}_{ct} \\ \underset{\sim}{v}_{rt} \end{bmatrix} , \quad \underset{\sim}{v}_\ell = \begin{bmatrix} \underset{\sim}{v}_{c\ell} \\ \underset{\sim}{v}_{r\ell} \end{bmatrix} \tag{II-12}$$

where the subscripts c and r refer to capacitor and resistor portions of the graph.

At this point we can begin the real procedure for setting up the state-variable equations [3]. If we partition the matrix $\underset{\sim}{T}$, of Eq. (II-11d), using $\underset{\sim}{C} = -\tilde{\underset{\sim}{T}}$, we find for Eq. (II-9), $\underset{\sim}{0} = \underset{\sim}{\mathcal{J}}\underset{\sim}{v}_b$, and for Eq. (II-7), $\underset{\sim}{0} = \underset{\sim}{\mathcal{C}}\,\underset{\sim}{i}_b$

$$\begin{matrix} c\ell \rightarrow \\ \\ r\ell \rightarrow \end{matrix} \begin{bmatrix} \underset{\sim}{0} \\ \\ \underset{\sim}{0} \end{bmatrix} = \left[\begin{array}{ccc:cc} \underset{\sim}{T}_{11} & \underset{\sim}{T}_{12} & \underset{\sim}{0} & \underset{\sim}{1}_{c\ell} & \underset{\sim}{0} \\ & & & & \\ \underset{\sim}{T}_{21} & \underset{\sim}{T}_{22} & \underset{\sim}{T}_{23} & \underset{\sim}{0} & \underset{\sim}{1}_{r\ell} \end{array}\right] \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{v}_{ct} \\ \underset{\sim}{v}_{rt} \\ \underset{\sim}{v}_{c\ell} \\ \underset{\sim}{v}_{r\ell} \end{bmatrix} \tag{II-13a}$$

$$\begin{matrix} \text{source} \rightarrow \\ ct \rightarrow \\ rt \rightarrow \end{matrix} \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix} = \left[\begin{array}{ccc:cc} \underset{\sim}{1}_n & \underset{\sim}{0} & \underset{\sim}{0} & -\tilde{\underset{\sim}{T}}_{11} & -\tilde{\underset{\sim}{T}}_{21} \\ \underset{\sim}{0} & \underset{\sim}{1}_{ct} & \underset{\sim}{0} & -\tilde{\underset{\sim}{T}}_{12} & -\tilde{\underset{\sim}{T}}_{22} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}_{rt} & \underset{\sim}{0} & -\tilde{\underset{\sim}{T}}_{23} \end{array}\right] \begin{bmatrix} \underset{\sim}{i}_s \\ \underset{\sim}{i}_{ct} \\ \underset{\sim}{i}_{rt} \\ \underset{\sim}{i}_{c\ell} \\ \underset{\sim}{i}_{r\ell} \end{bmatrix} \tag{II-13b}$$

In these equations $\underset{\sim}{T}_{13} = \underset{\sim}{0}$ since if there is a capacitor in a link then there is no resistor in the tree branches of the associated tie set. We also write $\underset{\sim}{i}_s$ for the source current and note $\underset{\sim}{i}_s = -\underset{\sim}{i}$, where $\underset{\sim}{i}$ is the port current.

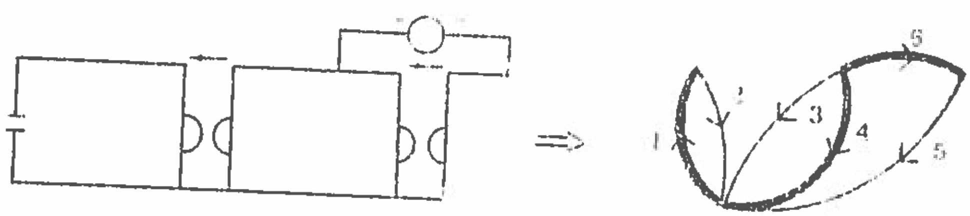
The circuit element constraints are next assumed to be of the form

$$\begin{bmatrix} \underset{\sim}{i}_{ct} \\ \underset{\sim}{i}_{c\ell} \\ \underset{\sim}{i}_{rt} \\ \underset{\sim}{i}_{r\ell} \end{bmatrix} = \begin{bmatrix} d\underset{\sim}{C}_t \underset{\sim}{v}_{ct}/dt \\ d\underset{\sim}{C}_\ell \underset{\sim}{v}_{c\ell}/dt \\ \underset{\sim}{G}_t \underset{\sim}{v}_{rt} \\ \underset{\sim}{G}_\ell \underset{\sim}{v}_{r\ell} \end{bmatrix} \qquad \text{(II-14a)}$$

In actual fact this form places some restrictions on the types of circuits allowed since no coupling between tree branch and link resistive (gyrator) elements is allowed; for example, the circuit of Fig. II-10 is ruled out. Of course a more general treatment is possible by using

$$\begin{bmatrix} \underset{\sim}{i}_{rt} \\ \underset{\sim}{i}_{r\ell} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{G}_t & \underset{\sim}{G}_{t\ell} \\ \underset{\sim}{G}_{\ell t} & \underset{\sim}{G}_\ell \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{rt} \\ \underset{\sim}{v}_{r\ell} \end{bmatrix} \qquad \text{(II-14b)}$$

but as we will see, the result is already complicated enough in notation.



note: 1 and 6 are required tree branches

Fig. II-10. EXAMPLE OF RESISTIVE COUPLING BETWEEN TREE BRANCHES.

Our next job is to make appropriate substitutions, etc. Through the various equations indexed as shown we can write the right side of Eq. (II-14a) as

$$
\begin{matrix} \text{identity} \rightarrow \\ \text{differentiate } c\ell \rightarrow \\ \text{identity} \rightarrow \\ r\ell \rightarrow \end{matrix}
\begin{bmatrix} d\mathbf{C}_t\mathbf{v}_{ct}/dt \\ d\mathbf{C}_\ell\mathbf{v}_{c\ell}/dt \\ \mathbf{G}_t\mathbf{v}_{rt} \\ \mathbf{G}_\ell\mathbf{v}_{r\ell} \end{bmatrix}
=
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{ct} & \mathbf{0} \\
-\frac{d\mathbf{C}_\ell}{dt}\mathbf{T}_{11} & -\frac{d\mathbf{C}_\ell\mathbf{T}_{12}\mathbf{C}_t^{-1}}{dt} & -\mathbf{C}_\ell\mathbf{T}_{11} & -\mathbf{C}_\ell\mathbf{T}_{12}\mathbf{C}_t^{-1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{rt} \\
-\mathbf{G}_\ell\mathbf{T}_{21} & -\mathbf{G}_\ell\mathbf{T}_{22}\mathbf{C}_t^{-1} & \mathbf{0} & \mathbf{0} & -\mathbf{G}_\ell\mathbf{T}_{23}\mathbf{G}_t^{-1}
\end{bmatrix}
$$

$$
\times \begin{bmatrix} \mathbf{v} \\ \mathbf{C}_t\mathbf{v}_{ct} \\ d\mathbf{v}/dt \\ d\mathbf{C}_t\mathbf{v}_{ct}/dt \\ \mathbf{G}_t\mathbf{v}_{rt} \end{bmatrix} \qquad \text{(II-15a)}
$$

while the left side can be expressed as

$$
ct \rightarrow \begin{bmatrix} \mathbf{1}_{ct} & -\tilde{\mathbf{T}}_{12} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{ct} \\ \mathbf{i}_{c\ell} \\ \mathbf{i}_{rt} \\ \mathbf{i}_{r\ell} \end{bmatrix} = \tilde{\mathbf{T}}_{22}\mathbf{i}_{r\ell} = \tilde{\mathbf{T}}_{22}\mathbf{G}_\ell\mathbf{v}_{r\ell} \qquad \text{(II-15b)}
$$

Substituting Eqs. (II-15a,b) into (II-14a) yields desirable equations with $\mathbf{v}_{c\ell}$ eliminated; but the presence of $\mathbf{v}_{rt}$ and $\mathbf{v}_{r\ell}$ is unwanted so we proceed to eliminate them also. We have

$$
r\ell \rightarrow \mathbf{v}_{r\ell} = -\begin{bmatrix} \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v}_{ct} \\ \mathbf{v}_{rt} \end{bmatrix} \qquad \text{(II-15c)}
$$

and

$$rt \rightarrow \underset{\sim}{i}_{rt} = \underset{\sim}{\tilde{T}}_{23}\underset{\sim}{i}_{r\ell} = \underset{\sim}{\tilde{T}}_{23}\underset{\sim}{G}_{\ell}\underset{\sim}{v}_{r\ell} = \underset{\sim}{G}_{t}\underset{\sim}{v}_{rt} \qquad \text{(II-15d)}$$

Combining these last two gives

$$\underset{\sim}{v}_{rt} = -\underset{\sim}{G}_{t}^{-1}\underset{\sim}{\tilde{T}}_{23}\underset{\sim}{G}_{\ell}[\underset{\sim}{T}_{21} \quad \underset{\sim}{T}_{22} \quad \underset{\sim}{T}_{23}]\begin{bmatrix}\underset{\sim}{v}\\ \underset{\sim}{v}_{ct}\\ \underset{\sim}{v}_{rt}\end{bmatrix}$$

which on solution for $\underset{\sim}{v}_{rt}$ gives

$$\underset{\sim}{v}_{rt} = -(\underset{\sim}{1}_{rt} + \underset{\sim}{G}_{t}^{-1}\underset{\sim}{\tilde{T}}_{23}\underset{\sim}{G}_{\ell}\underset{\sim}{T}_{23})^{-1}\underset{\sim}{G}_{t}^{-1}\underset{\sim}{\tilde{T}}_{23}\underset{\sim}{G}_{\ell}[\underset{\sim}{T}_{21} \quad \underset{\sim}{T}_{22}\underset{\sim}{C}_{t}^{-1}]\begin{bmatrix}\underset{\sim}{v}\\ \underset{\sim}{C}_{t}\underset{\sim}{v}_{ct}\end{bmatrix} \qquad \text{(II-15e)}$$

Equation (II-15c) then gives

$$\underset{\sim}{v}_{r\ell} = [\underset{\sim}{-1}_{r\ell} - \underset{\sim}{T}_{23}(\underset{\sim}{1}_{rt} - \underset{\sim}{G}_{t}^{-1}\underset{\sim}{\tilde{T}}_{23}\underset{\sim}{G}_{\ell}\underset{\sim}{T}_{23})^{-1}\underset{\sim}{G}_{t}^{-1}\underset{\sim}{\tilde{T}}_{23}\underset{\sim}{G}_{\ell}]$$

$$[\underset{\sim}{T}_{21} \quad \underset{\sim}{T}_{22}\underset{\sim}{C}_{t}^{-1}]\begin{bmatrix}\underset{\sim}{v}\\ \underset{\sim}{C}_{t}\underset{\sim}{v}_{ct}\end{bmatrix} \qquad \text{(II-15f)}$$

