A PASSIVE TRANSFER FUNCTION SYNTHESIS VALID FOR UNSTABLE SYSTEMS*

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ABSTRACT. — By a simple extension of a recent result of Silverman it is shown how general voltage transfer impulse response matrices can be synthesised by the use of passive circuit elements. The result is based upon state variable theory and allows the synthesis by purely passive components of unstable transfer functions.

I. Introduction.

For some time it has been believed that passive circuits could not exhibit unstable behaviour [1, p. 65]. However we show here in general and by example that there do exist circuits constructed completely from passive components, which, for example, can have poles of their transfer functions in the right half plane.

We begin by giving a general passive synthesis of voltage transfer impulse response matrices $H(t, \tau)$. The synthesis represents a simple, but far reaching and consequentially fascinating, extension of that of Silverman [2]. The method of Silverman essentially obtains a state-variable realization of $H(t, \tau)$; upon transformation of the realization matrices a passive resistor-gyrator network is obtained which upon loading with gyrators yields H. The extensions of Silverman's technique are in three directions. The first, and mathematically most important, allows for unbounded realizations by allowing a general lower limit on the transformation matrix, V of Eq. (4a). The second extension allows for impulsive terms in H trough incorporation of an added term D in the state-variable equations. Finally, we show how to use the ideas for the practically important mases of voltage transfer functions in place of the transfer admittances of Silverman.

Before giving the details of the method, section III, and a specific passive realization of an unbounded transfer function, section IV, we review slightly the general state-variable results needed.

II. State-Variable Review.

Given a dynamical n input m output system, it can often be represented in the state-variable (differential equation) form [3, p. 83].

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) \qquad (1a)$$

$$y(t) = C(t)\, x(t) + D(t)\, u(t) \qquad (1b)$$

where the n-vector u is the input, the m-vector y the output, and the k-vector x the state. By solving for the k.k fundamental matrix $\Phi(t, \tau)$ from the equations [1_k is the k.k identity matrix].

$$\dot{\Phi}(t, \tau) = A(t)\, \Phi(t, \tau),\ t \geqslant \tau,\ \Phi(t, t) = 1_k \qquad (1c)$$

we obtain the impulse response matrix

$$H(t, \tau) = D(t)\, \delta(t - \tau) + C(t)\, \Phi(t, \tau)\, B(\tau)\, 1(t - \tau) \qquad (1d)$$

Here $1(t)$ is the unit step function with $\delta(t)$ its derivative (the impulse) while

$$y(t) = \int_{-\infty}^{\infty} H(t, \tau)\, u(\tau)\, d\tau = H \,.\, u \qquad (1e)$$

Given an impulse response matrix

$$H(t, \tau) = \Lambda(t)\, \delta(t - \tau) + \Theta(t)\, \Psi(\tau)\, 1(t - \tau) \qquad (2a)$$

we can also reverse the process and find the state-variable equations, for a given realization $R = \{A, B, C, D\}$, for example through the identification :

$$A = O_k,\ B = \Psi,\ C = \Theta,\ D = \Lambda \qquad (2b)$$

where O_k is the k.k zero matrix ; k can be chosen as small as possible if it is taken as the minimum number of rows allowed in a specification of Ψ [4].

By introducing a new state $\bar{x}$ related by a non-singular transformation $T(t)$ to the old state x through

$$\bar{x}(t) = T(t)\, x(t) \qquad (3a)$$

we obtain a new set of realization matrices

$$\bar{A} = TAT^{-1} + \dot{T}T^{-1},\ \bar{B} = TB,\ \bar{C} = CT^{-1},\ \bar{D} = D \qquad (3b)$$

for which the impulse response matrix is preserved, that is $\bar{H} = H$.

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Finally we comment that any physical structure is constructed at some finite time t_c, with most t_c being quite recent. Consequently it is not only mathematically useful but practically significant to consider that there is some t_0, $t_0 > t_c$, for which the corresponding network voltages and currents are zero for $t < t_0$; that is $u(t) = 0$ for $t < t_0$ in the above. This convention allows us to use the standard passivity defenition [5]. That is, a network is passive if the total input energy is never negative for any time. We comment also that the assumption of $u(t) = 0$ for $t < t_0$ means that we are really just considering $t > t_0$ in our formulations.

With all these observations in mind we can turn to a development of the synthesis method.

III. General Synthesis Method.

Consider as given an m x n impulse response matrix $H(t, \tau)$ of the form of Eq. (2a) for which the input and output are voltages, $u = v_1$, $y = v_2$. Assume also that on hand is a state-variable realization $R = \{A, B, C, D\}$ for Eqs (1a, b), perhaps obtained through Eq. (2b).

