A Mathematical Basis for Linear and Nonlinear Analog Cochlear Models Using Kemp Echo Transients

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Abstract

Abstract here.

1 Introduction

The equations describing motion of the fluid and the basilar membrane will be derived from three considerations: 1) Conservation of mass, 2) Newton's law for force on a fluid, and 3) Basilar membrane motion equations. We idealize the cochlea as an exponentially tapered tube divided in two by a membrane that is stiff except for a portion that vibrates transverse to the fluid flow. The linear membrane taper is actually opposite to the exponential taper, the upper chamber is called the scala vestibuli and the lower one the scala tympani, the two are connected at the narrow end by an opening (the helicotrema), and at the wider end both chambers are sealed by a flexible membrane (the scala tympani by the round window and the scala vestibuli by the oval window). Sound inputs and outputs are at the oval window via the middle ear which acts as a transformer for signals transmitted from the outer ear at the ear drum.

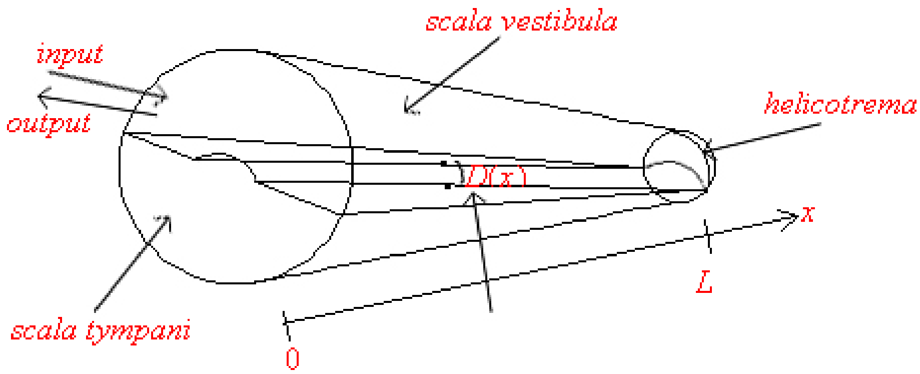


Figure 1: simplified cochlear system

We use subscripts v & t for the vestibular and tympanic scalae. Also t will denote time while x will be distance along the basilar membrane starting at the window end, a one-dimensional model being assumed. Thus

$$v_v(t,x),\ v_t(t,x) = \text{ velocities of fluid in } x \text{ direction} \tag{1}$$

$$p_v(t,x),\ p_t(t,x) = \text{ pressure in the fluid at } x \text{ \& } t \tag{2}$$

$$S_v(x) = S_t(x) = \text{ areas at } x \tag{3}$$

By virtue of the taper and the closed nature of the cochlea a velocity v_v in the scala vestibuli induces a velocity

$$v_v(t,x) = -v_v(t,x) \tag{4}$$

in the scala tympani, we take 4 as a basis assumption of the theory which allows us to work primarily with

v_v. Also it is the pressure difference

$$p = p_v - p_t \tag{5}$$

which causes the basilar membrane to vibrate so we will desire our equations in terms of p.

By way of electrical analogies, since p is measured at v with respect to t we will take p analogous to voltage and v_v analogous to current.

2 Macromechanical Basis

2.1 Conservation of Mass

We have for any fluid

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \overrightarrow{v} \cdot \nabla\rho \tag{6}$$

where ρ is the mass velocity and $\overrightarrow{v}$ is the velocity of fliud

But a decrease in mass in volume ΔV in time Δt

$$= \left(\frac{\partial}{\partial t} \iiint_{\Delta V} \rho \, dV \right) \Delta t \tag{7}$$

is the fluid inflow in time Δt through surface S bounding ΔV

$$= \left(- \iint_{S} p \overrightarrow{v} \cdot d\overrightarrow{S} \right) \Delta t \tag{8}$$

which by Gauss' law

$$= \left(- \iiint_{\Delta V} \nabla \cdot (\rho \overrightarrow{v}) dV \right) \Delta t \tag{9}$$

the minus sign coming from $d\overrightarrow{S}$ pointing out and $\overrightarrow{v}$ in. Equating terms under $\iiint$:

$$\frac{\partial\rho}{\partial t} = (-\nabla \cdot (\rho \overrightarrow{v}) dV) \Delta t \tag{10}$$

Substituting 10 into 6 gives $\frac{d\rho}{dt} = -\rho\nabla \cdot \overrightarrow{v}$ or

$$\nabla \cdot \overrightarrow{v} = -\frac{1}{\rho} \frac{d\rho}{dt} \tag{11}$$

But $\nabla\cdot\overrightarrow{v}$ is the divergence and divergence is calculated from

$$\nabla\cdot\overrightarrow{v}=\lim_{\Delta V\to 0}\left\{\frac{\iint_S \overrightarrow{v}\cdot d\overrightarrow{S}}{\Delta V}\right\} \tag{12}$$

Now consider a section of length Δx of the scala vestibuli:

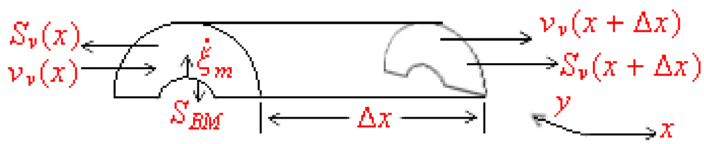


Figure 2: section of length Δx of scala vestibuli

Here the velocity perpendicular to the external surface is zero since this is a stiff boundary as it is over the stiff portion of the dividing membrane. For the basilar membrane, BM, we approximate the displacement $\xi(t,x,y)$ and velocity $\frac{\partial\xi(t,x,y)}{\partial t}=\dot{\xi}$ by triangular curves

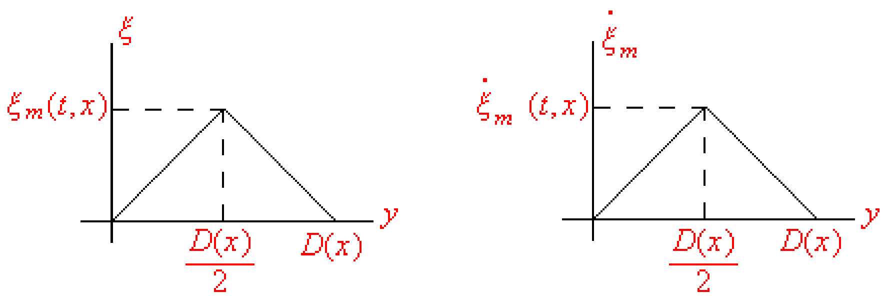


Figure 3: basilar membrane displacement and velocity functions

Then

$$\xi=\begin{cases}\frac{2\xi_m}{D}y & \text{if} \quad 0\le y\le\frac{D}{2}\\ -\frac{2\xi_m}{D}y+2\xi_m & \text{if} \quad \frac{D}{2}\le y\le D\end{cases},\ \dot{\xi}_m=\begin{cases}\frac{2\dot{\xi}_m}{D}y & \text{if} \quad 0\le y\le\frac{D}{2}\\ -\frac{2\dot{\xi}_m}{D}y+2\xi_m & \text{if} \quad \frac{D}{2}\le y\le D\end{cases} \tag{13}$$

Here $\xi_m(t,x)$ is the maximum displacement at time t and position x of the BM.