Now let us finally substitute Eqs. (II-15a,b) into (II-14a) to get

$$\underset{\sim}{\tilde{T}}_{22}\underset{\sim}{G}_{\ell}[\underset{\sim}{-1}_{r\ell} + \underset{\sim}{T}_{23}(\underset{\sim}{1}_{rt} + \underset{\sim}{G}_{t}^{-1}\underset{\sim}{\tilde{T}}_{23}\underset{\sim}{G}_{\ell}\underset{\sim}{T}_{23})^{-1}\underset{\sim}{G}_{t}^{-1}\underset{\sim}{T}_{23}\underset{\sim}{G}_{\ell}][\underset{\sim}{T}_{21} \quad \underset{\sim}{T}_{22}\underset{\sim}{C}_{t}^{-1}]\begin{bmatrix}\underset{\sim}{v}\\ \underset{\sim}{C}_{t}\underset{\sim}{v}_{ct}\end{bmatrix}$$

$$= [\underset{\sim}{\tilde{T}}_{12}\frac{d\underset{\sim}{C}_{\ell}}{dt}\underset{\sim}{T}_{11} \quad \underset{\sim}{\tilde{T}}_{12}\frac{d\underset{\sim}{C}_{\ell}\underset{\sim}{T}_{12}\underset{\sim}{C}_{t}^{-1}}{dt} \quad \underset{\sim}{\tilde{T}}_{12}\underset{\sim}{C}_{\ell}\underset{\sim}{T}_{11} \quad \underset{\sim}{1}_{ct} + \underset{\sim}{\tilde{T}}_{12}\underset{\sim}{C}_{\ell}\underset{\sim}{T}_{12}\underset{\sim}{C}_{t}^{-1}]\begin{bmatrix}\underset{\sim}{v}\\ \underset{\sim}{C}_{t}\underset{\sim}{v}_{ct}\\ d\underset{\sim}{v}/dt\\ d\underset{\sim}{C}_{t}\underset{\sim}{v}_{ct}/dt\end{bmatrix}$$

(II-15g)

Equation (II-15g) is the same as

$$\dot{s} = A\,s + B\,u + F\,\dot{u} \tag{II-16a}$$

where we have

$$s = C_t v_{ct}, \qquad u = v \tag{II-16b}$$

$$A = \left[1_{ct} + \tilde{T}_{12} C_\ell T_{12} C_t^{-1}\right]^{-1} \Big\{ -\tilde{T}_{12} \frac{dC_\ell T_{12} C_t^{-1}}{dt}$$

$$+ \tilde{T}_{22} G_\ell \left[-1_{r\ell} + T_{23}\left(1_{rt} + G_t^{-1} \tilde{T}_{23} G_\ell T_{23}\right)^{-1} G_t^{-1} T_{23} G_\ell \right] T_{22} C_t^{-1} \Big\}$$

$$B = \left[1_{ct} + \tilde{T}_{12} C_\ell T_{12} C_t^{-1}\right]^{-1} \Big\{ -\tilde{T}_{12} \frac{dC_\ell}{dt} T_{11}$$

$$+ \tilde{T}_{22} G_\ell \left[-1_{r\ell} + T_{23}\left(1_{rt} + G_t^{-1} \tilde{T}_{23} G_\ell T_{23}\right)^{-1} G_t^{-1} T_{23} G_\ell \right] T_{21} \Big\}$$

$$F = -\left[1_{ct} + \tilde{T}_{12} C_\ell T_{12} C_t^{-1}\right]^{-1} T_{12} C_\ell T_{11} \tag{II-16c}$$

For the output equations we can use the source portion of Eq. (II-13b) to get

$$-i = \tilde{T}_{11} i_{c\ell} + \tilde{T}_{21} i_{r\ell} \tag{II-17}$$

But, from Eq. (II-14a), $i_{c\ell} = dC_\ell v_{c\ell}/dt$ and $i_{r\ell} = G_\ell v_{r\ell}$; the $i_{c\ell}$ part can be evaluated from Eq. (II-15a), and the $i_{r\ell}$ part from Eq. (II-15f). Thus we find

$$y = C\,s + D\,u + E\,\dot{u} \tag{II-18a}$$

with

$$y = i \tag{II-18b}$$

$$C = \tilde{T}_{11} \frac{dC_\ell T_{12} C_t^{-1}}{dt} + \tilde{T}_{11} C_\ell T_{12} C_t^{-1} A - \tilde{T}_{21} G_\ell$$

$$\times \left[-1_{r\ell} + T_{23}\left(1_{rt} + G_t^{-1} \tilde{T}_{23} G_\ell T_{23}\right)^{-1} G_t^{-1} \tilde{T}_{23} G_\ell \right] T_{22} C_t^{-1}$$

$$\underline{D} = \tilde{\underline{T}}_{11}\frac{d\underline{C}_\ell}{dt}\underline{T}_{11} + \tilde{\underline{T}}_{11}\underline{C}_\ell\underline{T}_{12}\underline{C}_t^{-1}\underline{B} - \tilde{\underline{T}}_{21}\underline{G}_\ell$$

$$\cdot\left[-\underline{1}_{r\ell} + \underline{T}_{23}\left(\underline{1}_{rt} + \underline{G}_t^{-1}\tilde{\underline{T}}_{23}\underline{G}_\ell\underline{T}_{23}\right)^{-1}\underline{G}_t^{-1}\tilde{\underline{T}}_{23}\underline{G}_\ell\right]\underline{T}_{21}$$

$$\underline{E} = \tilde{\underline{T}}_{11}\underline{C}_\ell\underline{T}_{11} + \tilde{\underline{T}}_{11}\underline{C}_\ell\underline{T}_{12}\underline{C}_t^{-1}\underline{F} \tag{II-18c}$$

Thus we observe that the equations obtained are not the canonical set but the pair [4]

$$\dot{\underline{s}} = \underline{A}\,\underline{s} + \underline{B}\,\underline{u} + \underline{F}\,\dot{\underline{u}} \tag{II-16a}$$

$$\underline{y} = \underline{C}\,\underline{s} + \underline{D}\,\underline{u} + \underline{E}\,\dot{\underline{u}} \tag{II-18a}$$

Nevertheless if $\underline{C}_\ell = 0$ then $\underline{F} = \underline{0}$ and $\underline{E} = \underline{0}$, and thus, when there are no capacitor-source tie sets, we obtain the canonical equations. It should be observed that the results are valid for time-variable elements and that the only real restriction on the result is the requirement that there be no resistive coupling between tree branches and links, that is, zero $\underline{G}_{t\ell}$ and $\underline{G}_{\ell t}$ in Eq. (II-14b).

Even in the time-invariant case where there are no capacitor-source tie sets, where considerable simplification occurs, the equations still remain rather messy. Thus we observe that, although the formulation is important for illustrating the general nature of network state-space-like equations, the approach is not the most useful to be taken for normal analysis or synthesis.

As an example, let us reconsider the nonreciprocal Brune section of Fig. I-5. This is redrawn in Fig. II-11, where the appropriate tree is shown with the numbering requested by the theory.

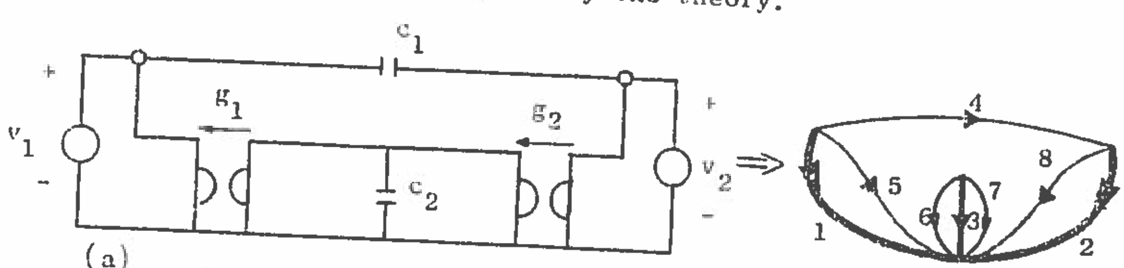


Fig. II-11. NONRECIPROCAL BRUNE SECTION GRAPH.

The tie set and cut set matrices are found as

$$\begin{matrix} 4 \rightarrow \\ 5 \rightarrow \\ 6 \rightarrow \\ 7 \rightarrow \\ 8 \rightarrow \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} \qquad \text{(II-19a)}$$

$$\begin{matrix} 1 \rightarrow \\ 2 \rightarrow \\ 3 \rightarrow \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{s1} \\ i_{s2} \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} \qquad \text{(II-19b)}$$

Next we have the element value constraints

$$\begin{bmatrix} i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} dc_2v_3/dt \\ dc_1v_4/dt \\ \begin{bmatrix} 0 & g_1 & 0 & 0 \\ -g_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_2 \\ 0 & 0 & -g_2 & 0 \end{bmatrix} \begin{bmatrix} v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} \end{bmatrix} \qquad \text{(II-19c)}$$

Then

$$\underset{\sim}{A} = -\frac{\tilde{\underset{\sim}{T}}_{22}\underset{\sim}{G}_\ell\underset{\sim}{T}_{22}}{c_2} = 0$$

$$\underset{\sim}{B} = -\tilde{\underset{\sim}{T}}_{22}\underset{\sim}{G}_\ell\underset{\sim}{T}_{21} = [g_1 \quad -g_2]$$

$$\underset{\sim}{F} = [0 \quad 0]$$

$$\underset{\sim}{C} = \frac{\tilde{\underset{\sim}{T}}_{21}\underset{\sim}{G}_\ell\underset{\sim}{T}_{22}}{c_2} = \begin{bmatrix} g_1/c_2 \\ -g_2/c_2 \end{bmatrix}$$

$$\underset{\sim}{D} = \tilde{\underset{\sim}{T}}_{11}\dot{c}_1\underset{\sim}{T}_{11} + \tilde{\underset{\sim}{T}}_{21}\underset{\sim}{G}_\ell\underset{\sim}{T}_{21} = \dot{c}_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underset{\sim}{E} = \tilde{\underset{\sim}{T}}_{11}c_1\underset{\sim}{T}_{11} = c_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

In this case the resulting equations are canonical and take the form

$$\dot{s} = 0\cdot s + [-g_1 \quad g_2]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad . \qquad \text{(II-19d)}$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_1/c_2 \\ -g_2/c_2 \end{bmatrix} s + \dot{c}_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} \qquad \text{(II-19e)}$$

The result is checked by calculating the transfer function matrix in the time-invariant case.

D. Transformation to Canonical Form

Because of the presence of $\underset{\sim}{F}$ in the resultant topological equations, it is of interest to find a transformation to eliminate the derivative of the input in the differential equation for the state. For this let us assume that we have on hand a set of equations

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \hat{\underset{\sim}{B}}\,\underset{\sim}{u} + \underset{\sim}{F}\,\dot{\underset{\sim}{u}} \qquad \text{(II-20a)}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x} + \hat{\underset{\sim}{D}}\,\underset{\sim}{u} + \underset{\sim}{E}\,\dot{\underset{\sim}{u}} \qquad \text{(II-20b)}$$

The transformation

$$\underset{\sim}{x} = \underset{\sim}{s} + \underset{\sim}{F}\,\underset{\sim}{u} \qquad \text{(II-20c)}$$

leads to the canonical set

$$\dot{\underset{\sim}{s}} = \underset{\sim}{A}\,\underset{\sim}{s} + (\hat{\underset{\sim}{B}} + \underset{\sim}{F} - \dot{\underset{\sim}{F}})\underset{\sim}{u} \qquad \text{(II-20d)}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{s} + (\hat{\underset{\sim}{D}} + \underset{\sim}{F})\underset{\sim}{u} + \underset{\sim}{E}\,\dot{\underset{\sim}{u}} \qquad \text{(II-20e)}$$

We observe that such a transformation, for which the input becomes part of the state, leaves the $\underset{\sim}{A}$, $\underset{\sim}{C}$, and $\underset{\sim}{E}$ matrices unchanged.