If we let a superscript tilde denote matrix transposition and define for any $t_0 > t_c$

$$V(t) = \int_{t_0}^{t} \Phi(t, \tau)\, \tilde{\Phi}(t, \tau)\, d\tau, \qquad t > t_0 \tag{4a}$$

then V is symmetric and positive definite for $t > t_0$. It possesses [6] a (symmetric) positive definite square root $V^{1/2}$ for which we define the transformation of Eq. (3a) as the inverse

$$T = V^{-1/2} \tag{4b}$$

Direct calculation of $\bar{A}$ using Eq. (3b) with $\dot{V} = 1_k + AV + V\tilde{A} = \dot{V}^{1/2} V^{1/2} + V^{1/2} \dot{V}^{1/2}$ yields, for $t > t_0$,

$$\bar{A} + \tilde{\bar{A}} = -V^{-1} \tag{4c}$$

At this point let us consider the following augmented equations in terms of the transformed state

$$\dot{\bar{x}} = \bar{A}\bar{x} + [\bar{B}, \tilde{\bar{C}}] \begin{bmatrix} u \\ u^* \end{bmatrix} \tag{5a}$$

$$\begin{bmatrix} y^* \\ y \end{bmatrix} = \begin{bmatrix} \tilde{\bar{B}} \\ \bar{C} \end{bmatrix} \bar{x} + \begin{bmatrix} O & -\tilde{D} \\ D & O \end{bmatrix} \begin{bmatrix} u \\ u^* \end{bmatrix} \tag{5b}$$

We observe that Eqs (1a, b) result if $u^* = O$ and if we ignore y^*. Likewise, if we premilinarily let

$$\begin{bmatrix} u \\ u^* \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2^* \end{bmatrix}, \quad \begin{bmatrix} y^* \\ y \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2' \end{bmatrix} \tag{6a}$$

then H for $i_2' = H \cdot v_1$ results by loading the coupling admittance

$$Y_c(t) = \left[\begin{array}{cc|c} O & -\tilde{D} & -\tilde{\bar{B}} \\ D & O & -\bar{C} \\ \hline \bar{B} & \tilde{\bar{C}} & -\bar{A} \end{array}\right] \tag{6b}$$

in k unit capacitors and then shorting (for $u^* = v_2^* = O$) the final ports. By virtue of Eq. (4c) with $\dotplus$ denoting the direct sum, we readily find

$$Y_c + \tilde{Y}_c = [O \dotplus O] \dotplus [\tilde{V}^{-1/2} V^{-1/2}];$$

hence Y_c is easily realized by using time variable (passive) gyrators and k unit (passive) resistors [7]. To recover $y = v_2$ we insert unit cascade gyrators at the last set of ports such that $v_2 = i_2'$ while $u^* = -i_2 = O$ is retained by open circuit loads on these final n ports. The synthesis is then complete and summarized by the circuit diagram of Fig. 1. It is worth obesrving that if the realization of Eq. (2b) is used then $V = (t - t_0\, 1_k$ and

$$Y_c = \begin{bmatrix} O & -\Lambda & \dfrac{-\tilde{\Psi}}{\sqrt{t - t_0}} \\ \Lambda & O & -\sqrt{t - t_0}\,\Theta \\ \dfrac{\Psi}{\sqrt{t - t_0}} & \sqrt{t - t_0}\,\tilde{\Theta} & \dfrac{1}{2(t - t_0)}\, 1_k \end{bmatrix}$$

Consequently the gyrators for A_{sk} are absent and the gyrator loaded in unit resistors can be replaced by the uncoupled resistors of admittance matrix

$$1_k / [2(t - t_0)].$$

We comment that there is nothing in the method which limits it to purely stable systems. By example we next show some of the consequences.

IV. Unstable example.

Let it be desired to synthesize by the above method the scalar impulse response

$$H(t, \tau) = e^{t - \tau}\, 1(t - \tau) \tag{7a}$$

such results from the (Laplace transform) unstable transfer function

$$\frac{V_2}{V_1}(p) = \frac{1}{p-1} \qquad (7b)$$

We can use Eq. (2b) to obtain a realization but it is also clear that a time-independent realization exists, this being given by :

$$A = 1, \ B = 1, \ C = 1, \ D = 0 \qquad (7c)$$

for which :

$$\Phi(t,\tau) = e^{t-\tau},\ V(t) = \int_{t_0}^{t} e^{2t}\, e^{-2\tau}\, d\tau = 1/2[e^{2(t-t_0)} - 1] \qquad (7d)$$

Note that $V(t) > 0$ for $t > t_0$ as expected. Thus

$$T = V^{-1/2} = \frac{\sqrt{2}}{[e^{2(t-t_0)} - 1]^{1/2}} \qquad (7e)$$

Equations (3b) yield

$$\bar{A} = \frac{-1}{e^{2(t-t_0)} - 1} = -1/2\, T^2,\ \bar{B} = T\ \ \bar{C} = T^{-1}$$

$$D = 0 \qquad (7f)$$

The coupling admittance matrix is given by

$$Y_c = \begin{bmatrix} 0 & 0 & -T \\ 0 & 0 & -T^{-1} \\ T & T^{-1} & T^2/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T^2/2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -T \\ 0 & 0 & -T^{-1} \\ T & T^{-1} & 0 \end{bmatrix} \qquad (7g)$$