For the right side of 12 for the section

$$\iint_S \overrightarrow{v}\cdot d\overrightarrow{S}=\iint_x \overrightarrow{v}\cdot d\overrightarrow{S}+\iint_{x+\Delta x}\overrightarrow{v}\cdot d\overrightarrow{S}+\iint_{BM}\overrightarrow{v}\cdot d\overrightarrow{S} \tag{14}$$

for which

$$\iint\limits_{BM} \overrightarrow{v} \cdot d\overrightarrow{S} = \left(-\int_0^{D(x)} \dot{\xi}(t,x,y)ds \right) \Delta x = -2\int_0^{\frac{D(x)}{2}} \dot{\xi}(t,x,y)ds\Delta x \tag{15}$$

where ds is the differential arc length of BM:

$$ds = \sqrt{dy^2 + \left(\frac{\xi_m}{D/2} dy \right)^2} \tag{16}$$

or

$$\iint\limits_{BM} \overrightarrow{v} \cdot d\overrightarrow{S} = -2\int_0^{\frac{D(x)}{2}} \frac{2\dot{\xi}_m(t,x)}{D(x)} y \sqrt{1 + \left(\frac{2\xi_m(t,x)}{D(x)})^2 \right)^2} dy \cdot \Delta x \tag{17}$$

$$= \left[-2\Delta x \cdot \frac{2\dot{\xi}_m}{D} \sqrt{1 + \left(\frac{2\xi_m}{D} \right)^2} \cdot \frac{1}{2} y^2 \right]_0^{D/2} \tag{18}$$

$$= -\frac{D}{2} \cdot \dot{\xi}_m \sqrt{1 + \left(\frac{2\xi_m}{D} \right)^2} \Delta x \tag{19}$$

Since

$$D(x) >> 2\xi_m(t,x) \tag{20}$$

using $\sqrt{1+x} \approx 1 + \frac{1}{2}x$, $|x|$ small

$$\iint\limits_{BM} \overrightarrow{v} \cdot d\overrightarrow{S} \approx -\frac{D(x)}{2} \dot{\xi}_m(t,x) \left[1 + 2\left(\frac{\xi_m(t,x)}{D(x)} \right)^2 \right] \Delta x \tag{21}$$

For $\nabla \cdot \overrightarrow{v}$ we then have for 12

$$\nabla \cdot \overrightarrow{v} = \lim_{\Delta x \to 0} \left\{ \frac{-S_v(x)v_v(t,x) + S_v(x+\Delta x)v_v(x+\Delta x) - \frac{D}{2}\dot{\xi}_m(t,x)\left[1 + 2\left(\frac{\xi_m}{D} \right)^2 \right] \Delta x}{S_v(x)\Delta x} \right\} \tag{22}$$

$$= \frac{1}{S_v(x)} \left\{ \frac{\partial\left(S_v(x)v_v(t,x) \right)}{\partial x} - \frac{D(x)}{2} \dot{\xi}_m(t,x) \left[1 + 2\left(\frac{\xi_m(t,x)}{D(x)} \right)^2 \right] \right\}$$

$$= -\frac{1}{\rho}\frac{d\rho}{dt} \text{ (by 11)}$$

We next assume that the fluid in the cochlea is incompressible, i.e.

$$\frac{d\rho}{dt} = 0 \tag{23}$$

in which case we find

$$\nabla(S_v v_v) = \frac{\partial(S_v v_v)}{\partial x} = \frac{D}{2}\dot{\xi}_m \left[1 + 2\left(\frac{\xi_m}{D}\right)^2\right] \tag{24}$$

This is one of our main equations. By symmetry and the fact that $S_t + S_v$ is constant in time

$$\frac{\partial S_t v_t}{\partial x} = -\frac{D}{2}\dot{\xi}_m \left[1 + 2\left(\frac{\xi_m}{D}\right)^2\right] \tag{25}$$

2.2 Newton's Law

By $d[\cdot]/dt = \partial[\cdot]/\partial t + \overrightarrow{v} \cdot \nabla[\cdot]$ and using subscirpt $i = v$ or t

$$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + (v_i \cdot \nabla)v_i = -\frac{1}{\rho} f_i \tag{26}$$

where

$$\begin{aligned} -f_i = &\text{ force on } S_i/\text{volume exerted by the medium on a particle of fluid} \\ = &-(\nabla p_i + R_i v_i) \end{aligned} \tag{27}$$

and R_i is the coefficient of frictional force. We have

$$R_v = R_t \tag{28}$$

Therefore, using $v_t = -v_v$

$$\frac{\partial v_v}{\partial t} + v_v \Delta v_v = -\frac{1}{\rho}(\nabla p_v + R_v v_v) \tag{29}$$

$$\frac{\partial v_t}{\partial t} + v_t \nabla v_t = -\frac{\partial v_v}{\partial t} + v_v \nabla v_v = -\frac{1}{\rho}(\nabla p_t - R_v v_v) \tag{30}$$

Subtracting 30 from 29 and using $p = v_v - p_t$

$$2\frac{\partial v_v}{\partial t} = -\frac{1}{\rho}(\nabla p + 2R_v v_v) \tag{31}$$

or

$$\nabla p = -\frac{2}{S_v}\left[R_v(S_v v_v) + \rho\frac{\partial(S_v v_v)}{\partial t}\right] \tag{32}$$

which is our second main equation.

2.3 Basilar Membrane Motion

We consider the basilar membrane to be a second order system with an equivalent mass, $\mu(x)$, friction, $\sigma(x)$, and spring constant $\phi(x)$, all per unit area. Then, using the average displacement over $D(x)$ as $\frac{\xi_m}{2}$

$$-p = \mu \frac{\partial^2 \left(\frac{\xi_m}{2}\right)}{\partial t^2} + \sigma \frac{\partial \left(\frac{\xi_m}{2}\right)}{\partial t} + \phi \left(\frac{\xi_m}{2}\right) \tag{33}$$

where the minus sign results since p is the force down on the membrane. We rewrite this as

$$-p = \frac{1}{D}\left[\mu \frac{\partial^2 \left(\frac{D\xi_m}{2}\right)}{\partial t^2} + \sigma \frac{\partial \left(\frac{D\xi_m}{2}\right)}{\partial t} + \phi \left(\frac{D\xi_m}{2}\right)\right] \tag{34}$$

which is the final main equation desired.

2.4 Summary

To summarize, we let

$$u_v = S_v v_v \tag{35}$$

$$s = \frac{\partial[\cdot]}{\partial t} \tag{36}$$

Then 24, 32, and 34 are

$$\nabla u_v = \left[1 + \frac{8}{D^4}\left(\frac{D\xi_m}{2}\right)^2 \cdot s\left(\frac{D\xi_m}{2}\right)\right] \tag{37}$$

$$\nabla p = \frac{-2}{S_v}\left[R_v + \rho s\right] u_v \tag{38}$$

$$-p = \frac{\left[\mu s^2 + \sigma s + \phi\right]}{D}\left(\frac{D\xi_m}{2}\right) \tag{39}$$

