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Real Computation for Real Roots of Real Cubics

Robert W. Newcomb

Abstract

By using invertible real transformations a cubic polynomial with real coefficients is transformed to a useful one-parameter form for obtaining real roots. From that canonical form, real roots are shown to be calculated by real equations. Examples are given to illustrate the process.

I. INTRODUCTION. Over the years cubic polynomials with real coefficients have been extensively studied [5] with an intriguing case, known as the "casus irreducibilis" requiring the use of complex numbers in the case of a positive discriminant [1][7]. Indeed it has been proven that one can not escape the use of an extension field in the positive discriminant case [4]. However there do exist methods to solve such cubics by non-analytic methods, for example by the graphical means of "turtle and bullet paths" [3] so one would expect to be able to find real roots using only real calculations. Indeed this is possible as seen in Mathematica [8]. Here we also express a real root analytically in a little different form.

II. A CANONICAL FORM. Given a cubic polynomial P(x) with real coefficients, a_0, a_1, a_2, $a_3 \neq 0$,

$$P(z) = a_3z^3 + a_2z^2 + a_1z + a_0 \qquad (1)$$

For which the interest is to calculate a real root using only real numbers.

We assume $a_3 \neq 0$ as well as $a_0 \neq 0$ since, if $a_0=0$ then $z=0$ is a real root. Of interest since it expresses the nature of roots, z_1, z_2, z_3, is the discriminant [2]

$$\Delta_P = a_2^2a_1^2 - 4a_3a_1^3 - 4a_2^3a_0 - 27a_3^2a_0^2 + 18a_3a_2a_1a_0 = (z_1-z_2)(z_1-z_3)(z_2-z_3) \qquad (2)$$

If $\Delta<0$ there is only one real root, if $\Delta=0$ there are three real roots, at least two of them equal, and if $\Delta>0$ there are three distinct real roots, the latter being the "casus irreducibilis" of most interest.

Then we transform P(x) into the canonical form

$$p(x) = x^3 + Cx + 1 \qquad \{\text{giving } \Delta_p = \Delta = -(4C^3 + 27)\} \qquad (3a,b)$$

by the following two invertible transformations using the real cube root of a real number (thus only real operations on real numbers).

a) $z = y - a_2/(3a_3)$ giving $\underline{P}(y) = a_3y^3 + A_1y + A_0$ for which $\Delta_{\underline{P}}=\Delta_P = -4a_3A_1^3-27a_3^2A_0^2$ (4a,b,c)

where $A_0=\{a_0+[2a_2^3/(27a_3^2)]-[a_1a_2/(3a_3)]\}$ and $A_1=\{a1-a_2^2/(3a_3)\}$ (4d,e)
If also $\underline{a_0}=0$ then $z=-a_2/(3a_3)$ is a real root and if further $A_1=0$ it is a triple real root. (4f)

b) $x = (a_3/A_0)^{(1/3)}y$ if $A_0\neq0$ giving p(x) of (3a) and $\Delta_p=(a_3/A_0)^6\Delta_P=-4C^3-27$ with $C=A_1/(a_3^{1/3}A_0^{(2/3)})$ (4h,i)

Except if $\Delta=i$ with three equal roots in which case (3a) is replaced by $p(x)=x^3$.

If a root x is found for (3a) then working backwards a root for (1) is found by

$$z = y-a_2/(3a_3) = (A_0/a_3)^{1/3}x - a_2/(3a_3) \qquad (5)$$

It is of interest that p(x) shows that there is a large class of cubics described by the same parameter C while the C contains fundamental information about this class. The figure following the first example below gives a comparison between a typical P(z) and p(x) showing how the root pattern is preserved.