E. Combination of Methods

If one applies the topological method to a purely resistive structure, the results are considerably simplified. In the cases where there is no coupling between tree branches and links, one merely has that the admittance is given by $\underset{\sim}{D}$ of Eq. (II-18c). We point out that the operations to obtain $\underset{\sim}{D}$ are in this resistive case relatively easy to set up on a computer. Hence if capacitor extractions are first made and then a topological analysis carried out on the resulting resistive structure, a very convenient method of setting up state-variable equations via the computer results. The method is also quite easily extended to cover those cases where there is resistive coupling between links and tree branches.

By first setting up the graph of the circuit, the topological approach can be used to check the circuit for capacitor-source tie sets to establish the existence of the $\underset{\sim}{E}$ matrix. If there are such tie sets, the topological formulation to calculate $\underset{\approx}{E}$ can actually be carried out--the last of Eq. (II-18c)--since this calculation in itself is not too difficult.

F. Discussion

Because we feel it important to understand somewhat more fully how state-variable equations can arise, as well as more of their meaning, we have presented two convenient methods of setting up the canonical equations. Although both methods cover most situations of interest and have been presented for the time-varying case, neither one is in itself completely general. The capacitor extraction method is lacking in that there can be no capacitor-source tie sets in the circuit, while the topological method needs to be extended to cover the case where non-dynamical (that is, resistive) portions have coupling between the tree branches and the links. The capacitor extraction method has the advantage of simplicity while the topological method has the advantage of proceeding directly from the circuit structure. When the two methods are combined by applying the topological techniques to the nondynamic portions resulting from the capacitor extractions, an excellent method appropriate for computer analysis of networks results.

To this point we have not commented upon the existence of various inverses needed in the topological approach. To investigate these would cause an inappropriate diversion so we merely mention that in the case of passive time-invariant circuit elements all inverses are known to exist [3, p. 511].

In many applications, especially for integrated circuits, one meets voltage-controlled voltage sources. By changing somewhat the theory, these can be handled directly, but for our purposes it is worth observing that the topological theory presented applies if one is willing to use the equivalence of Fig. II-12, for which each of the cascade portions posseses a conductance matrix.

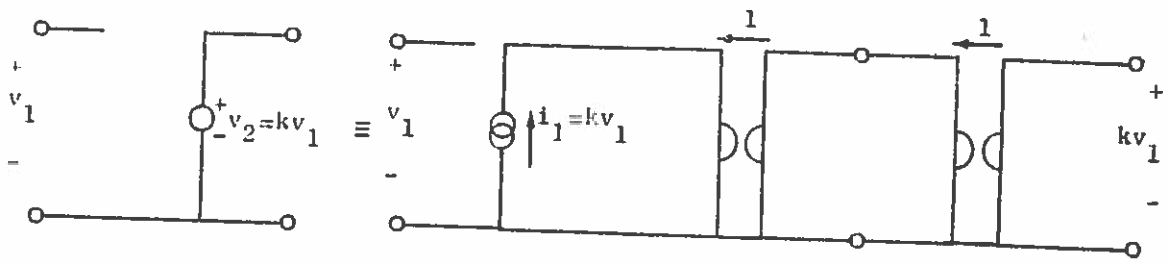


Fig. II-12. CONTROLLED SOURCE EQUIVALENT.

Since the topological method is in itself a bit complicated in end results, it is of interest to note that the results are almost identical to those obtained by Bryant [4] by very similar means.

Our next step will be to reverse the procedure and set up a physical realization from a state-variable realization.

G. References

1. Miller, J. A., and R. W. Newcomb, "A Computer-Oriented Technique for Determining State-Variable Equations for Admittance Descriptions," to appear.

2. Huelsman, L. P., "Circuits, Matrices, and Linear Vector Spaces," McGraw-Hill, New York, 1963.

3. Brown, D. P., "Derivative-Explicit Differential Equations for RLC Graphs," J. Franklin Institute, vol. 275, no. 6, June, 1963, pp. 503-514.

4. Bryant, P. R., "The Explicit Form of Bashkows A Matrix," IRE Trans. on Circuit Theory, vol. CT-9, no. 3, September, 1962, pp. 303-306.

5. Bailey, E., Private discussions, Stanford, summer 1967.

H. Exercises

1. Set up canonical state-variables equations for the filter circuit of Fig. EII-1.

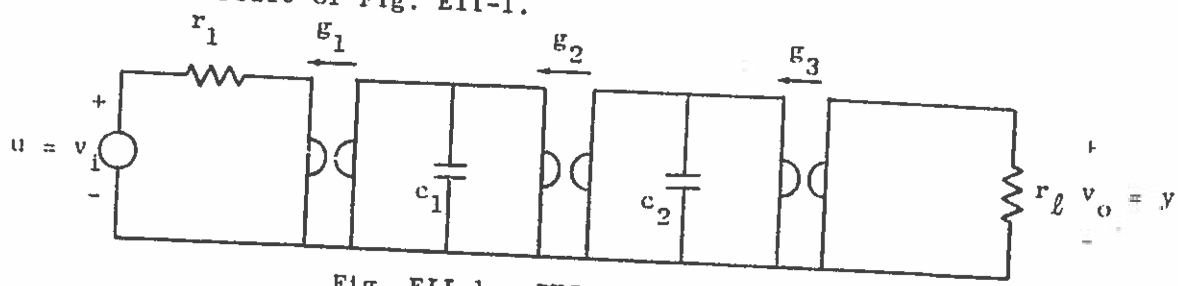


Fig. EII-1. DEGREE TWO FILTER.

2. Set up the canonical state-variable equations for the classical degree two feedback section of Fig. EII-2.

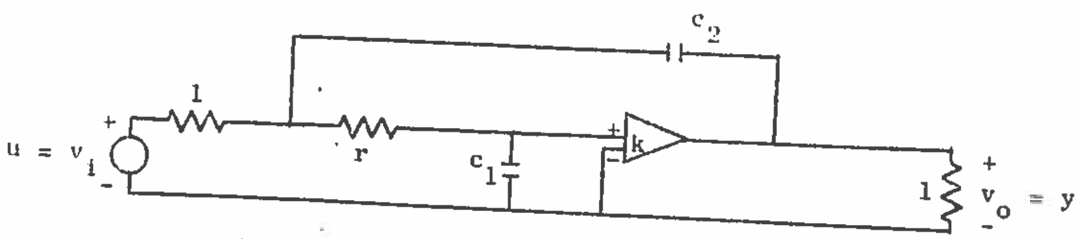


Fig. EII-2. DEGREE TWO FEEDBACK SECTION.

*3. Develop a method for the analysis by topological means of the general resistive structure coming from the capacitor extraction method such has been proposed by E. Bailey [5].

4. Set up the canonical state-variable equations for the integrated circuit integrator of Fig. III-4b and investigate various transformations on the resultant equations.

*5. Investigate the existence of the inverses needed to form $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{F}$ of Eq. (II-16c). From such an investigation, exhibit an example of a circuit with no canonical set of state-variable equations. Further, investigate the set of equations needed to be discussed such that all circuits, active or passive but with differential equation descriptions, are covered.

6. Set up the canonical equations by the topological method without using gyrator replacements when only inductors and capacitors (as well as sources) are present.

A la hâte, sagement et minutieusement, ils se préparent à vivre.
Et puis, vers la vingtième année, s'eloignent à la hâte, en étouffant leurs pas, comme s'ils venaient de découvrir qu'ils s'étaient trompés de demeure et qu'ils allaient passer leur vie parmi des hommes qu'ils ne connaissaient pas.

M. Maeterlinck
"Les Avertis" du "Trésor des Humbles"

CHAPTER III

INTEGRATED AND ANALOG CIRCUIT CONFIGURATIONS

A. Summary

The canonical equations are convenient for system simulation, especially through the use of integrated circuits. Here we discuss the concepts of interest in terms of appropriate integrated circuit configurations. In the development special operational amplifier circuits are considered to illustrate some of the points associated with integrated circuit structures.

B. Canonical Equation Simulation - Block Diagram

Let us consider the canonical equations of the form

$$\dot{\underset{\sim}{s}} = \underset{\sim}{A}\, \underset{\sim}{s} + \underset{\sim}{B}\, \underset{\sim}{u} \qquad \text{(I-7a)}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\, \underset{\sim}{s} + \underset{\sim}{D}\, \underset{\sim}{u} \qquad \text{(I-7b)}$$

where the dot has been used to denote time differentiation. If we integrate these canonical equations while denoting the (zero state) integral operator as $1/p$, that is,

$$\frac{1}{p} = \int_{t_o}^{t} [\;\;]\, dt \qquad \text{(III-1a)}$$

then we arrive at the useful equations for analog simulation

$$\underset{\sim}{s} = \left(\frac{1}{p}\,\underset{\sim}{1}_k\right)\,[\underset{\sim}{A}\,\underset{\sim}{s} + \underset{\sim}{B}\,\underset{\sim}{u}] \tag{III-1b}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{s} + \underset{\sim}{D}\,\underset{\sim}{u} \tag{III-1c}$$

where, as before, $\underset{\sim}{1}_k$ is the $k \times k$ identity, the state $\underset{\sim}{s}$ being a k vector. For any input $\underset{\sim}{u}$ the system can be simulated from a given realization $R = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}\}$, such that the output $\underset{\sim}{y}$ is determined by the block diagram of Fig. III-1. Note that since the various subsystems are multidimensional, the separate blocks have, in general, multiple inputs and outputs.

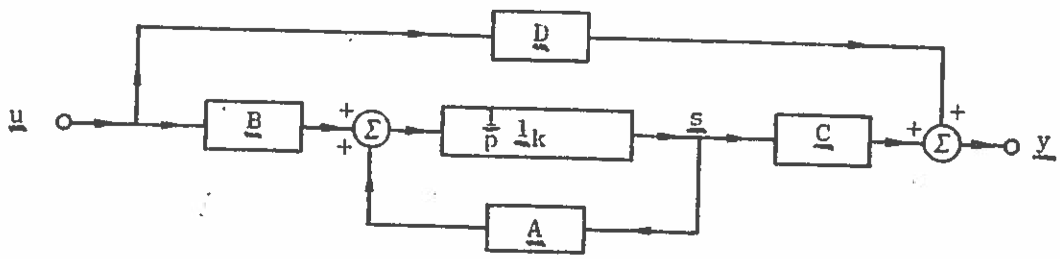


Fig. III-1. BLOCK DIAGRAM FOR CANONICAL EQUATIONS.