We can eliminate $\left(\frac{D\xi_m}{2}\right)$ by solving for it from 39 and substituting in 37. Thus

$$\frac{D\xi_m}{2} = \frac{-D}{\left[\mu s^2 + \sigma s + \phi\right]} \cdot p = \frac{-1}{sQ(x,s)} p(t,x) \tag{40}$$

where

$$Q(x,s) = \frac{\mu(x)s^2 + \sigma(x)s + \phi(x)}{D(x)s} \tag{41}$$

Let

$$P(x,s) = \frac{2}{S_v(x)}\left[Rv + \rho s\right] \tag{42}$$

Then we have

$$\nabla p = -P(x,s)u_v \tag{43}$$

$$\nabla u_v = -\left[1 + \frac{8}{D^4}\left(\frac{1}{sQ}\cdot p\right)^2\right]\cdot\frac{1}{Q(x,s)}p \tag{44}$$

Linearizing by assuming

$$1 >> \frac{8}{D^4}\left(\frac{1}{sQ}p\right)^2 \tag{45}$$

gives

$$\nabla p = -Pu_v \tag{46}$$

$$\nabla u_v \approx -\frac{1}{Q}p \tag{47}$$

3 Scattering Treatment - Linear Case

Convert to force scattering-type variables by

$$\begin{bmatrix} p \\ u_v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ G(x,s) & -G(x,s) \end{bmatrix}\begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{48}$$

where

$$\begin{bmatrix} p^i \\ p^n \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & R(x,s) \\ 1 & -R(x,s) \end{bmatrix}\begin{bmatrix} p \\ u_v \end{bmatrix} \tag{49}$$

and

$$R = \frac{1}{G} \tag{50}$$

Then

$$\nabla\begin{bmatrix} p \\ u_v \end{bmatrix} = -\begin{bmatrix} 0 & P \\ \frac{1}{Q} & 0 \end{bmatrix}\begin{bmatrix} p \\ u_v \end{bmatrix} \tag{51}$$

becomes, using $' = \frac{\partial[]}{\partial x} = \nabla$,

$$\begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix} \nabla \begin{bmatrix} p^i \\ p^n \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G' & -G' \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} = - \begin{bmatrix} 0 & P \\ \frac{1}{Q} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{52}$$

or

$$\nabla \begin{bmatrix} p^i \\ p^n \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & R \\ 1 & -R \end{bmatrix} \left\{ \begin{bmatrix} 0 & 0 \\ G' & -G' \end{bmatrix} + \begin{bmatrix} 0 & P \\ \frac{1}{Q} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix} \right\} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{53}$$

$$= -\frac{1}{2} \left\{ RG' \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{R}{Q} & P \\ -\frac{R}{Q} & P \end{bmatrix} \begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix} \right\} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{54}$$

$$= -\frac{1}{2} \begin{bmatrix} RG' + \frac{R}{Q} + GP & -RG' + \frac{R}{Q} - GP \\ -RG' - \frac{R}{Q} + GP & RG' - \frac{R}{Q} - GP \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{55}$$

$$= -\frac{R}{2} \begin{bmatrix} G' + P\left(\frac{1}{PQ} + G^2\right) & -G' + P\left(\frac{1}{PQ} - G^2\right) \\ -G' - P\left(\frac{1}{PQ} - G^2\right) & G' - P\left(\frac{1}{PQ} + G^2\right) \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{56}$$

On observing the $(1,2)$ and $(2,1)$ terms, choose

$$G^2 = \frac{1}{PQ} \tag{57}$$

which has

$$2GG' = \nabla \left(\frac{1}{PQ} \right) \tag{58}$$

meaning

$$G' = \frac{1}{2G} \nabla \left(\frac{1}{PQ} \right) \tag{59}$$

Thus,

$$\frac{R}{2} G' = \frac{1}{4G^2} \nabla \left(\frac{1}{PQ} \right) = \frac{PQ}{4} \nabla \left(\frac{1}{PQ} \right) = -\frac{1}{4PQ} \nabla (PQ) \tag{60}$$

and 56 gives

$$\nabla \begin{bmatrix} p^i \\ p^n \end{bmatrix} = \begin{bmatrix} -\frac{RG'}{2} - \frac{R}{Q} & \frac{RG'}{2} \\ \frac{RG'}{2} & \frac{RG'}{2} + \frac{R}{Q} \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{61}$$

for which we define

$$\rho(x,s) = -\frac{RG'}{2} = \frac{1}{4PQ}\nabla(PQ) \tag{62}$$

$$\gamma(x,s) = \frac{R}{Q} = RG^2P = GP = \sqrt{\frac{P}{Q}} \tag{63}$$

This allows 61 to be written as

$$\nabla \begin{bmatrix} p^i \\ p^n \end{bmatrix} = \begin{bmatrix} \rho-\gamma & -\rho \\ -\rho & \rho+\gamma \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{64}$$

In 64 γ is like a propogation function, yielding delay through the section of length dx.

4 Space Discretization - Linear Case

We consider a section of length Δx where for a total of N sections we have

$$\Delta x = \frac{L}{N},\ x_k = \frac{kL}{N} = x_{k-1} + \Delta x \tag{65}$$

For the k^{th} section we evaluate γ and ρ at $x = x_k$. We break the section into two portions, one in terms of γ with $\rho = 0$ and the other in cascase in terms of ρ with $\gamma = 0$.

In the first case where $\rho = 0$ we have

$$\nabla \begin{bmatrix} p^i \\ p^n \end{bmatrix} = \begin{bmatrix} -\gamma & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{66}$$

or

$$\begin{bmatrix} p^i(t, x_k+\Delta x) \\ p^n(t, x_k+\Delta x) \end{bmatrix} = \begin{bmatrix} e^{-\gamma\Delta x} & 0 \\ 0 & e^{\gamma\Delta x} \end{bmatrix} \begin{bmatrix} p^i(t, x_k) \\ p^n(t, x_k) \end{bmatrix} \tag{67}$$

Now for $\gamma = 0$

$$\nabla \begin{bmatrix} p^i \\ p^n \end{bmatrix} = \begin{bmatrix} \rho & -\rho \\ -\rho & \rho \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix} \tag{68}$$

which we write as

$$\begin{bmatrix} p^i(t, x_k+\Delta x) \\ p^n(t, x_k+\Delta x) \end{bmatrix} - \begin{bmatrix} p^i(t, x_k) \\ p^n(t, x_k) \end{bmatrix} = \begin{bmatrix} \rho_k\Delta x p^i(t, x_k+\Delta x) - \rho\Delta x p^n(t, x_k) \\ -\rho_k\Delta x p^i(t, x_k) + \rho\Delta x p^n(t, x_k+\Delta x) \end{bmatrix} \tag{69}$$

or

$$\begin{bmatrix} 1-\rho\Delta x & 0 \\ 0 & 1-\rho\Delta x \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix}_{x_{k+1}} = \begin{bmatrix} 1 & -\rho\Delta x \\ -\rho\Delta x & 1 \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix}_{x_k} \tag{70}$$

or

$$\begin{bmatrix} p^i \\ p^n \end{bmatrix}_{x_{k+1}} = \frac{1}{1-\rho\Delta x} \begin{bmatrix} 1 & -\rho\Delta x \\ -\rho\Delta x & 1 \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix}_{x_k} \tag{71}$$