III. REAL ROOTS. One gets the following real solutions for (3) with $\Delta = \Delta_p = -4C^3-27$, using real cube roots of real numbers,

$$x_r = \begin{cases} -2^{-\frac{1}{3}}[\left(1+\sqrt{-\Delta/27}\right)^{\frac{1}{3}} + \left(1-\sqrt{-\Delta/27}\right)^{\frac{1}{3}}] \quad if\ \Delta < 0 & (6a) \\ -2^{\frac{2}{3}},\ 2^{\frac{-2}{3}},\ \ 2^{\frac{-2}{3}} \quad if\ \ \Delta = 0 & (6b) \\ -2^{\frac{2}{3}}3^{-1/2}(\Delta+27)^{\frac{1}{6}}\cos\left(\frac{2\mathrm{k}\pi+\arctan(\sqrt{\Delta/27})}{3}\right), k=0,1,2 \quad if\ \Delta > 0 & (6c) \end{cases}$$

Note that, that the triple real root case is covered at (4f). Also since $\Delta + 27 = -4C^3$, in terms of C, (6c) takes the form

$$x_r = -2\cdot 3^{-\frac{1}{2}}(-C)^{\frac{1}{2}}\cos\left(\frac{2k\pi+atan\left(\sqrt{\left(\frac{-C}{3}\right)\cdot\left(2\frac{C^2}{3}\right)^2-1}\right)}{3}\right) \ where\ \ C < -3\cdot 4^{-1/3} \qquad (6d)$$

Proof of (6):

We follow Wolfrom Math World [8, Eq.(31)] which gives a short proof that a zero of $x^3 + 3Qx - 2R$ is

$$x = [R+\sqrt{Q^3+R^2}]^{1/3} + [R-\sqrt{Q^3+R^2}]^{1/3} \qquad (7)$$

Interestingly this is always real (when Q^3+R^2= <0 the two cube root terms are complex conjugates). For the case of p(x) of (3), (7) is evaluated as, with Q=C/3 & R=-1/2 as $Q^3+R^2 = (4C^3+27)/(4x27) = -\Delta/(4x27)$ and using the real cube root of -1,

$$x_r = -2^{-1/3}[\left(1-\sqrt{-\Delta/27}\right)^{\frac{1}{3}} + \left(1+\sqrt{-\Delta/27}\right)^{\frac{1}{3}}] \qquad (8a)$$

Although x_r is always real it does use complex numbers in the case that Δ>0. In the case of (6a) is just a rewrite of (8a). For Δ=0, which is $C=-3x2^{-2/3}$, (8a) gives $x_r=-2^{2/3}$; dividing by $x+2^{2/3}$one gets for (3a) $p(x)=(x+2^{2/3})(x-2^{-2/3})^2$ when there are two equal roots. But when Δ>0 the two main bracketed terms are complex conjugates. So we can write in terms of magnitude and angle

$$1+\ i\sqrt{\Delta/27} = \left(\sqrt{1+\Delta/27}\right)e^{i[2k\pi+\theta]}\ where\ k = 0,1,2, and\ \theta = \arctan(\sqrt{\Delta/27}) \qquad (8b,c)$$

Taking the real cube roots, $[1+\ i\sqrt{\Delta/27}]^{1/3}$=$(1+\Delta/27)^{1/6}e^{i[2k\pi=\theta]/3}$ (8d)

$$\left(1+\ i\sqrt{\Delta/27}\right)^{\frac{1}{3}} +\ (1-i\sqrt{\Delta/27})^{\frac{1}{3}} = (1+\Delta/27)^{\frac{1}{6}}\left[e^{\frac{i[2k\pi+\theta]}{3}} + e^{-\frac{i[2k\pi+\theta]}{3}}\right] = \frac{(\Delta+27)^{1/6}}{3^{1/2}}2\cos(\frac{[2k\pi+\theta]}{3}) \qquad (8e)$$

Where $\cos\left(\frac{[2k\pi+\theta]}{3}\right) = \cos\left(\frac{[2k\pi+arctan\left(\sqrt{\Delta/27}\right)]}{3}\right) for\ k = 0,1,2$ (8f))