Several things can be noted concerning Fig. III-1:

1. Positive feedback is used and hence for (asymptotic) stability we require $\underset{\sim}{A}$ to have all of its eigenvalues negative.
2. Except for the integrators, all blocks consist simply of gain elements. Such multidimensional gain blocks can be constructed by interconnecting one-dimensional gain blocks, as shown for example in Fig. III-2 for the 2-input, 3-output case. We shall later see a method of summing, with gain, many inputs using a single amplifier, but at this point remark that the gain blocks as well as summers need consist only of operational amplifiers and resistors.
3. All integrators are uncoupled and of unity gain. In practice, and especially with integrated circuits, nonunity gain integrators must be used, necessitating a scale change. Since it is most convenient to construct all components identical with integrated circuits, it is practically more useful to simulate the system through the equations

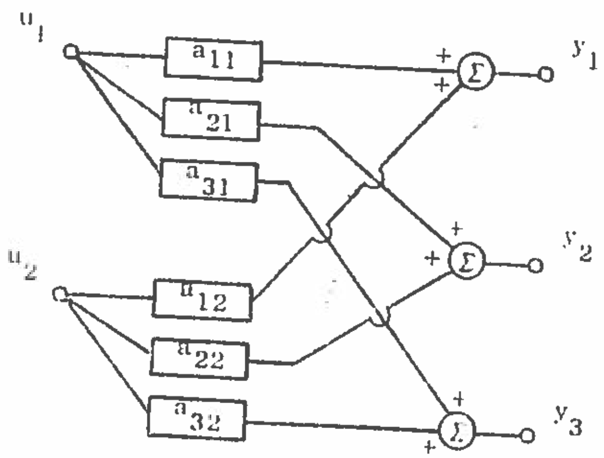
$$\underset{\sim}{s} = \left(\frac{\Lambda}{p}\underset{\sim}{1}_k\right)\left[\frac{\underset{\sim}{A}}{\lambda}\underset{\sim}{s} + \frac{\underset{\sim}{B}}{\lambda}\underset{\sim}{u}\right] \tag{III-2a}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{s} + \underset{\sim}{D}\,\underset{\sim}{u} \tag{III-2b}$$

where λ is an appropriate gain constant. A simulation of these latter equations is just as for the previous ones except that the integrator and $\underset{\sim}{A}$, $\underset{\sim}{B}$ blocks are scaled in gain.

4. For most practical simulations it is customary to use voltages as variables, in which case all gains are for voltage transfer elements.

5. Time-variable realizations R are allowed, in which case it is of interest to observe that our use of p is as a differential operator and not as the Laplace transform variable.



$$\Leftrightarrow \quad \underset{\sim}{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \underset{\sim}{u}$$

Fig. III-2. THE 2-INPUT, 3-OUTPUT GAIN BLOCK.

C. Integrators and Summers

In order to simulate the canonical equations we see that it is of interest to have gain blocks, integrators, and summers. A glance at Fig. III-2, as well as the manner in which summation occurs in Fig. III-1, shows that the gain portions can be incorporated in the summers. Consequently, we concentrate upon one-dimensional integrators and multiple-input, single-output summers with emphasis upon structures suitable for integrated circuits.

The basic building block is the operational amplifier. For integrated circuits one likes to use symmetrical structures with equal resistors, with quantities of interest determined by ratios of resistors in place of absolute values, where possible. Likewise one generally avoids pnp transistors where possible because of processing problems associated with making both npn and pnp transistors simultaneously. One is therefore led to consider the basic operational amplifier structure of Fig. III-3, on which many refinements are made to obtain various types of improvements, as higher gain by cascading of input amplifiers. For reasonable values of R, larger than the T_2 emitter-base resistance (say $R \approx 3$ kΩ), the gain of the device is roughly [1, p.]

$$K \approx \frac{q}{4kT} V_b \; [\approx 10\, V_b \text{ at room temperature}] \qquad \text{(III-3)}$$

where q = electron charge, k = Boltzmann's constant, T = absolute temperature. We observe that a differential amplifier is obtained, this being convenient for summers which both add and subtract. On the circuit diagram some of the dc voltages have been indicated for convenience with the input voltages v_+ and v_- assumed held at zero volts dc by external circuitry under the application of no signal. The zener diode is inserted in order to allow proper bias of T_2.

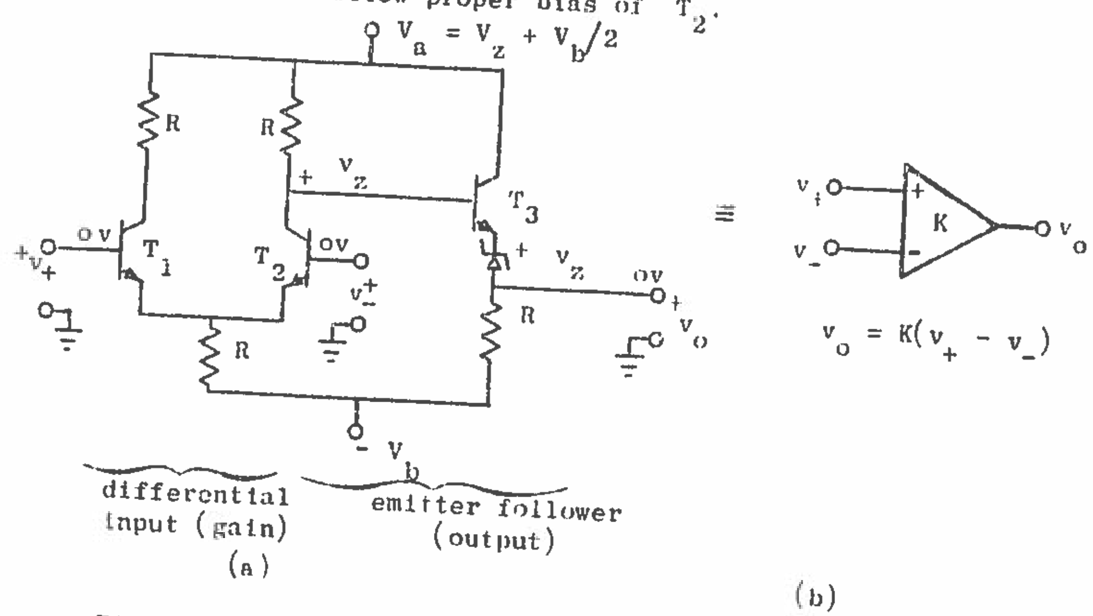
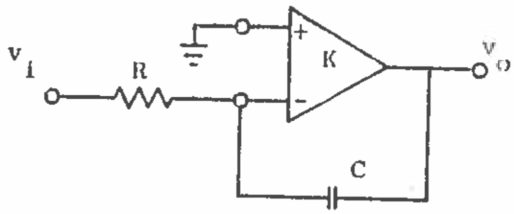


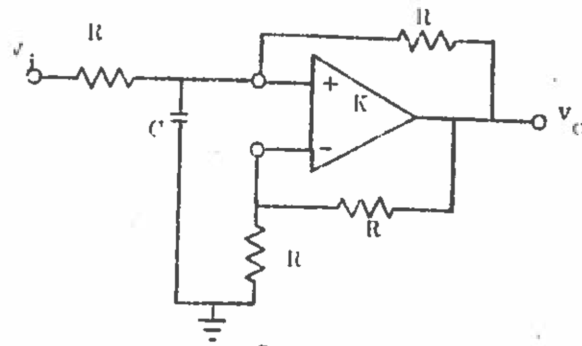
Fig. III-3. BASIC DIFFERENTIAL OPERATIONAL AMPLIFIER.

One can, of course, use the standard capacitor feedback structure for integration, as shown in Fig. III-4a, but if a completely integrated device is desired, which includes integrated capacitors, then it is most



$$v_o = \frac{-1}{RpC\left(1 + \frac{1}{K}\right) + \frac{1}{K}} v_i$$

(a)



$$v_o = \frac{2}{RpC\left(1 + \frac{2}{K}\right) + \frac{4}{K}} v_i$$

(b)

Fig. III-4. POSSIBLE INTEGRATORS.

convenient to use the integrator of Fig. III-4b, which in fact gives a slightly larger gain constant also. Note that as with most such operational amplifier circuits we desire infinite gain, $K = \infty$, in the basic amplifier itself, in which case the grounded amplifier configuration gives

$$v_o = \frac{2}{RpC} v_i \qquad \text{(III-4)}$$

Concerning summation, the diagram of Fig. III-5 yields a convenient circuit for integration which has, for $K = \infty$, the input-output relationship [1, p.]

$$v_o = \sum_{j=1}^{m_+} \frac{RG^- + 1}{G^+} G_j^+ v_j^+ - \sum_{j=1}^{m_-} RG_j^- v_j^- \qquad \text{(III-5a)}$$

Through this relationship any values of the coefficients can be obtained through a solution of simultaneous equations since, for the resistance R_j^+ and R_j^- we have the necessary conductances defined as

$$G^+ = \sum_{j=0}^{m_+} G_j^+ , \quad G^- = \sum_{j=0}^{m_-} G_j^- , \quad G_j^+ = \frac{1}{R_j^+} , \quad G_j^- = \frac{1}{R_j^-} \qquad \text{(III-5b)}$$

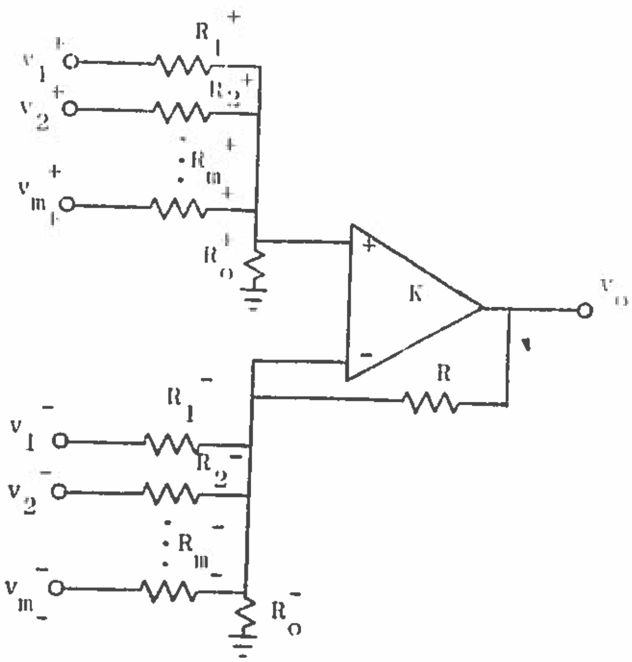


Fig. III-5. SUMMER.

However, it should be observed that inconvenient values for construction through the use of integrated circuits can occur and thus a cascade of components may sometimes be necessary.

D. Scalar Degree Two Realizations

The most practically met situations are those of scalar transfer functions. In such cases the transfer function can be written as the product of degree one and two factors, having real coefficients if we assume that the original transfer function is rational with real coefficients. For sensitivity reasons it is most useful to construct the transfer function through its factors instead of in one complete form. Thus we exhibit a structure for the transfer function

$$T(p) = d + \frac{c_2 p + c_1}{p^2 + 2\zeta\omega_n p + \omega_n^2} \qquad \text{(III-6a)}$$

where we assume for stability reasons that the undamped natural frequency ω_n and the damping ratio ζ are nonnegative. We remark that the quality factor Q can be defined by

$$Q = \frac{1}{2\zeta} \qquad \text{(III-6b)}$$

and that degree one transfer functions are simply realized (and hence left as an exercise).

We claim that a realization of the general degree two transfer function is given by [application of Eq. (I-10c) gives Eq. (III-6a)]

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad \underset{\sim}{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underset{\sim}{C} = [c_1 \quad c_2], \quad \underset{\sim}{D} = d \tag{III-6c}$$

Assuming nonnegative c_1, c_2, d, a circuit diagram suitable for integration would be as shown in Fig. III-6 where the values of resistance can be adjusted for available capacitance ranges. The presence of feedback loops can readily be seen, as well as an appreciation gained for the complications attendant on going to the complete simulation of higher degree transfer functions (without the initial factorization). We observe that the minimum number of capacitors, two, is used for Fig. III-6.