Putting these two sections in cascade yields

$$\begin{bmatrix} p^i \\ p^n \end{bmatrix}_{k+1} = \frac{1}{\tau_k} \begin{bmatrix} 1 & -\rho_k \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} e^{-\gamma_k} & 0 \\ 0 & e^{\gamma_k} \end{bmatrix} \begin{bmatrix} p^i \\ p^n \end{bmatrix}_{k} \tag{72}$$

where

$$\gamma_k = \gamma(x_k, s)\Delta x \tag{73}$$

$$\rho_k = \rho(x_k, s)\Delta x \tag{74}$$

$$\tau_k = 1 - \rho_k \tag{75}$$

Schematically this k^{th} section in signal-flow graph form is

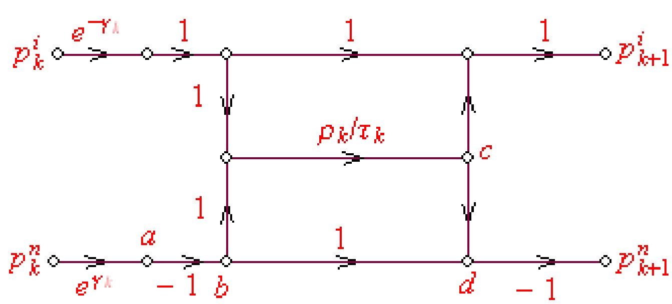


Figure 4: cochlear section signal flow graph

However we are interested in the reflected signal p_k^n as an output and p_{k+1}^n as an input, so express these equations with input vector $\begin{bmatrix} p_k^i \\ p_{k+1}^n \end{bmatrix}$ and output vector $\begin{bmatrix} p_{k+1}^i \\ p_k^n \end{bmatrix}$. For this 72 is

$$p_{k+1}^i = \frac{1}{\tau_k} e^{-\gamma_k} p_k^i - \frac{\rho_k}{\tau_k} e^{\gamma_k} p_k^n \tag{76}$$

$$p_{k+1}^n = -\frac{\rho_k}{\tau_k} e^{-\gamma_k} p_k^i + \frac{1}{\tau_k} e^{\gamma_k} p_k^n \tag{77}$$

The latter is

$$p_k^n = \rho_k e^{-2\gamma_k} p_k^i + \tau_k e^{-\gamma_k} p_{k+1}^n \tag{78}$$

which in the former gives

$$p_{k+1}^i = \frac{1}{\tau_k}\left(1-\rho_k^2\right) e^{-\gamma_k} p_k^i - \frac{\rho_k}{\tau_k^2} p_{k+1}^n \tag{79}$$

Thus

$$\begin{bmatrix} p_{k+1}^i \\ p_k^n \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau_k}\left(1-\rho_k^2\right) e^{-\gamma_k} & -\frac{\rho_k}{\tau_k^2} \\ \rho_k e^{-2\gamma_k} & \tau_k e^{-\gamma_k} \end{bmatrix} \begin{bmatrix} p_k^i \\ p_{k+1}^n \end{bmatrix} \tag{80}$$

By inverting signals in the last figure according to the labelled signals as

$$a = e^{\gamma_k} p_k^n \text{ or } p_k^n = e^{-\gamma_k} a \tag{81}$$

$$b = -a \text{ or } a = -b \tag{82}$$

$$d = b + c \text{ or } b = d - c \tag{83}$$

$$p_{k+1}^n = -d \text{ or } d = -p_{k+1}^n \tag{84}$$

we get the following useful signal-flow graph

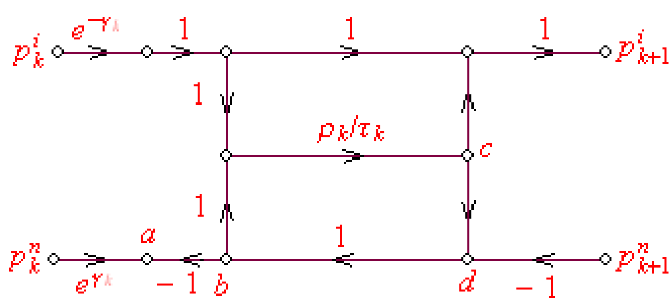


Figure 5: new signal flow graph with reflected signal as input

For this we note at the k^{th} junction of sectoins $R(x_k, s)$ is the reference, this being different for the two ends of the same section

5 Terminations

The desired representation is a cascade of N sections of the kind just exhibitied along with terminations. At the helicotrema end

$$v_v = 0\ ,\ u_v = 0 \tag{85}$$

which from the transformation to scattering variables of 49 is

$$p_N^i = p_N^n \tag{86}$$

which in a signal-flow graph is a direct connection:

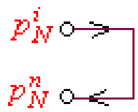


Figure 6: helicotrema signal flow graph

In a circuit diagram, since v_v is analogous to a current, this is an open circuit.

At the soruce end, we excite the outer ear with a speaker-like transducer which we take as a pressure source; in a circuit diagram this is

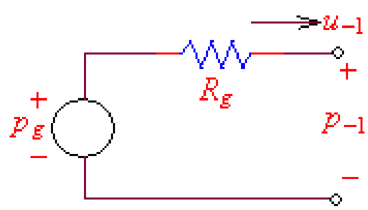


Figure 7: pressure source input to the cochlea

The coupling to the round window we represent by an ideal transformer in a circuit diagram

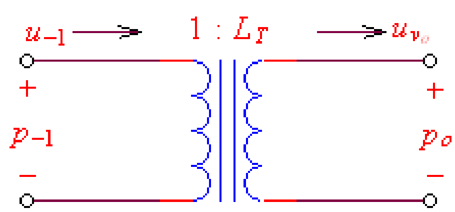


Figure 8: middle ear circuit representation

where

$$p_o = L_T p_{-1} \tag{87}$$

$$u_{-1} = L_T u_{v_o} \tag{88}$$

and $L_T > 1$ since the middle ear acts as a lever arm to increase the force (and decrease velocity).

Here R_g is the impedance of the outer ear, which in the literature taken to be real.

We take

$$R_{-1} = R_g \tag{89}$$

in which case, from 49

$$p^i_{-1} = \frac{1}{2}\left(p_{-1} + R_{-1}u_{-1}\right) = \frac{1}{2}\left([p_g - R_g u_{-1}] + R_{-1}u_{-1}\right) \tag{90}$$

$$= \frac{1}{2}p_g \tag{91}$$

That is the incident pressure at stage zero is $\frac{1}{2}$ the input transducer pressure while the reflected pressure is

$$p^n_{-1} = \frac{1}{2}\left(p_{-1} - R_{-1}u_{-1}\right) \tag{92}$$

$$= \frac{1}{2}p_g - R_g u_{-1} \tag{93}$$

In a signal-flow graph representation

Figure 9: signal flow relation between incident and transduced pressure

Note that if the load (cochlea) as reflected into the primary side of the transformer were to equal R_g, then p_{-1} would equal $R_{-1}u_{-1}$ and the output p^n_{-1} would be zero. However, such matching does not occur, giving a reflected pressure to be measured.

