MAA "The College Mathematics Journal" Problem 1171:

For $x^3-2x^2-x+1=0$ with roots $a<b<c$ find $g = (a/b)^2 + (b/c)^2 + (c/a)^2$

Solution: Using standard cubic equation solutions one finds a = -0.80193774 , b = 0.55495813,

c= 2.24697961 as the roots giving g = 10. {Note: g uses squares, so for $a<b<c$ in this MAA problem g uses $b^2<a^2<c^2$}

More generally for the cubic with real coefficients and all real roots

$$f(x) = x^3 + a_2x^2 + a_1x + a_0 = 0 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$$

on defining the alternate of g, with inverted entries [h=17 for the Journal problem],

$$h = (b/a)^2 + (c/b)^2 + (a/c)^2$$

g and h can be found in terms of a given cubic's coefficients (assuming $a_0=-abc\neq0$) through the following formulas:

$$\mathrm{Max}\{g,h\} = \{-B + \mathrm{SquareRoot}[B^2-4C]\}/(2a_0^2) \quad \text{where}$$

$$B = -(2G+(a_1a_2-a_0)^2);\ \ C=G^2-(a_1a_2-a_0)^2(-5a_0^2+2a_0a_1a_2) \text{ with } G=-a_0a_2^3-a_1^3-2a_0^2+3a_0a_1a_2$$

While Min{g,h} goes with the minus sign on the square root. The sign is determined by

$$\mathrm{Sign}(g - h) = \mathrm{Sign}\{a^2(b^4-c^4) + b^2(c^4-a^4) + c^2(a^4-b^4)\}$$

This is proven as follows:

Define g_0 (using $a_0 = -abc$)

(1a,b) $g_0 = a_0^2g = (-abc)^2[(a/b)^2 + (b/c)^2 + (c/a)^2]$

And similarly for $h_0 = a_0^2h$; h is introduced for determining the end result sign and its calculations following directly those of g. Direct calculation gives

(1c,d) $g_0 = (a^4c^2+b^4a^2+c^4b^2) = (a^2c+ab^2+bc^2)^2 -2(abc)(a^2b+ac^2+b^2c)$

Next define for use in (1d)

(1e,f) $X = a^2c+ab^2+bc^2$; $Y = a^2b+ac^2+b^2c$

X and Y are related by

(1g,h) $-a_1a_2 = (ab+ac+bc)(a+b+c) = (X+Y) -3a_0 \implies Y = (3a_0-a_1a_2) -X$

Then inserting (1h) into (1d), written as $g_o=X^2+2a_0Y$ (and similarly $h_0=Y^2+2a_0X$) gives the quadratic equation for X, now in terms of g_o,

(2a) $X^2 -2a_0X +[2a_0(3a_0-a_1a_2)-g_o] =0$

with solution

(2b) $X = a_0 \pm \mathrm{SqRt}(-5a_0^2+2a_0a_1a_2+g_o)$

Similarly Y satisfies (2a,b) with g_0 replaced by h_0. In order to find g_o we expand the first term in (1d), which is X^2, in terms of roots, keeping the second term which includes X in terms of coefficients and g_o. For the first term we have

(3a) $(a^2c+ab^2+bc^2)^2= (a^4c^2+a^2b^4+b^2c4)+2(a^3b^2c+a^2bc^3+ab^3c^2)$

To proceed we evaluate the 3rd and 4th power terms by noting that a,b,c satisfy f(x)=0 so $x^3=-a_2x^2-a_1x-a_0$ and with it $x^4=-a_2x^3-a_1x^2-a_0x=(a_2^2-a_1)x^2+(a_1a_2-a_0)x+a_0a_2$. Inserting these expressions into (3a) evaluated at roots, x=a,b,c, gives, using the relations between coefficients and roots [a_1=ab+ac+bc, a_2=-a-b-c],

(3b,c,) $(a^2c+ab^2+bc^2)^2 = [(a_2^2-a_1)[a^2c^2+a^2b^2+b^2c^2) -2a_2(a^2b^2c+a^2bc^2+ab^2c^2)]$

$+[(a_1a_2-a_0)(ac^2+a^2b+b^2c) -2a_1(ab^2c+a^2bc+abc^2)] +[a_0a_2(c^2+a^2+b^2)-2a_0(b^2c+a^2b+ac^2)]$

$= [(a_2^2-a_1)[a^2c^2+a^2b^2+b^2c^2) -2a_2(abc)a_{1)}]+[(a_1a_2-a_0)Y-2a_1(abc)(-a_2)]+[a_0a_2(c^2+a^2+b^2)-2a_0Y]$