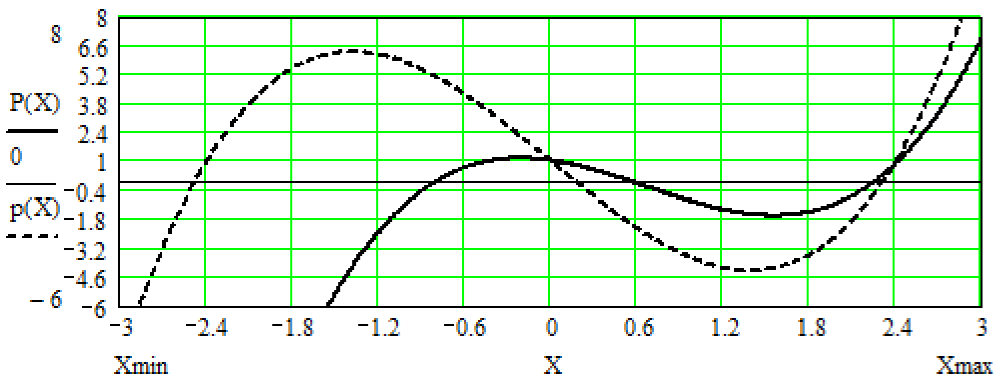
Combining equations (8) gives (6c), =(6d), in the casus irreducibilis case.

Although this has used complex numbers, only real numbers are needed to evaluate (6).

IV. EXAMPLES. (numerically evaluated with MathCad and displayed to three figures)

1) [8] Applying the above technique to $P(z)=z^3 - 2z^2 - z + 1$ which has $\Delta_P = 49$. we have $y=z+2/3$ and $A_0=1-(2/3)-16/27)=(9-16)/27=-7/27$; $A_1=-1-(-2)^2/3=-1-4/3=-7/3 \Rightarrow y^3-(7/3)y+(-7/27)$; $C=(A_1/A_0^{2/3}) = -3*7^{1/3}$ so $p(x)=x^3-(3*7^{1/3})x+1$, $\Delta=\Delta_p=(27)^2=(a_3/A_0)^{6/3}\Delta_P$ showing there are three real distinct roots. All three are found from (6c) $x_r=-2^{2/3}\,3^{1/2}(27^2+27)^{1/6}\cos(2k\pi+\arctan(27/3)$. Then $y_x=(A_0/a_3)^{1/3}x_r$ giving $z_r=y_r-(a_2/3a_0)= (2-7^{1/3})/3$ as the three real roots of $z^3- 2z^2-z+1$ these being 2.247, -0.902 and 0.555. As a check $z_r^3-2z_r^2-z_r+1$ evaluates on the order of 10^{-15}.

The figure illustrates p(x) compared to P(x)



2) In trisecting an angle ψ the equation for $z=\cos(\psi/3)$ is, when given $\cos(\psi)$, $P(z)=4z^3-3z-\cos(\psi)$. This becomes, with $x=-(\cos(\psi)/4)^{1/3}z$, $p(x)=x^3+Cx+1$ with $C=-3x4^{-1/3}\cos^{-2/3}(\psi)$ and for which $\Delta=27 >0$. Then by (6c), $x_r=-2^{2/3}\,3^{-1/2}\cos([2k\pi+\psi]/3)$. When $\psi=\pi/4$ in which case $\cos(\psi)=0.707$, this gives the three x roots: -1.721, 1.26, 0.461. From these the three z cubic roots are 0.966, -0.707, -0.259.

V. CONCLUSIONS. If in the conversion of a given polynomial P(z) to $p(x)=x^3+Cx+1$ either $a_0=0$ or $A_0=0$ then 0 is a (possibly multiple) solution. Otherwise all the solutions are nicely expressed in terms of the discriminant Δ as per equations (6) which give the real solutions when the coefficients are real. This would seem to contradict the need for use of an extension field but such is used since the result is based upon equation (7). Still, one would expect that real solutions should be expressed directly in terms of a real polynomial's coefficients since only real operations are used in the geometric constructions of Lill's and other methods discussed for example in reference [3]. Various other one-parameter forms exist. For example, If one sets $w=1/x$ the parameter reverts to the square term, $p_w(x)=x^3+Cx^2+1$; other analytic forms are seen to be developed for Mathematica using hyperbolics in [8, (70)-(84)].

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