The scattering description of the transformer is needed: use 48 with 87.

$$p^i_o = \frac{1}{2}\left(p_o + R(0,s)u_{v_o}\right) = \frac{1}{2}\left(L_T p_{-1} + \frac{R_o u_{-1}}{L_T}\right);\ R_o = R(0,s) \tag{94}$$

$$= \frac{1}{2}\left(L_T\left[p^i_{-1} + p^n_{-1}\right] + \frac{R_o}{L_T}\left[G_{-1}p^i_{-1} - G_{-1}p^n_{-1}\right]\right) \tag{95}$$

$$\frac{1}{2}\left[L_T + \frac{R_o G_{-1}}{L_T}\right]p^i_{-1} + \frac{1}{2}\left[L_T - \frac{R_o G_{-1}}{L_T}\right]p^n_{-1} \tag{96}$$

and

$$p_o^n = \frac{1}{2}(p_o - R_o u_{v_o}) = \frac{1}{2}\left(L_T p_{-1} - \frac{R_o}{L_T}u_{-1}\right) \tag{97}$$

$$= \frac{1}{2}\left(L_T\left[p^i_{-1} + p^n_{-1}\right] - \frac{R_o}{L_T}\left[G_{-1}p^i_{-1} - G_{-1}p^n_{-1}\right]\right) \tag{98}$$

$$= \frac{1}{2}\left[L_T - \frac{R_o G_{-1}}{L_T}\right]p^i_{-1} + \frac{1}{2}\left[L_T + \frac{R_o G_{-1}}{L_T}\right]p^n_{-1} \tag{99}$$

or

$$\begin{bmatrix} p_o^i \\ p_o^n \end{bmatrix} = \frac{1}{2L}\begin{bmatrix} L_T^2 + R_o G_{-1} & L_T^2 - R_o G_{-1} \\ L_T^2 - R_o G_{-1} & L_T^2 + R_o G_{-1} \end{bmatrix}\begin{bmatrix} p^i_{-1} \\ p^n_{-1} \end{bmatrix} \tag{100}$$

which in signal-flow graph form is

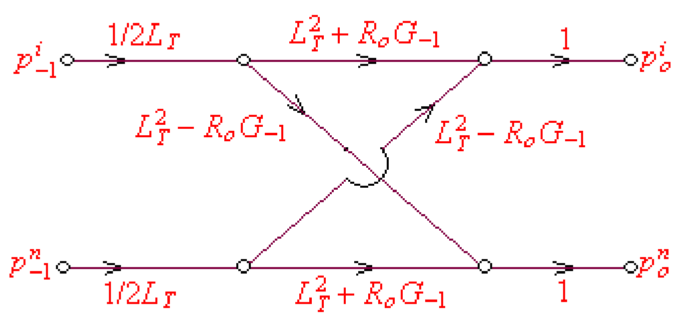


Figure 10: scattering treatment of transformer

On taking p_o^n as an input and p^n_{-1} as an output this is

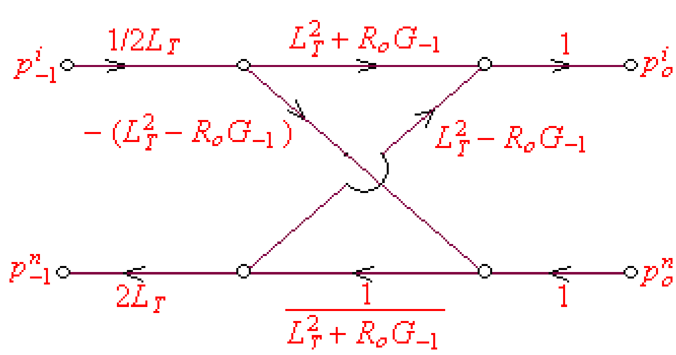


Figure 11: transformer signal flow graph with p_o^n as input

The full system cascades the various sections

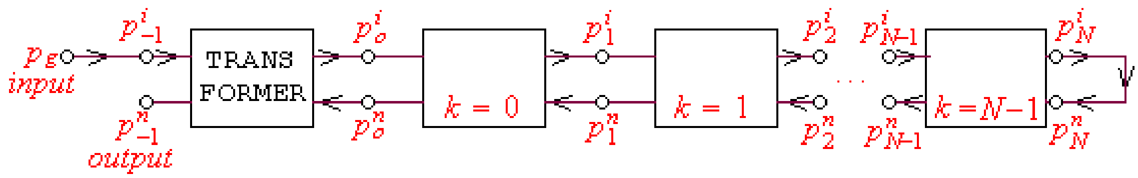


Figure 12: signal flow of full cochlear system

6 Circuit Realization - Linear Case

For a circuit realization we return to 46 and 47 to get the internal sections

$$\frac{p(x+\Delta x)-p(x)}{\Delta x} = -P(x,s)u_v \tag{101}$$

$$\frac{u_v(x+\Delta x)-u_v(x)}{\Delta x} = -\frac{1}{Q(x,s)}p \tag{102}$$

or

$$p(x_{k+1}) = p(x_k) - \frac{2\Delta x}{S_v(x_k)}\left[Rv+\rho s\right]u_v(x_k) \tag{103}$$

$$u_v(x_{k+1}) = u_v(x_k) - \frac{D(x_{k+1})\Delta x}{\mu(x_{k+1})s+\sigma(x_{k+1})+\frac{\phi(x_{k+1})}{s}}p(x_{k+1}) \tag{104}$$

These latter come from the following circuit

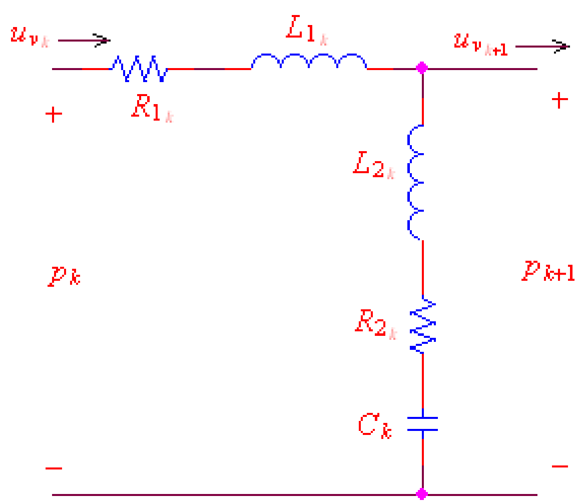


Figure 13: linear circuit realization

where

$$R_{1_k} = \frac{2R_v}{S_v(x_k)}\Delta x \tag{105}$$

$$L_{1_k} = \frac{2\rho}{S_v(x_k)}\Delta x \tag{106}$$

$$L_{2_k} = \frac{\mu(x_{k+1})}{D(x_{k+1})\Delta x} \tag{107}$$

$$R_{2_k} = \frac{\sigma(x_{k+1})}{D(x_{k+1})\Delta x} \tag{108}$$

$$C_k = \frac{D(x_{k+1})\Delta x}{\phi(x_{k+1})} \tag{109}$$

The full system as a circuit then becomes:

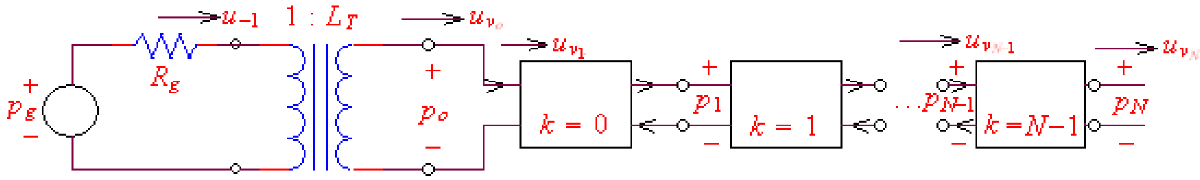


Figure 14: full system circuit diagram

Here the input is p_g and the output is u_{-1}.