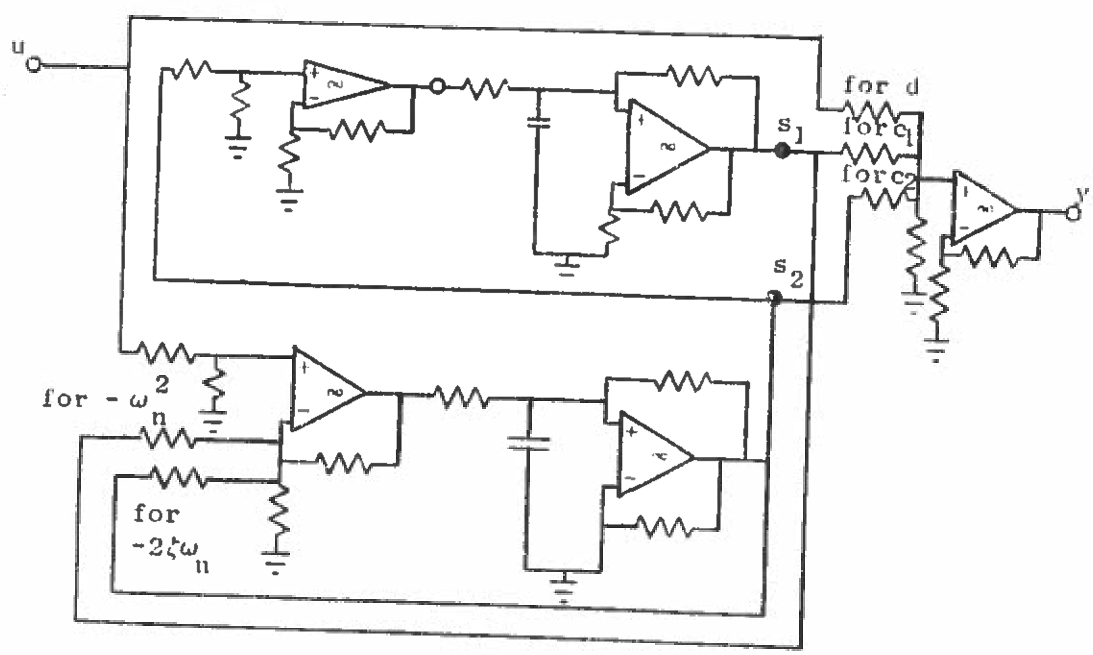


Fig. III-6. POSSIBLE DEGREE TWO SCALAR SIMULATION.

E. Canonical Equation Simulation - Admittances

In Section II-b we saw that the state-variable equations could be set up for admittance "transfer" function (matrices) by extracting capacitors. Here we can reverse the procedure. Thus consider the resistive (n+c)-port, assumed time-invariant, described by

$$\underset{\sim}{Y}_c = \begin{bmatrix} \underset{\sim}{y}_{11} & \underset{\sim}{y}_{12} \\ \underset{\sim}{y}_{21} & \underset{\sim}{y}_{22} \end{bmatrix} \tag{III-7a}$$

and loaded in its final c-ports by c unit capacitors, as shown in Fig. III-7. We calculate for the input admittance

$$\underset{\sim}{y}_{in} = \underset{\sim}{y}_{11} - \underset{\sim}{y}_{12}(p\underset{\sim}{1}_c + \underset{\sim}{y}_{22})^{-1}\underset{\sim}{y}_{21} \tag{III-7b}$$

If we compare the result with that of the transfer function

$$\underset{\sim}{T}(p) = \underset{\sim}{D} + \underset{\sim}{C}(p\underset{\sim}{1}_k - \underset{\sim}{A})^{-1}\underset{\sim}{B} \tag{I-10c}$$

we see that the identification

$$\underset{\sim}{Y}_c = \begin{bmatrix} \underset{\sim}{D} & -\underset{\sim}{C} \\ \underset{\sim}{B} & -\underset{\sim}{A} \end{bmatrix} \tag{III-7c}$$

is possible, with the dimension of the state chosen as the number of capacitors, k = c. Consequently, given a minimal (or even nonminimal)

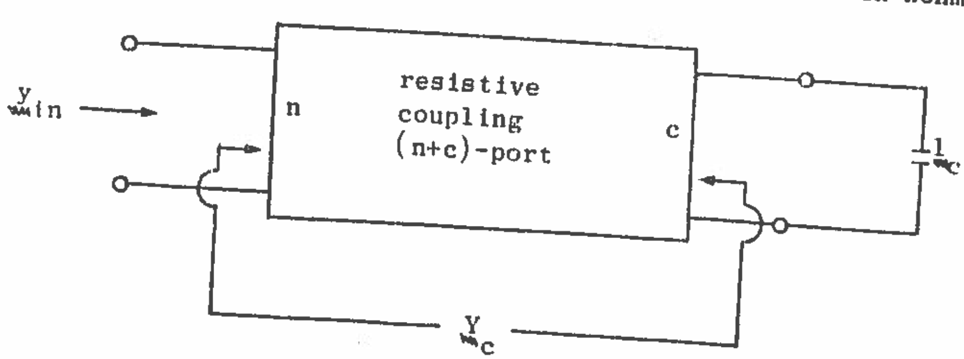


Fig. III-7. CAPACITOR LOADED STRUCTURE.

realization $R = (\underline{A}, \underline{B}, \underline{C}, \underline{D})$, we can construct a circuit, when the transfer function is an admittance matrix, by synthesizing $\underline{Y}_c$ of Eq. (III-7c) and loading in $k = c$ unit capacitors. But $\underline{Y}_c$ being a constant matrix is realized through the use of (positive and negative) resistors and gyrators. Later we will show how $\underline{Y}_c$ can be transformed to become positive-real, if the original transfer function admittance, $\underline{y}_{in}$, is positive-real but such requires the development of more theory. We can remark, however, that if the state-variable equations have a term $\underline{E}\,\dot{\underline{u}}$ added to the output equations, this term can be synthesized by a transformer network (constructed from gyrators if desired) loaded in unit capacitors with the result connected in parallel with that of Fig. III-7.

To synthesize $\underline{Y}_c$ itself, we can proceed by decomposing it into its symmetric and skew-symmetric parts,

$$\underline{Y}_c = \underline{Y}_{c\ sy} + \underline{Y}_{c\ sk} \qquad \text{(III-8a)}$$

where

$$2\underline{Y}_{c\ sy} = \underline{Y}_c + \tilde{\underline{Y}}_c\,, \qquad 2\underline{Y}_{c\ sk} = \underline{Y}_c - \tilde{\underline{Y}}_c \qquad \text{(III-8b)}$$

and again, the super tilde ~ denotes transposition. The skew-symmetric part is immediately constructed from gyrators, one for each nonzero entry for example. The symmetric part can be further decomposed as

$$\underline{Y}_{c\ sy} = \underline{G}[\underline{1}_{r_+} \dot{+} (-\underline{1}_{r_-})]\tilde{\underline{G}} \qquad \text{(III-8c)}$$

where $\dot{+}$ denotes the direct sum of two matrices. The right side of Eq. (III-8c) can be synthesized by loading a gyrator coupling network of admittance matrix

$$\underline{Y}_g = \begin{bmatrix} \underline{0} & \underline{G} \\ -\tilde{\underline{G}} & \underline{0} \end{bmatrix} \qquad \text{(III-8d)}$$

in r_+ unit positive resistors and r_- unit negative resistors, as shown in Fig. III-8 [recall that a formula similar to Eq. (III-7b) applies]. The coupling structure itself results as a parallel connection of the circuits for the symmetric and skew-symmetric parts of $\underset{\sim}{Y}_c$.

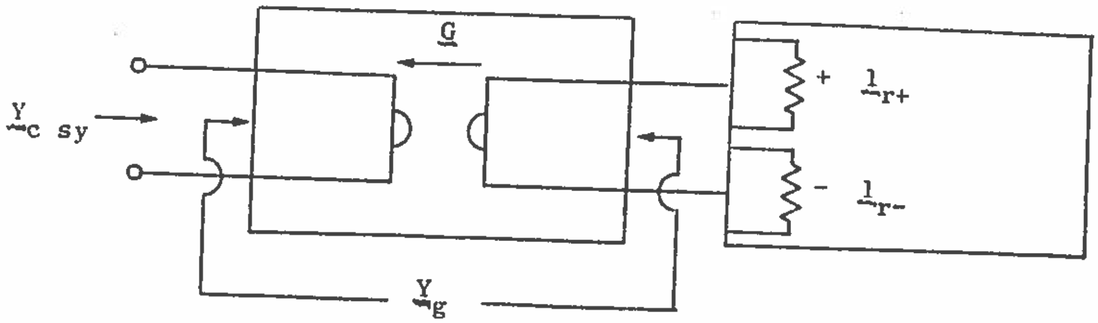


Fig. III-8. CONFIGURATION FOR SYMMETRIC PART OF Y_c.

As an example of the method, let us consider the degree two lowpass admittance

$$y_{in}(p) = \frac{1}{p^2 + 2\zeta p + 1} \qquad \text{(III-9a)}$$

We observe that this admittance is not positive real (as $1/y_{in}$ has a double pole at infinity) in which case active devices must be incorporated. Combining the realization of Eqs. (III-6c) with $\underset{\sim}{Y}_c$ of Eq. (III-7c) yields

$$\underset{\sim}{Y}_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 2\zeta \end{bmatrix} \qquad \text{(III-9b)}$$

which has the symmetric and skew-symmetric parts

$$\underset{\sim}{Y}_{c\,sy} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ -1/2 & 0 & 0 \\ 1/2 & 0 & 2\zeta \end{bmatrix}, \qquad \underset{\sim}{Y}_{c\,sk} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ 1/2 & 0 & -1 \\ 1/2 & 1 & 0 \end{bmatrix} \qquad \text{(III-9c)}$$

To diagonalize the symmetric part we can add $-1/4\zeta$ times the last row to the first and then add -4ζ times the first row to the second. In terms of elementary matrices this gives

$$\begin{bmatrix} 1 & 0 & 0 \\ -4\zeta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/4\zeta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 1/2 \\ -1/2 & 0 & 0 \\ 1/2 & 0 & 2\zeta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/4\zeta & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4\zeta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/8\zeta & 0 & 0 \\ 0 & 2\zeta & 0 \\ 0 & 0 & 2\zeta \end{bmatrix}$$

On multiplying out the inverses of the transformation matrices (which are easily found by changing sign on the off-diagonal terms), we arrive at

$$\underset{\sim}{Y}_{c\ sy} = \begin{bmatrix} 1 & 0 & 1/4\zeta \\ 4\zeta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/8\zeta & 0 & 0 \\ 0 & 2\zeta & 0 \\ 0 & 0 & 2\zeta \end{bmatrix} \begin{bmatrix} 1 & 4\zeta & 0 \\ 0 & 1 & 0 \\ 1/4\zeta & 0 & 1 \end{bmatrix} \tag{III-9d}$$

We observe that the diagonal matrix is not quite in the form used in Eq. (III-8c), but this is not crucial since we merely use nonunit resistors with the negative one placed first (the other form can easily be obtained by using some additional steps). We then wish to load a gyrator 6-port described by

$$\underset{\sim}{Y}_{g} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1/4\zeta \\ 0 & 0 & 0 & 4\zeta & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -4\zeta & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1/4\zeta & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{III-9e}$$

in one negative and two positive resistors to obtain $\underset{\sim}{Y}_{c\ sy}$. The result is shown in Fig. III-9a.