To further evaluate these we also use the relations between coefficients and roots via

(3c,d) $a_1^2=(ab+ac+bc)^2=(a^2b^2+a^2c^2+b^2c^2)+2(a^2bc+ab^2c+abc^2) \implies (a^2b^2+a^2c^2+b^2c^2)= (a_1^2-2a_0a_2)$

(3e,f) $a_2^2=(-a-b-c)^2=(a^2+b^2+c^2)+2(ab+ac+bc) \implies (a^2+b^2+c^2)=(a_2^2-2a_1)$

Thus, using (3c), (1c) becomes, with Y given by (1h),

(4a,b) $g_o=[(a_2^2-a_1)(a_1^2-2a_0a_2)-2a_2(-a_0)a_1]+[(a_1a_2-a_0)Y-2a_1(-a_0)(-a_2)]+[a_0a_2(a_2^2-2a_1)] -2a_0Y] +2a_0Y$

$= [(a_2^2-a_1)(a_1^2-2a_0a_2)+a_0a_2(a_2^2-2a_1)] +(a_1a_2-a_0)((3a_0-a_1a_2) -X)$

Using the solution for X from (2b) gives on rearranging to isolate the square root

(4c) $(a_1a_2-a_0)[\pm \mathrm{SqRt}(-5a_0^2+2a_0a_1a_2+g_o)]$

$= [(a_2^2-a_1)(a_1^2-2a_0a_2)+a_0a_2(a_2^2-2a_1)]-(a_1a_2-a_0)(3a_0-a_1a_2)-a_0(a_1a_2-a_0) -g_o$

Squaring this gives the final quadratic equation for g_o for which the ± in X disappears:

(4d) $(a_1a_2-a_0)^2(-5a_0^2+2a_0a_1a_2+g_o)$

$= \{[(a_2^2-a_1)(a_1^2-2a_0a_2)+a_0a_2(a_2^2-2a_1)]-(a_1a_2-a_0)(3a_0-a_1a_2) -a_0(a_1a_2-3a_0) -g_0 \}^2$

Which is

(5a) $g_o^2 + Bg_o+C=0 \implies g_0=(-B\pm \mathrm{SqRt}(B^2-4C))/2$

With

(5b) $B = -2\{[(a_2^2-a_1)(a_1^2-2a_0a_2)+a_0a_2(a_2^2-2a_1)]-(a_1a_2-a_0)(3a_0-a_1a_2) -a_0(a_1a_2-3a_0)\} -(a_1a_2-a_0)^2$

(5c) $C=\{[(a_2^2-a_1)(a_1^2-2a_0a_2)+a_0a_2(a_2^2-2a_1)]-(a_1a_2-a_0)(3a_0-a_1a_2)-a_0(a_1a_2-3a_0)\}^2$

$-(a_1a_2-a_0)^2(-5a_0^2+2a_0a_1a_2)$

This is now in terms of the coefficients of f(x), as in the solution statement. Written in terms of G

(5d,e) $B = -2G -(a_1a_2-3a_0)^2$ and $C = G^2-(a_1a_2-a_0)^2(-5a_0^2+2a_0a_1a_2)$

(5f,g) $G = \{[(a_2^2-a_1)(a_1^2-2a_0a_2)+a_0a_2(a_2^2-2a_1)] -(a_1a_2-a_0)(3a_0-a_1a_2) -a_0(a_1a_2-3a_0)\}$

$=-a_0a_2^3-a_1^3-2a_0^2+3a_0a_1a_2$

There remains the ± ambiguity on the solution. Following the above reasoning h_0 identically satisfies (5a). Since there are two solutions, one must be g_0 and the other h_0. In the case of all real roots these are both positive so the largest of the two goes with the + in (5a) giving the result. Using the definitions directly one finds

(6a) $g_0 - h_0 = a^2(b^4-c^4) + b^2(c^4-a^4) + c^2(a^4-b^4)$

which can have either sign for all real roots even when $|a|<|b|<|c|$. Evaluating the difference using (4b) for g_0 and its h_0 version

(6b,c) $g_0 - h_0 = [a_1a_2-a_0](Y-X) = [a_1a_2-a_0][a^2(b-c) + b^2(c-a) + c^2(a-b)]$

To obtain this totally in terms of the coefficients of f(x) requires more study but can be obtained by using known equations for thr roots..

It is also noted that all of the foregoing applies to complex coefficients and roots.