7 Improvement in Nonlinearities

In the previous $S_v(x,t) \approx S_v(x)$ was assumed at 32. Here the corrections are incorporated when the time variation of $S_v(t,x)$ is included. Recall

$$S_v(t,x) = S_v(x) - \frac{1}{2}D(x)\xi_m(t,x) \tag{110}$$

Thus

$$\frac{\partial(S_v v_v)}{\partial t} = S_v\frac{\partial v_v}{\partial t} + v_v\frac{\partial S_v}{\partial t} = S_v\frac{\partial v}{\partial t} - v_v\left(\frac{D}{2}\dot{\xi}_m\right) \tag{111}$$

or

$$\frac{\partial v_v}{\partial t} = \frac{1}{S_v}\frac{\partial u_v}{\partial t} + \frac{v_v}{S_v}\frac{1}{2}D\dot{\xi}_m \tag{112}$$

and 31,

$$\nabla p = -2R_v v_v - 2\rho\frac{\partial v_v}{\partial t} = -\frac{2R_v}{S_v}u_v - 2\rho\left[\frac{1}{S_v}su_v + \frac{u_v}{S_v^2}\left(\frac{D}{2}\dot{\xi}_m\right)\right] \tag{113}$$

or

$$\nabla p = -\frac{2}{S_v}\left[R_v u_v + \rho\frac{u_v}{S_v}\frac{1}{2}D\dot{\xi}_m + \rho\frac{\partial u_v}{\partial t}\right] \tag{114}$$

$$= -\frac{2}{S_v}\left[Rv + \rho s\right]u_v - \frac{2\rho}{S_v^2}u_v s\left(\frac{1}{2}D\xi_m\right) \tag{115}$$

Using 40, 43 is

$$\nabla p = -P(x,s)u_v + \frac{2\rho}{S_v^2}u_v\frac{1}{Q}p \tag{116}$$

The final nonlinear equations would then be

$$\nabla p = -Pu_v + \frac{2\rho}{S_v^2}u_v\frac{1}{Q}p \tag{117}$$

$$\nabla u_v = -\left[1 + \frac{8}{D^4}\left(\frac{1}{sQ}p\right)^2\right]\frac{1}{Q}p \tag{118}$$

8 Circuit Realization - Nonlinear Case

From 39 we have

$$\frac{D\xi_m}{2}\Delta x = \frac{-D\Delta x}{s\left(\mu s + \sigma + \phi/s\right)}p = \frac{1}{s}i_s = -\frac{\Delta x}{sQ}p \quad (= Cv_c) \tag{119}$$

which is seen to be the integral of the current i_s going up in the short arm branch. And by 37

$$u_v(x_{k+1}) = u_v(x_k) + \left[1 + \frac{8}{D^4\Delta x^2}\left(\frac{D\xi_m}{2}\Delta x\right)^2\right]\cdot s\left(\frac{D\xi_m}{2}\Delta x\right) \tag{120}$$

while 117 is written as

$$p(x_{k+1}) = p(x_k) - P(x_k,s)\Delta x u_v(x_k) - \frac{2\rho}{S_v^2}u_v(x_k)\cdot s\left(\frac{D\xi_m}{2}\Delta x\right) \tag{121}$$

Here, since $C_k = D\Delta x/\phi$ by 109,

$$\frac{8}{D^4\Delta x^2}\left(\frac{D\xi_m}{2}\Delta x\right)^2 = \frac{8}{D^4\Delta x^2}\left(\frac{1}{s}i_s\right)^2 = \frac{8}{D^4\Delta x^2}\left(C_k V_{c_k}\right)^2 = \frac{8}{D^2\phi^2}V_{c_k}^2 \tag{122}$$

The terminations are as in the linear case so the figure before this remains valid

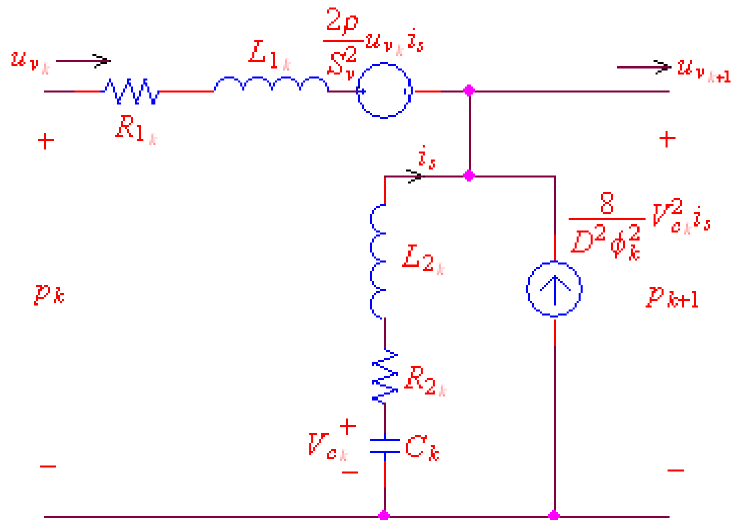


Figure 15: nonlinear circuit realization

9 Table of Values

To implement the previous circuits we need parameter values for a typical ear, as is collected here.

$$L = 35\ mm \tag{123}$$

$$D(x) = ax + b;\ a = 0.086,\ b = 0.2\ mm \tag{124}$$

$$\rho = 1\frac{g}{cm^3} \tag{125}$$

$$R_v = 0.056\frac{g}{cm \cdot s^{1/2}} \tag{126}$$

$$S_v(x) = ce^{-dx};\ c = 0.025\ cm,\ d = 0.5\ cm^{-1} \tag{127}$$

$$R_g = 193\frac{dyne \cdot s}{cm^3} \tag{128}$$

$$\mu(x) = 0.1\frac{g}{cm^2} \tag{129}$$

$$\sigma(x) = fe^{-gx};\ f = 600\frac{dyne \cdot s}{cm^3},\ g = 1.7\ cm^{-1} \tag{130}$$

$$\phi(x) = he^{-kx};\ h = 2 \cdot 10^9\frac{dyne}{cm^3},\ k = 3.4\ cm^{-1} \tag{131}$$

$$L_T = 1.3 \tag{132}$$

Note that $S_v(L) = 0.00869\ cm^2$, $D(L) = 0.5\ mm$; since $S_v(L) = \frac{1}{2}\left(\pi r_L^2\right)$, $r_L = 0.744\ mm$ and $2r_L = 1.49 > 0.5 = D(L)$

10 Justification of $v_t = -v_v$

In actual fact this result should be

$$u_t = -u_v \tag{133}$$

where

$$u_t = S_t v_t;\ u_v = S_v v_v \tag{134}$$

or

$$v_t = -\frac{S_v}{S_t} v_v \tag{135}$$

But if $S_t = S_v$ then $v_t = -v_v$ as we have used.

To see that $u_t = -u_v$ pass a closed surface S that cuts the cochlea by the transverse plane at x and surrounds the helicotrema end.

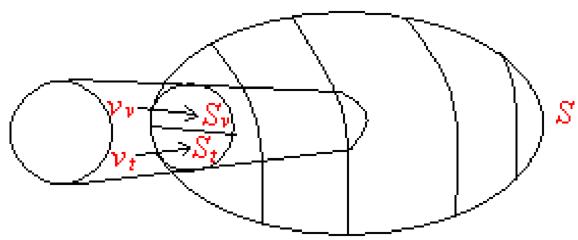


Figure 16: Closed surfaced passing through cochlea

Since the velocity over S_v is to be taken uniform it is in one direction as in that over S_t. By conservation of mass

$$\iint_S p \overrightarrow{v} \cdot d\overrightarrow{S} = 0 = \rho \left(v_v S_v + v_t S_t \right) \tag{136}$$

which is 133.