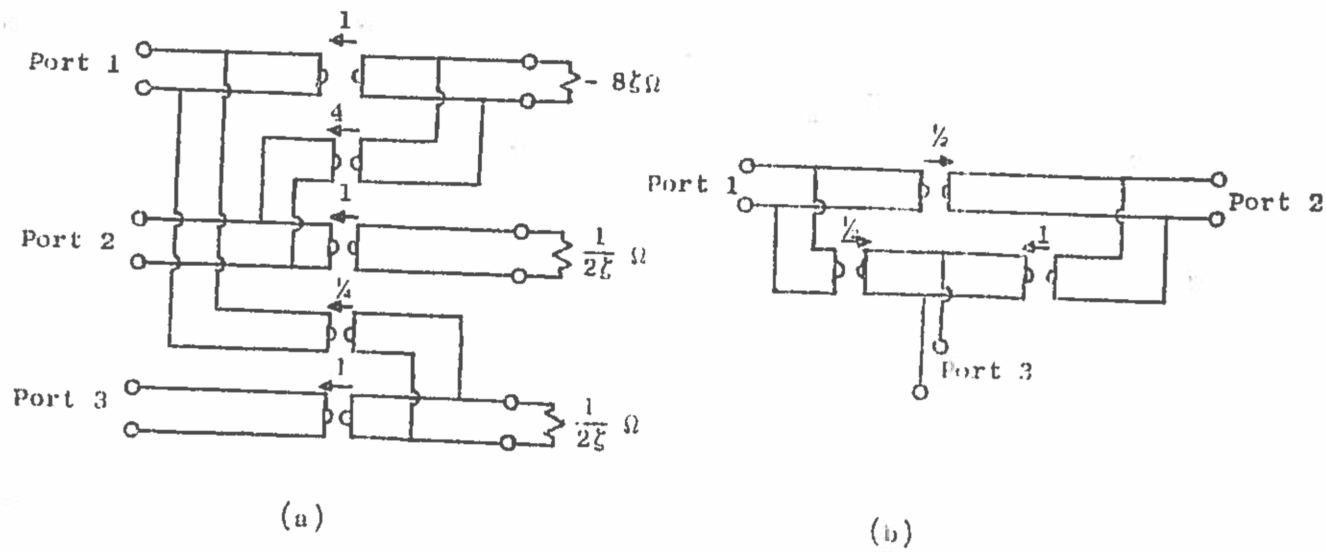


Fig. III-9. CIRCUITS FOR $\underset{\sim}{Y}_{c\ sy}$ (a) AND $\underset{\sim}{Y}_{c\ sk}$ (b).

The circuit for $\underset{\sim}{Y}_{c\ sk}$ is similarly obtained and is shown in the (b) portion of Fig. III-9. The two portions of this figure are connected in parallel with the final two ports loaded in unit capacitors to obtain the desired input admittance at port 1.

F. Discussion

Using the canonical state-variable equations, analog configurations can easily be set up using a block diagram representation of the equations; the resulting components are realized through summers and integrators, the latter being obtained through the use of operational amplifier circuits. Since integrated operational amplifiers have proven extremely practical and since the only other elements needed are resistors and capacitors, both of which can be integrated, the method is quite useful for integrated circuit designs.

It is of interest to observe that exactly as many capacitors are used as there are state variables, and in fact no fewer can ever be used. Since, of the components required here, capacitors are the most difficult elements to make in integrated circuits, the method is about as convenient as could ever be hoped for. As a consequence we have introduced some basic configurations particularly suited for integration.

It should be mentioned that in integrated circuits the ratios of resistors are rather accurately obtained, whereas absolute values are extremely hard to fix accurately. If we observe the coefficients for the summer multipliers, Eq. (III-5a), we see that indeed these coefficients depend only upon ratios of resistances. The situation is somewhat different for the integrator where both resistance and capacitance are involved. In fact, since RC products only quite far away from unity are available in integrated circuit form, it is important to introduce an integrating scale constant in the state-variable equations, the λ of Eq. (III-2b).

We observe that although the equations allow time-variable coefficients and, in fact, the circuit representations hold for such coefficients, it is practically quite difficult to perform time variations on the operational amplifier structures.

Although we have not discussed the possibility, it is actually more convenient to perform time variations by use of the capacitance extraction method. But we have discussed how the previous analysis method, through capacitor extraction, can be carried over to synthesis to create a resistive coupling structure by specifying the admittance coupling matrix $\underset{\sim}{Y}_c$ in terms of the realization $R = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$. In conjunction with this we have given one method of synthesis of $\underset{\sim}{Y}_c$ in terms of gyrators, which can be integrated [2], and positive and negative resistors. Since the negative resistors cause some concern for practical integration, it is of perhaps more practical interest to point out that $\underset{\sim}{Y}_c$ can be obtained as an interconnection of voltage-controlled current sources and that such sources are relatively easy to integrate [1, p.].

Of the two methods presented, the first probably has the advantage in scalar situations of allowing for smaller sensitivities. To obtain these sensitivities of small size it is important to decompose the transfer function into degree one or two portions and cascade the resulting sections. However it is worth mentioning that a good sensitivity analysis of the second (capacitor extraction) method has as yet not been made.

Here we really only treated the synthesis of voltage transfer functions (by the operational amplifier techniques) or of admittance matrices (by the capacitor extraction methods). However, by the use of voltage-to-current or current converters, other specifications can equally well be realized.

G. References

1. Newcomb, R. W., "Active Integrated Circuit Synthesis," Prentice-Hall, Englewood Cliffs, N.J., 1968.

2. Chua, H. T., and R. W. Newcomb, "Integrated Direct-Coupled Gyrator," Electronic Letters, vol. 3, no. 5, May 1967, pp. 182-184.

H. Exercises

1. Set up the state-variable configuration using integrated operational amplifiers for the degree one transfer function. Compare with results obtainable with simple RC circuits.
2. Discuss modifications needed in the theories if terms of the form $\underset{\approx}{E}\,\dot{\underset{\approx}{u}}$ are present. Explain why these are avoided, where possible, in the operational amplifier techniques.
3. Complete the example of Section E by drawing the final overall circuit. Compare with alternate methods and discuss advantages and disadvantages of the method.
4. Synthesize

$$y_{in}(p) = \frac{2p}{p^2 + 2\zeta p + 1}$$

5. Discuss circuits for obtaining the gyrators and negative resistors needed in Fig. III-9.
6. Investigate methods of obtaining practical realizations for the case of time-variable structures.

Ils sont étranges. Ils semblent plus près de la que les autres et ne rien soupçonner, et cependant leurs yeux ont une certitude si profonde qu'il faut qu'ils sachent tout et qu'ils aient eu plus d'un soir le temps de se dire leur secret.

M. Maeterlinck
"Les Avertis" du "Trésor des Humbles"

IV. MINIMAL REALIZATION CREATION

A. Summary

By conversion of a high order differential equation to a set of first degree ones a minimal realization is relatively easily obtained in the scalar case. For matrix transfer functions the algebraic method of Ho is presented for obtaining minimal realizations.

B. Scalar Minimal Realizations

Previously we have seen how a given circuit can be analyzed to obtain an appropriate set of canonical equations. Likewise we have seen how a circuit can be obtained when a realization is on hand, that is when the canonical equations are on hand. Here we complete the picture for time-invariant structures by giving an algebraic procedure for finding a minimal realization from a given transfer function. We begin with the scalar case for which the result can be easily given.

We therefore first begin by assuming as given the scalar transfer function

$$T(p) = \frac{dp^{\delta} + d_{\delta}p^{\delta-1} + \ldots + d_2 p + d_1}{p^{\delta} + a_{\delta}p^{\delta-1} + \ldots + a_2 p + a_1} \tag{IV-1}$$

of degree δ. If we treat p as the differential operator d/dt this transfer function defines the differential equation

$$\left[p^{\delta} + a_{\delta}p^{\delta-1} + \ldots + a_2 p + a_1\right]y = \left[dp^{\delta} + d_{\delta}p^{\delta-1} + \ldots + d_2 p + d_1\right]u$$

(IV-2a)

We can now introduce some changes of variables beginning with

$$y = s_1 + du \qquad \text{(IV-2b)}$$

which results in

$$\left[p^{\delta-1} + a_{\delta}p^{\delta-2} + \ldots + a_2\right]ps_1 + a_1 s_1 = \left[(d_{\delta}-a_{\delta}d)p^{\delta-1} + \ldots + (d_1-a_1 d)\right]u$$

Next letting

$$ps_1 = s_2 + (d_{\delta}-a_{\delta}d) \qquad \text{(IV-2c)}$$

results in

$$\left[p^{\delta-2} + a_{\delta}p^{\delta-3} + \ldots + a_3\right]ps_2 + a_2 s_2 + a_1 s_1 =$$

$$\left[\{(d_{\delta-1}-a_{\delta-1}d) - a_{\delta}(d_{\delta}-a_{\delta}d)\}p^{\delta-2} + \ldots + \{(d_1-a_1 d) - a_2(d_{\delta}-a_{\delta}d)\}\right]u$$

Continuing by letting

$$ps_2 = s_3 + \{(d_{\delta-1}-a_{\delta-1}d) - a_{\delta}(d_{\delta}-a_{\delta}d)\} \qquad \text{(IV-2d)}$$

etc., results in the final equation

$$ps_{\delta} + a_{\delta}s_{\delta} + a_{\delta-1}s_{\delta-1} + \ldots + a_2 s_2 + a_1 s_1 = b_{\delta}u \qquad \text{(IV-2e)}$$

where b_{δ} is a combination of the a_i and d_i coefficients. We have then obtained the canonical equations which can be summarized as

$$
\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \\ \dot{s}_3 \\ \vdots \\ \dot{s}_{\delta-1} \\ \dot{s}_{\delta} \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-a_1 & -a_2 & -a_3 & \cdots & -a_{\delta-1} & -a_{\delta}
\end{bmatrix}
\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{\delta-1} \\ s_{\delta} \end{bmatrix}
+
\begin{bmatrix} d_{\delta} - a_{\delta} d \\ \cdots \\ \\ \\ \cdots \\ b_{\delta} \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{\delta-1} \\ s_{\delta} \end{bmatrix}
+ du \qquad \text{(IV-3)}
$$

We observe that the realization is minimal since $\underset{\sim}{A}$ is $\delta \times \delta$ and $T(p)$ has degree δ. Also, the same procedure holds for the time-varying case with these however being additional derivatives of coefficients in the $\underset{\sim}{B}$ matrix.

From Eq. (IV-3) many other (in fact all) minimal realizations can be obtained by use of nonsingular transformations on the state, that is by introducing

$$
\hat{\underset{\sim}{s}} = \underset{\sim}{T}\, \underset{\sim}{s}
$$

C. Matrix Minimal Realizations

The matrix case is much more difficult to pursue. We follow the algebraic procedure of Ho [1] by first introducing a nonminimal realization which is reduced to be minimal.