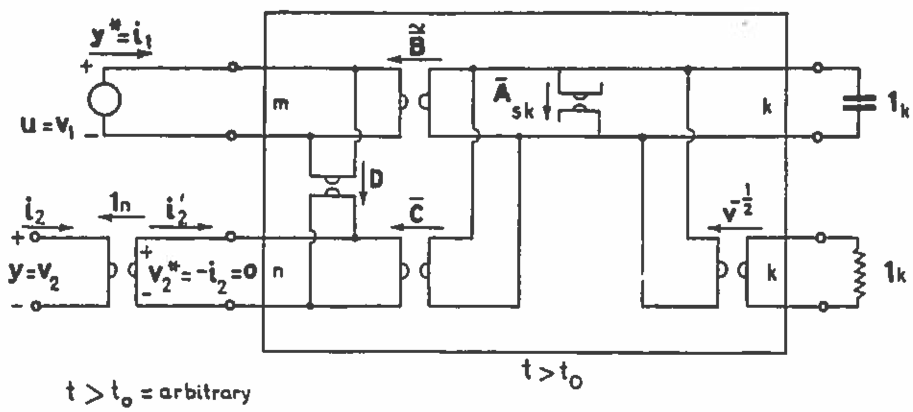

Fig. 1. — Passive Synthesis of Voltage Transfer Impulse Response Matrices.

which is clearly passive. By combining the resistor loaded gyrator of Fig. 1 with its resistor, the circuit of Fig. 2 results. Note how the restriction $t > t_0$ enters into the passivity of the resistor ; note also that any finite t_0 is allowed.

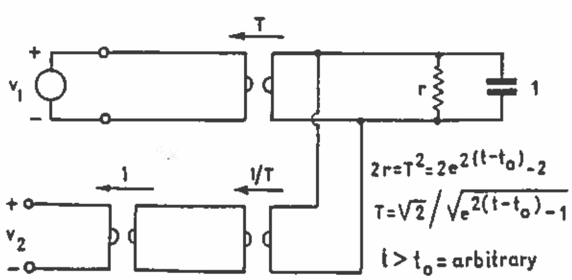

Fig. 2. — Example Synthesis of $\mathcal{L}\,[H(t,0)] = \dfrac{1}{p-1}$.

V. Discussion.

By a set of simple extensions of a beautifully conceived synthesis of Silverman a general synthesis of time-variable networks to yield prescribed voltage transfer characteristics has been given. As shown by the example, the method allows the synthesis of unstable transfer functions by purely passive structures.

However a close examination shows that the synthesis is valid only over a semi-infinite time range $t_0 < t < \infty$. Practically this is no restriction since all physical structures of the type under consideration have been, and will continue to be, constructed at some finite time $t_c < t_0$. Nevertheless there are situations where $t_0 = -\infty$ is allowed in the theory, in which case the $V(t)$ treated at Eq. (4a) reduces to that of Silverman ; in turn all of his results apply. In particular if a structure having bounded components exists then it can be found using $t_0 = -\infty$.

Some further comment on the passivity seems required. To define a network one must give the domain of definition of the port voltages and currents as well as the constraints involved among these. Passivity then requires that the total input energy is never negative for the allowed voltages and currents [8, p. 115]. Here our assumption has been that all voltages and currents are zero before time t_0, this being reasonable on physical grounds with t_0 slightly larger than the construction time (in fact early theories and most textbooks would assume $t_0 = 0$). We could have proceeded differently and considered a " new " concept, that of " passivity for t_0 (time) limited functions ", but this seems an unnecessary complication. However it should be observed that the example circuit can not be used on all the conceivable inputs to which $1/(p-1)$ can be applied, but only on those for which $v_1(t)$ is " t_0 limited ", that is for which $v_1(t) = 0$ when $t < t_0$. Nevertheless t_0 is completely arbitrary so a proper choice will allow use of the circuit for any specific v_1 having support bounded on the left. We also point out that the resistor of Fig. 2 is unbounded at $t = t_0$ and negative for $t < t_0$, showing physically why we have only considered the structure for $t > t_0$ where $r > 0$ and all components are passive. Of course the interconnection of passive networks can only lead to further passive networks, but Fig. 1 shows that all instabilities can be conceived of as being due to unbounded gyration conductances. Because of passivity of the elements the state, that is the set of capacitor voltages, is seen to be stable in the sense of Lyapunov [9] ; this is equally verified by the example for which any initial capacitor voltage (with $v_1 = 0$) decays (for $t > t_0$). A physical reason for the possibility of instabilities in passive structures can perhaps be seen by noting that the application of a dc voltage to an ideal transformer with an exponentially varying turns ratio yields an unbounded response. We comment that the transformation of Eq. (4b) does not in general preserve the Lyapunov stability of the system.

On the practical side we would point out the importance of the general synthesis result to the area of integrated circuits where it allows transfer voltage ratio synthesis because of the ease of integration of all components involved.

It is observed, specifically from the example, that time-variable components are used ; an alternate synthesis based upon similar ideas exists to give synthesis of stable transfer functions using time-invariant elements ; it is hoped to discuss this later.

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