If the two areas S_v and S_t are not equal (which strictly they will not be under perturbation) then 30 becomes

$$\frac{\partial v_t}{\partial t} + v_t \nabla v_t = -\frac{\partial \left(\frac{S_v}{S_t} v_v \right)}{\partial t} + \left(\frac{S_v}{S_t} v_v \right) \nabla \left(\frac{S_v}{S_t} v_v \right) \tag{137}$$

$$= -\frac{1}{\rho} \left(\nabla p_t - R_v \frac{S_v}{S_t} v_v \right) \tag{138}$$

or

$$\nabla p_t = R_v \frac{S_v}{S_t} v_v + \rho \frac{\partial \left(\frac{S_v}{S_t} v_v \right)}{\partial t} - \rho \left(\frac{S_v}{S_t} v_v \right) \nabla \left(\frac{S_v}{S_t} v_v \right) \tag{139}$$

while, from 29

$$\nabla p_v = -R_v v_v - \rho \frac{\partial v_v}{\partial t} - \rho v_v \nabla v_v \tag{140}$$

Then, as $p = p_v - p_t$,

$$\nabla p = -R_v \left(1 + \frac{S_v}{S_t}\right) v_v - \rho \frac{\partial \left[\left(1 + \frac{S_v}{S_t}\right) v_v\right]}{\partial t} - \rho \left\{ v_v \nabla v_v - \frac{S_v}{S_t} v_v \nabla \left(\frac{S_v}{S_t} v_v\right)\right\} \tag{141}$$

$$= -R_v \left(\frac{1}{S_v} + \frac{1}{S_t}\right) u_v - \rho \frac{\partial \left[\left(\frac{1}{S_v} + \frac{1}{S_t}\right) u_v\right]}{\partial t} - \rho \left\{ \frac{u_v}{S_v} \nabla \left(\frac{u_v}{S_v}\right) - \frac{u_v}{S_t} \nabla \left(\frac{u_v}{S_t}\right)\right\} \tag{142}$$

For which we will define the parallel area as

$$\frac{1}{S_p} = \frac{1}{S_v} + \frac{1}{S_t} = \frac{S_v + S_t}{S_v S_t} = \frac{2S_v(x)}{S_v(t,x) S_t(t,x)} \quad \left(\approx \frac{2}{S_v} \ if \ S_v = S_t\right) \tag{143}$$

and note that

$$\nabla \left(\frac{u_v}{S_v}\right) = \frac{1}{S_v} \nabla u_v - \frac{u_v}{S_v^2} \nabla S_v; \ \nabla \left(\frac{u_v}{S_t}\right) = \frac{1}{S_t} \nabla u_v - \frac{u_v}{S_t^2} \nabla S_t \tag{144}$$

making 142

$$\nabla p = -\frac{R_v}{S_p} u_v - \frac{\rho}{S_p} \frac{\partial u_v}{\partial t} - \rho \left\{ \left(\frac{1}{S_v^2} - \frac{1}{S_t^2}\right) u_v \nabla u_v \right\} - \rho u_v \frac{\partial \left(1/S_p\right)}{\partial t} - \rho \left\{ \frac{u_v^2}{S_t^3} \nabla S_t - \frac{u_v^2}{S_v^3} \nabla S_v \right\} \tag{145}$$

Now, from 143,

$$\frac{\partial \left(1/S_p\right)}{\partial t} = 2S_v(x) \cdot \frac{-1}{S_v^2(t,x) S_t^2(t,x)} \left\{ S_t \frac{\partial S_v}{\partial t} + S_v \frac{\partial S_t}{\partial t} \right\} \tag{146}$$

$$= -\frac{2S_v(x)}{S_v S_t} \left\{ \frac{1}{S_v} \cdot \left(-\frac{D}{2} \dot{\xi}_m\right) + \frac{1}{S_t} \left(\frac{D}{2} \dot{\xi}_m\right)\right\} \tag{147}$$

$$= -\frac{2S_v(x)}{S_v S_t} \left(\frac{1}{S_t} - \frac{1}{S_v}\right) \left(\frac{D}{2} \dot{\xi}_m\right) \tag{148}$$

$$= \frac{4S_v(x)}{(S_v S_t)^2} \left(\frac{D}{2} \xi_m\right) \left(\frac{D}{2} \dot{\xi}_m\right) \tag{149}$$

since

$$S_v(t,x) = S_v(x) - \frac{D(x)}{2} \xi_m(t,x) \tag{150}$$

$$S_t(t,x) = S_t(x) + \frac{D(x)}{2} \xi_m(t,x) \tag{151}$$

Further

$$\nabla S_v = S_v'(x) - \frac{D'}{2}\xi_m - \frac{D}{2}\nabla\xi_m \tag{152}$$

$$\nabla S_t = S_t'(x) + \frac{D'}{2}\xi_m + \frac{D}{2}\nabla\xi_m \tag{153}$$

giving

$$S_v^3\nabla S_t - S_t^3\nabla S_v = \left[S_v^3 S_t'(x) - S_t^3 S_v'(x)\right] + \left(S_v^3 + S_t^3\right)\nabla\left(\frac{D}{2}\xi_m\right) \tag{154}$$

and

$$\frac{1}{S_v^2} - \frac{1}{S_t^2} = \frac{S_t - S_v}{S_v S_t}\cdot\frac{S_t + S_v}{S_v S_t} = \frac{S_t(x) - S_v(x) + D\xi_m}{S_v S_t}\cdot\frac{S_v(x) + S_t(x)}{S_v S_t} \tag{155}$$

Equations 146-155 allow the evaluation of the nonlinear terms in 145. When $S_v(x) = S_t(x)$ we get for 145

$$\nabla p = -\frac{2R_v}{S_v}u_v - \frac{2\rho}{S_v}\frac{\partial u_v}{\partial t} - \frac{2\rho}{S_v^3}\left[\nabla\left(\left(\frac{D}{2}\xi_m\right)u_v^2\right) + u_v\frac{\partial\left(\frac{D}{2}\xi_m\right)}{\partial t}\right] \tag{156}$$

where the last two terms are nonlinear terms. In this the nonlinear terms appear to be quite small. In any event the complete set of described equations is, from 37, 39, and 156

$$\nabla p = -\frac{2\rho}{S_v}\left[\frac{R_v}{\rho} + s\right]u_v - \frac{2\rho}{S_v^3}\left[\nabla\left(\left(\frac{D}{2}\xi_m\right)u_v^2\right) + u_v\frac{\partial\left(\frac{D}{2}\xi_m\right)^2}{\partial t}\right] \tag{157}$$

$$\nabla u_v = \left[1 + \frac{8}{D^4}\left(\frac{D}{2}\xi_m\right)^2\right]\left(\frac{D}{2}\xi_m\right) \tag{158}$$

$$-p = \frac{\left[\mu s^2 + \sigma s + \phi\right]}{D}\left(\frac{D}{2}\xi_m\right) \tag{159}$$

Note that assuming $S_v = S_t$ before taking the derivatives gave $\frac{-2\rho u_v}{S_v^2}\left(\frac{D}{2}\dot{\xi}_m\right)$, see 114, while the more accurate caclulation for 157, gave for the same term $-\frac{2\rho u_v}{S_v^2}\left(\frac{D}{2}\dot{\xi}_m\right)\frac{\left(\frac{D}{2}\xi_m\right)}{S_v}$ which has the factor $\left(\frac{D}{2}\xi_m\right)/S_v$ different. This seems to be a significant difference.