We begin by observing the form of the transfer function matrix in terms of the realization matrices. Assuming the realization to be minimal, that is the state of minimal dimension δ, $k = \delta$, we obtain on expanding the inverse of $p\underset{\sim}{1}_\delta - \underset{\sim}{A}$ about $p = \infty$,

$$\underset{\sim}{T}(p) = \underset{\sim}{D} + \underset{\sim}{C}\,(p\underset{\sim}{1}_\delta - \underset{\sim}{A})^{-1}\underset{\sim}{B}$$

$$= \underset{\sim}{D} + \sum_{i=0}^{\infty} \underset{\sim}{C}\,\frac{\underset{\sim}{A}^i}{p^{i+1}}\,\underset{\sim}{B} \qquad \text{(IV-4a)}$$

where $\underset{\sim}{A}^i$ is the i^{th} power of $\underset{\sim}{A}$. By making a direct expansion of $\underset{\sim}{T}(p)$ itself about $p = \infty$ yields the coefficients $\underset{\sim}{A}_i$ for the series

$$\underset{\sim}{T}(p) = \underset{\sim}{A}_{-1} + \sum_{i=0}^{\infty} \frac{\underset{\sim}{A}_i}{p^{i+1}} \qquad \text{(IV-4b)}$$

Since $\underset{\sim}{T}(p)$ is rational we can equate term by term in the last two expressions to obtain that $R = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}\}$ is a realization if and only if

$$\underset{\sim}{D} = \underset{\sim}{T}_{-1} = \underset{\sim}{T}(\infty) \qquad \text{(IV-4c)}$$

$$\underset{\sim}{A}_i = \underset{\sim}{C}\,\underset{\sim}{A}^i\underset{\sim}{B}\ , \qquad i = 0,\ 1,\ \ldots \qquad \text{(IV-4d)}$$

Our job is to hunt for an $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ which satisfy, Eq. (IV-4d); we comment that this last equation holds no matter if the realization is minimal or not, but that we are actually searching for a minimal one.

Since $\underset{\sim}{T}(p)$ is rational there is a relationship among the $\underset{\sim}{A}_i$ of Eq. (IV-4b). To obtain this relationship we can find the least common denominator polynomial

$$g(p) = p^r + a_r p^{r-1} + \ldots + a_1 \qquad \text{(IV-5a)}$$

of $\underset{\sim}{T}(p)$ which next allows us to write the transfer function as a matrix polynomial divided by $g(p)$. Thus

$$\underset{\sim}{T}(p) = \frac{p^r \underset{\sim}{B}_{r+1} + p^{r-1}\underset{\sim}{B}_r + \dots + p\underset{\sim}{B}_2 + \underset{\sim}{B}_1}{p^r + a_r p^{r-1} + \dots + a_1} \tag{IV-5b}$$

As a consequence the product $g(p)\underset{\sim}{T}(p)$ is polynomial and on using the expansion of Eq. (IV-4b) we have

$$\left(\sum_{j=1}^{r+1} a_j p^{j-1}\right)\left(\sum_{k=-1}^{\infty} \underset{\sim}{A}_k / p^{k+1}\right) = \sum_{i=1}^{r+1} \underset{\sim}{B}_i p^{i-1}$$

Equating those coefficients, $\sum_{j=1}^{r+1} a_j \underset{\sim}{A}_{\ell+j-2}$, of $p^{-\ell}$ to zero we find

$$\underset{\sim}{A}_k = -\sum_{j=1}^{r} a_j \underset{\sim}{A}_{k-r+j-1} \quad , \qquad k \geq r \tag{IV-5c}$$

As we saw in Eq. (IV-3) the $\underset{\sim}{A}$ matrix was the companion matrix determined solely by $g(p)$. As a consequence we introduce its generalization, for which we recall that $\underset{\sim}{T}(p)$ is an $n \times m$ matrix. Thus, the generalized $(rn \times rn)$ companion matrix for $g(p)$ is defined by

$$\underset{\sim}{\Omega}_n = \left[\begin{array}{c|c|c|c|c} \underset{\sim}{0}_n & \underset{\sim}{1}_n & \underset{\sim}{0} & \underset{\sim}{0} & \\ \hline & & \underset{\sim}{1}_n & & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & & \ddots & \\ & & \underset{\sim}{0} & & \\ & & & \underset{\sim}{0} & \underset{\sim}{1}_n \\ \hline -a_1\underset{\sim}{1}_n & -a_2\underset{\sim}{1}_n & -a_3\underset{\sim}{1}_n & \cdots & -a_r\underset{\sim}{1}_n \end{array}\right] \tag{IV-6a}$$

where, as before, $\underset{\sim}{1}_n$ is the $n \times n$ identity matrix. To accompany this companion matrix we need the generalized $(rn \times rm)$ Hankel matrix.

$$
S_r = \begin{bmatrix} A_0 & A_1 & \cdots & A_{r-1} \\ A_1 & A_2 & \cdots & A_r \\ \vdots & & & \\ A_{r-1} & A_r & \cdots & A_{2r-2} \end{bmatrix} \tag{IV-6b}
$$

From Eq. (IV-5c) we observe that Ω_n acts to shift rows, or columns, of S_r when the two are multiplied, that is

$$
\Omega_n S_r = S_r \tilde{\Omega}_m = \begin{bmatrix} A_1 & A_2 & \cdots & A_r \\ A_2 & A_3 & \cdots & A_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ A_r & A_{r+1} & \cdots & A_{2r-1} \end{bmatrix} \tag{IV-6c}
$$

where the superscript tilde denotes matrix transposition. As a consequence premultiplication of S_r by Ω_n^i brings A_i to the (1,1) position of the result. In order to isolate this position we define the $\rho \times \gamma$ matrix

$$
1_{\rho,\gamma} = [1_\rho \;\vdots\; 0_{\rho\times\gamma-\rho}] \tag{IV-6d}
$$

for which the first ρ columns are the identity matrix with the remaining columns zero. Then

$$
A_i = 1_{n,rn} (\Omega_n^i S_r) \tilde{1}_{m,rm} \tag{IV-6e}
$$

A possible realization is

$$
A = \Omega_n , \quad B = S_r \tilde{1}_{m,rm} , \quad C = 1_{n,rn} , \quad D = T(\infty) \tag{IV-7}
$$

for note that Eq. (IV-6e) is just $\underset{\sim}{A}_i = \underset{\sim}{C}\,\underset{\sim}{A}^i\underset{\sim}{B}$ which is as required by Eq. (IV-4d). This realization however is not generally minimal, having $k = rn$ which is generally larger than the minimum size, δ, required; as we will see, Eq. (IV-13), this latter is given by $\delta = \text{rank}\ \underset{\sim}{S}_r$. As a consequence let

$$\delta = \text{rank}\ \underset{\sim}{S}_r \tag{IV-8a}$$

in which case one can readily find nonsingular matrices $\underset{\sim}{M}$ and $\underset{\sim}{N}$ to bring $\underset{\sim}{S}_r$ to diagonal form

$$\underset{\sim}{M}\,\underset{\sim}{S}_r\,\underset{\sim}{N} = \tilde{\underset{\sim}{1}}_{\delta,rn}\,\underset{\sim}{1}_{\delta,rm} \tag{IV-8b}$$

In terms of the matrices defined to this point we can now exhibit a minimal realization. Our result is: a rational $n \times m$ transfer function matrix $\underset{\sim}{T}(p)$, finite at infinity, has a minimal realization given by

$$\underset{\sim}{A} = \underset{\sim}{1}_{\delta,rn}\,\underset{\sim}{M}\,\underset{\sim}{\Omega}_n\,\underset{\sim}{S}_r\,\underset{\sim}{N}\,\tilde{\underset{\sim}{1}}_{\delta,rm}\ , \qquad \underset{\sim}{B} = \underset{\sim}{1}_{\delta,rn}\,\underset{\sim}{M}\,\underset{\sim}{S}_r\,\tilde{\underset{\sim}{1}}_{m,rm}$$

$$\underset{\sim}{C} = \underset{\sim}{1}_{n,rn}\,\underset{\sim}{S}_r\,\underset{\sim}{N}\,\tilde{\underset{\sim}{1}}_{\delta,rm}\ , \qquad \underset{\sim}{D} = \underset{\sim}{T}(\infty) \tag{IV-9}$$

To see that Eqs. (IV-9) do define a minimal realization we can proceed as follows. First we observe that

$$\underset{\sim}{S}_r^{\#} = \underset{\sim}{N}\,\tilde{\underset{\sim}{1}}_{\delta,rm}\,\underset{\sim}{1}_{\delta,rn}\,\underset{\sim}{M} \tag{IV-10a}$$

acts as a pseudo-inverse for $\underset{\sim}{S}_r$ since direct calculation gives

$$\underset{\sim}{S}_r = \underset{\sim}{S}_r\,\underset{\sim}{S}_r^{\#}\,\underset{\sim}{S}_r\ , \qquad \underset{\sim}{S}_r^{\#} = \underset{\sim}{S}_r^{\#}\,\underset{\sim}{S}_r\,\underset{\sim}{S}_r^{\#} \tag{IV-10b}$$

Next consider the following sequence of operations which begins from Eq. (IV-6e).

$$\underset{\sim}{A}_i = \underset{\sim}{1}_{n,rn} (\underset{\sim}{\Omega}_n^i \underset{\sim}{S}_r) \underset{\sim}{\tilde{1}}_{m,rm} = \underset{\sim}{1}_{n,rn} \underset{\sim}{\Omega}_n^i \underset{\sim}{S}_r \underset{\sim}{S}_r^{\#} \underset{\sim}{S}_r \underset{\sim}{\tilde{1}}_{m,rm}$$

$$= \underset{\sim}{1}_{n,rn} \underset{\sim}{S}_r \underset{\sim}{\tilde{\Omega}}_m^i \underset{\sim}{S}_r^{\#} \underset{\sim}{S}_r \underset{\sim}{\tilde{1}}_{m,rm} = \underset{\sim}{1}_{n,rn} \underset{\sim}{S}_r \underset{\sim}{S}_r^{\#} \underset{\sim}{S}_r \underset{\sim}{\tilde{\Omega}}_m^i \underset{\sim}{S}_r^{\#} \underset{\sim}{S}_r \underset{\sim}{\tilde{1}}_{m,rm}$$

$$= (\underset{\sim}{1}_{n,rn} \underset{\sim}{S}_r \underset{\sim}{N} \underset{\sim}{\tilde{1}}_{\delta,rm})(\underset{\sim}{1}_{\delta,rn} \underset{\sim}{M} \underset{\sim}{S}_r \underset{\sim}{\tilde{\Omega}}_m^i \underset{\sim}{N} \underset{\sim}{\tilde{1}}_{\delta,rm})(\underset{\sim}{1}_{\delta,rn} \underset{\sim}{M} \underset{\sim}{S}_r \underset{\sim}{\tilde{1}}_{m,rm})$$

$$= \underset{\sim}{C} (\underset{\sim}{1}_{\delta,rn} \underset{\sim}{M} \underset{\sim}{\Omega}_n \underset{\sim}{S}_r \underset{\sim}{N} \underset{\sim}{\tilde{1}}_{\delta,rm})^i \underset{\sim}{B} = \underset{\sim}{C} \underset{\sim}{A}^i \underset{\sim}{B}$$

Here the next to the last step is justified by iteration of the result

$$\underset{\sim}{1}_{\delta,rn} \underset{\sim}{M} \underset{\sim}{S}_r \underset{\sim}{\tilde{\Omega}}_m^2 \underset{\sim}{N} \underset{\sim}{\tilde{1}}_{\delta,rm} = \underset{\sim}{1}_{\delta,rn} \underset{\sim}{M} \underset{\sim}{\Omega}_n \underset{\sim}{S}_r \underset{\sim}{S}_r^{\#} \underset{\sim}{S}_r \underset{\sim}{\tilde{\Omega}}_m \underset{\sim}{N} \underset{\sim}{\tilde{1}}_{\delta,rm}$$

$$= \underset{\sim}{1}_{\delta,rn} \underset{\sim}{M} \underset{\sim}{\Omega}_n \underset{\sim}{S}_r \underset{\sim}{N} \underset{\sim}{\tilde{1}}_{\delta,rm} \underset{\sim}{1}_{\delta,rn} \underset{\sim}{M} \underset{\sim}{\Omega}_n \underset{\sim}{S}_r \underset{\sim}{N} \underset{\sim}{\tilde{1}}_{\delta,rm}$$

As a consequence a realization has been obtained and it only remains to show that it is minimal.