The structure of a circuit for equations 157-159 is as on Figure 15 except that the nonlinear series arm voltage source depends on V_{c_k} from the same section and $V_{c_{k+1}}$ of the next section since

$$p(x_{k+1}) = p(x_k) - \frac{2\Delta x}{S_v}\left(R_v + \rho s\right)u_v - \frac{2\rho}{S_v^3}\left[2u_{v_k}\Delta x i_{s_k} C_k V_{c_k}\right] - \frac{2\rho}{S_v^3}\left[u_{v_{k+1}}^2 C_{k+1}V_{k+1} - u_{v_k}^2 C_k V_k\right] \tag{160}$$

which is obtained from 157 with $i_s = s\left(\frac{D}{2}\xi_m\right) = -\frac{1}{sQ}p = sV$

$$K = \frac{8}{D^4\Delta x^2}C_k^2 = \frac{8}{D^2\phi^2} \tag{161}$$

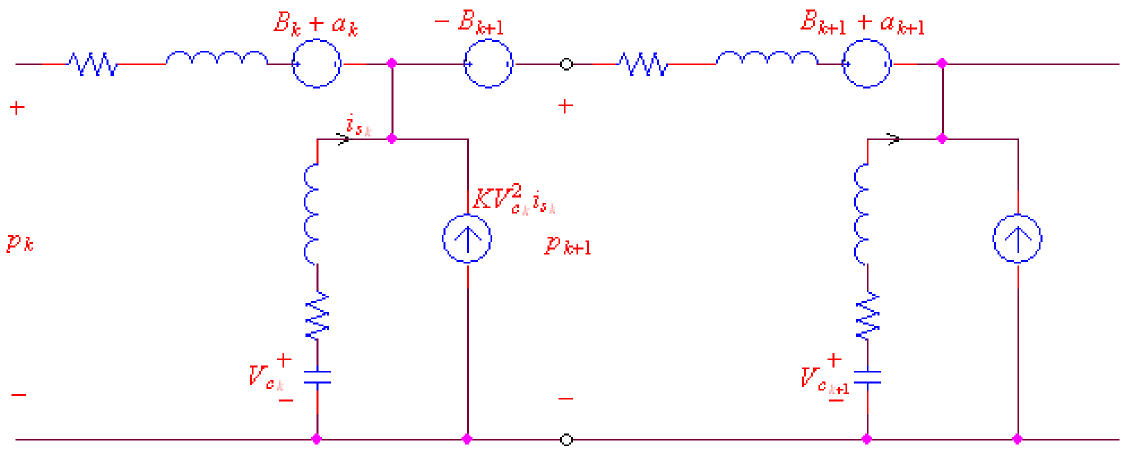
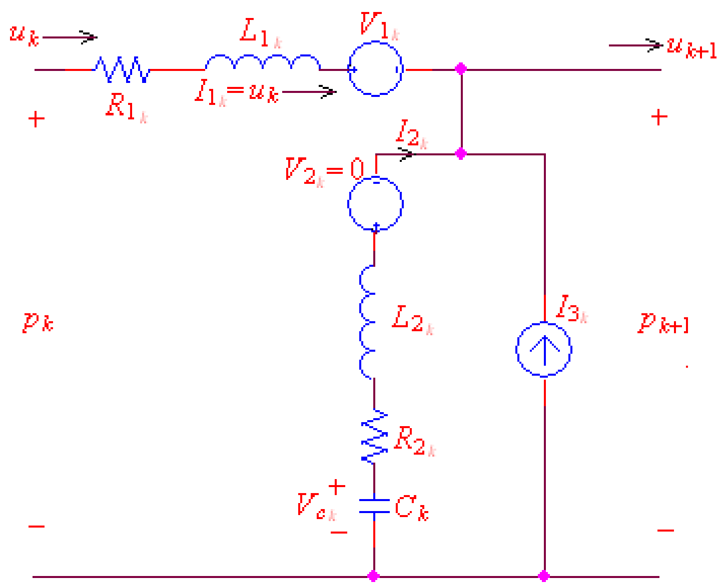


Figure 17: cancellation of sources between adjacent sections

Putting $u_{v_{k+1}}^2 C_{k+1} V_{k+1}$ on the right of the kth section and $u_{v_k}^2 C_k V_k$ on the left we see the former can be cancelled with the latter of the $(k+1)$st section. Thus the $\nabla\left(\left(\frac{D}{2}\xi_m\right)u_v^2\right)$ term can be ignored and the circuit in Figure 15 holds with the nonlinear voltage source being

$$a_k = 4\frac{\rho u_{v_k}}{S_v^2}\Delta x i_{s_k} C_k V_k \tag{162}$$

Therefore a useful representation of the kth section for use on Spice, etc, is



k^{th} cochlear section circuit diagram where

$$V_{1_k} = \frac{4\rho\Delta x C_k}{S_v^3(x_k)} I_{1_k} I_{2_k} V_{c_k} \tag{163}$$

$$I_{3_k} = \frac{8C_k}{D^4(x_k)\Delta x^2} I_{2_k} V_{c_k}^2 \tag{164}$$

and

$$R_{1_k} = \frac{2R_v}{S_v(x_k)}\Delta x \tag{165}$$

$$L_{1_k} = \frac{2\rho}{S_v(x_k)}\Delta x \tag{166}$$

$$L_{2_k} = \frac{\mu(x_{k+1})}{D(x_{k+1})\Delta x} \tag{167}$$

$$R_{2_k} = \frac{\sigma(x_{k+1})}{D(x_{k+1})\Delta x} \tag{168}$$

$$C_k = \frac{D(x_{k+1})\Delta x}{\phi(x_{k+1})} \tag{169}$$

$$\Delta x = L/N$$

where N is the number of sections, $x_k = k\Delta x$, and $k = [0, N-1]$ with parameters as given preivously. Note that for the linearized representation the sources present in the above circuit are absent.

11 Time Discretization - Linear Case

For time discretizatoin we use backward differencing

$$sf = \frac{\partial f}{\partial t} \approx \frac{f(t) - f(t-\Delta t)}{\Delta t} = \frac{\left(1 - \frac{1}{z}\right)}{\Delta t} f(t) \tag{170}$$

where the derivative operator is replaced via the delay operator $1/z$. Thus, we use the replacement

$$s = \frac{\left(1 - \frac{1}{z}\right)}{\Delta t} \tag{171}$$

where Δt is the sampling interval.

For the section of the scattering treatment, we need

$$\rho_k(z) = \rho(x_k, s)\Delta x = \frac{1}{4PQ}\frac{\partial PQ}{\partial x} \tag{172}$$

$$\gamma_k(z) = \gamma_k(x_k, s)\Delta x = \sqrt{\frac{P}{Q}} \tag{173}$$

where, 41 and 42,

$$P(x, s) = \frac{2}{S_v(x)}\left[R_v + \rho s\right] \tag{174}$$

$$Q(x, s) = \frac{\mu(x)s^2 + \sigma(x)s + \phi(x)}{D(x)s} \tag{175}$$

We see that ρ_k is of degree 3, in either s or z, and γ_k is irrational of degree $(2)^{1/2}$ and occurs as an exponent. Thus, we desire to do a ratoinal approximation of $e^{-\gamma_k(z)}$