For this latter demonstration let us introduce the ordinary observability and controllability matrices

$$\underset{\sim}{P} = [\underset{\sim}{\tilde{C}}, \underset{\sim}{\tilde{A}} \underset{\sim}{\tilde{C}}, \ldots, \underset{\sim}{\tilde{A}}^{r-1} \underset{\sim}{\tilde{C}}] , \quad \underset{\sim}{Q} = [\underset{\sim}{B}, \underset{\sim}{A} \underset{\sim}{B}, \ldots, \underset{\sim}{A}^{r-1} \underset{\sim}{B}] \qquad \text{(IV-11)}$$

Then for any realization, since $\underset{\sim}{A}_i = \underset{\sim}{C} \underset{\sim}{A}^i \underset{\sim}{B}$, we find from direct multiplication that

$$\underset{\sim}{S}_r = \underset{\sim}{\tilde{P}} \underset{\sim}{Q} \qquad \text{(IV-12)}$$

Now suppose that there exists a realization having $\underset{\sim}{A}$ of size $k \times k$ with $k < \delta = \text{rank } \underset{\sim}{S}_r$. We have a contradiction since

$$\text{rank } \underset{\sim}{S}_r \leq \min [\text{rank } \underset{\sim}{P}, \text{ rank } \underset{\sim}{Q}] \leq k < \delta = \text{rank } \underset{\sim}{S}_r \qquad \text{(IV-13)}$$

where the middle inequality follows from $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ being of sizes $k \times rm$. We conclude that the realization is the smallest possible with δ being what we have previously called the degree.

D. Examples

Consider the transfer function

$$\underset{\sim}{T}(p) = \begin{bmatrix} \frac{1}{p+1} \\ \\ \frac{1}{(p+1)(p+2)} \end{bmatrix} \qquad \text{(IV-14a)}$$

One procedure would be to connect a degree one realization between the input and first output and a degree two realization between the input and second output. However the final result would have a 3-dimensional state, which would not be minimal since, as we next show, two dimensions suffice. Hence we proceed to apply the theory of the previous section.

The least common denominator is

$$g(p) = p^2 + 3p + 2 = p^2 + a_2 p + a_1 \qquad \text{(IV-14b)}$$

Thus we have

$$m = 1 \quad , \qquad n = 2 \quad , \qquad r = 2 \qquad \text{(IV-14c)}$$

and for $\underset{\sim}{S}_r$ we must calculate the expansion of $\underset{\sim}{T}(p)$ about infinity up to $\underset{\sim}{A}_2$. We find by simply dividing the denominators into the numerators beginning with the highest powers of p

$$\underset{\sim}{T}(p) = \frac{1}{p}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{p^2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{p^3}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \ldots$$

$$= \underset{\sim}{A}_{-1} + \frac{1}{p}\underset{\sim}{A}_0 + \frac{1}{p^2}\underset{\sim}{A}_1 + \frac{1}{p^3}\underset{\sim}{A}_2 + \ldots \qquad \text{(IV-14d)}$$

The Hankel matrix can then be formed

$$\underset{\sim}{S}_r = \begin{bmatrix} \underset{\sim}{A}_0 & \underset{\sim}{A}_1 \\ \underset{\sim}{A}_1 & \underset{\sim}{A}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -3 \end{bmatrix} \end{bmatrix} \qquad \text{(IV-14e)}$$

and one finds by the use of elementary operations that

$$\underset{\sim}{M}\,\underset{\sim}{S}_r\,\underset{\sim}{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(IV-14-f)}$$

We also have

$$\underset{\sim}{1}_{\delta,rn} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \underset{\sim}{1}_{\delta,rm} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underset{\sim}{1}_2 \qquad \text{(IV-14-g)}$$

The final matrix necessary for Eqs. (IV-9) is the companion matrix associated with $g(p)$.

$$\underset{\sim}{\Omega}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} \qquad \text{(IV-14h)}$$

We can then calculate the minimal realization using Eq. (IV-9)

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \qquad \text{(IV-15a)}$$

$$\underset{\sim}{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(IV-15b)}$$

$$\underset{\sim}{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(IV-15c)}$$

$$\underline{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(IV-15d)}$$

One can easily check that $\underline{T}(p)$ results from this realization through the calculation of $\underline{D} + \underline{C}(p\,\underline{1}_2 - \underline{A})^{-1}\underline{B}$.

By physically constructing, as in Chapter III, the canonical state variable equations

$$\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \text{(IV-15e)}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad \text{(IV-15f)}$$

one can obtain a device with the given transfer function and which uses the minimum number of dynamical elements (capacitors, say, for integrated circuits).

Next let us consider the minimal realization of the general degree two scalar

$$T(p) = d + \frac{c_2 p + c_1}{p^2 + 2\zeta\omega_n p + \omega_n^2} \quad \text{(III-6a)}$$

which was previously considered (Fig. III-6). We have

$$m = n = 1 \quad , \qquad r = \delta = 2 \quad \text{(IV-16a)}$$

and Eqs. (IV-7) already give a minimal realization, as do Eqs. (IV-3) as well as Eqs. (III-6c). For Eqs. (IV-7) we have

$$\underline{\Omega}_1 = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \quad \text{(IV-16b)}$$

which follows on identification of terms from

$$g(p) = p^2 + 2\zeta\omega_n p + \omega_n^2 = p^2 + a_2 p + a_1 \quad \text{(IV-16c)}$$

Likewise

$$\underset{\sim}{S}_r = \begin{bmatrix} c_2 & c_1 - 2\zeta\omega_n c_2 \\ c_1 - 2\zeta\omega_n c_2 & 2\zeta\omega_n c_1 - (1+4\zeta^2)\omega_n^2 c_2 \end{bmatrix} \qquad \text{(IV-16d)}$$

which follows from the expansion of T(p) about infinity

$$T(p) = d + \frac{c_2}{p} + \frac{c_1 - 2\zeta\omega_n c_2}{p^2} + \frac{2\zeta\omega_n c_1 - (1+4\zeta^2)\omega_n^2 c_2}{p^3} + \ldots \qquad \text{(IV-16e)}$$

Equations (IV-7) give

$$\underset{\sim}{A} = \underset{\sim}{\Omega}_1 = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \qquad \underset{\sim}{B} = \underset{\sim}{S}_r \tilde{\underset{\sim}{I}}_{m,rm} = \begin{bmatrix} c_2 \\ c_1 - 2\zeta\omega_n c_2 \end{bmatrix}$$

$$\underset{\sim}{C} = \underset{\sim}{I}_{n,rn} = [1 \quad 0], \qquad \underset{\sim}{D} = d \qquad \text{(IV-16f)}$$

We observe that the calculations for Eq. (IV-9) are sometimes unnecessarily burdensome, as for example, in this case $\underset{\sim}{M}$ and $\underset{\sim}{N}$ are not even needed. Also from the simplicity of Eq. (III-6c) which has $\underset{\sim}{C} = [c_1, c_2]$ we see that perhaps there is a more convenient method (as yet undiscovered) for finding minimal realizations.

E. Discussion

Using a basic equation, (IV-6c), for a decomposition of the matrices $\underset{\sim}{A}_i$ obtained by expanding the transfer function $\underset{\sim}{T}(p)$ about infinity a generally nonminimal realization, Eq. (IV-7), is easily found from which simple but ingenious manipulations lead to a minimal realization, Eq. (IV-9). The matrix case is seen to be somewhat a generalization of the scalar situation where a minimal realization is relatively easily obtained by converting a higher order differential equation to a set of first order ones. Because the method proceeds in an algebraic manner directly from the transfer function it is quite suitable for computer synthesis of systems, although as yet we are unaware of such a program being carried out. In fact it appears that it is worthwhile looking for improved methods, since, as the last example has shown, there are sometimes situations when easier calculations than those called for by the general theory can be used.

There are of course other methods of obtaining minimal realizations. One such is to augment $\underline{T}$ such that $m = n$, make appropriate frequency shifts and constant additions such that it is positive or bounded-real, and then give a minimal reactive synthesis of the result [2]. Other methods exist which work in the time domain from impulse response matrices [3][4]. But for the time-invariant case the procedure of Ho, given here, presents the most promising because of its possibilities for computer synthesis of systems. Nevertheless we will later, Chapter IX, briefly look at the time-domain for time-variable synthesis procedures.

At this point we have on hand the basic portions of the important theories. We have seen how to set up the canonical equations from a circuit, and now from a transfer function, and we have shown how to obtain a circuit from the canonical equations and thus from a transfer function. As a consequence our remaining topics are all associated with improvements and extensions of the basic results. We first look into methods of finding equivalents, which require more knowledge of the concepts of observability and controllability.

F. References

1. Ho, B. L., "On Effective Construction of Realizations from Input-Output Descriptions," Ph.D. Dissertation, Stanford University, March 1966.
2. Anderson, B. D. O., and R. W. Newcomb, "A Canonical Simulation of a Transfer Function Matrix," IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966, pp. 319-320.
3. Youla, D. C., "The Synthesis of Linear Dynamical Systems from Prescribed Weighting Patterns," SIAM Journal, Vol. 14, No. 3, May 1966, pp. 527-549.
4. Desoer, C. A., and P. P. Varaiya, "The Minimal Realization of a Nonanticipative Impulse Response Matrix," SIAM Journal, Vol. 15, No. 3, May 1967, pp. 754-764.

G. Exercises

1. Find $\underset{\sim}{B}$ in compact form for Eq. (IV-3).
2. For the general degree two scalar transfer function, exhibit completely Eq. (IV-3) and compare with the several other results available.
3. Insert the modifications required for Eq. (IV-3) to hold for time-variable circuits.
4. Find a realization for

$$\underset{\sim}{T}(p) = \left[\frac{1}{p+1} \;,\quad \frac{1}{(p+1)(p+2)}\right]$$

and compare with the results of Eq. (IV-15).

5. Find a realization for

$$T(p) = \begin{bmatrix} \frac{p-1}{p+1} \\ \frac{2}{p+a} \end{bmatrix}$$

for an arbitrary. What is the nature of the result when $a = 1$?

*6. Investigate the realization of $\underset{\sim}{T}(p)$ by factorization into degree one or two parts and the realization in minimal form of each part.

7. For Eq. (IV-15d) find $\underset{\sim}{M}$ and $\underset{\sim}{N}$ and determine a minimal realization using the general theory associated with this $\underset{\sim}{M}$ and $\underset{\sim}{N}$. Compare with the realization of Eq. (IV-15f).

8. Find a realization for $T(p) = 1/(p+1)$ and one for $T(p) = 1/(p+1)(p+2)$ and "connect" the two to obtain a realization for the text example of Eq. (IV-14a). Compare the result with that of the text and discuss with specific reference to minimality.