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NEW SYNTHESIS METHOD FOR POSITIVE-REAL MATRIXES*

Using an even-and-odd-part split of the numerator and denominator of a positive-real matrix in one variable, a two-variable lossless matrix is formed. Recent synthesis methods are applied to give a lossless synthesis of the lossless two-variable matrix, with the original one-variable matrix resulting after the replacement of one type of reactive element by resistors.

Several methods exist for the synthesis of passive *RLC* transformer gyrator networks with prescribed impedance matrixes (References 2 and 3 and pp. 240–329 of Reference 1). Here we show how another such synthesis method results from recent developments of synthesis techniques for two-variable lossless matrixes.[4,5,6] Two-variable-reactance synthesis methods have been developed for the synthesis of passive networks with variable elements[7] and networks consisting of certain classes of lumped and distributed elements.[8,9] Their use in *RLC* analysis has been previously hinted at (p. 252 of Reference 10), but synthesis in terms of them, which is no more complicated than by most existing methods, has only recently been explored.[11]

The method can be summarised as follows. Given a rational $n \times n$ positive-real impedance matrix $Z(p)$, a dummy variable s is introduced to form a new matrix $W(p, s)$ such that $W(p, s)$ is a two-variable lossless matrix with the property

$$W(p, 1) = Z(p) \qquad (1)$$

We shall later detail the construction of $W(p, s)$. But once $W(p, s)$ is formed, the next step in the procedure is the synthesis, by known techniques, of the two-variable lossless matrix $W(p,s)$ as the impedance matrix seen at the first $n$ports of an $(n+k)$port lossless $p$plane network, loaded with unit $s$plane inductors at the last $k$ports (p. 32 of Reference 4 and p. 20 of Reference 5), as shown in Fig. 1*a*. k is a suitable

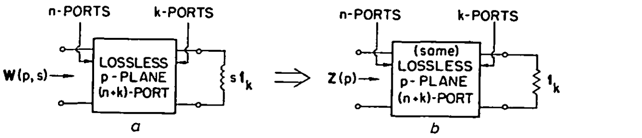

Fig. 1 *Element replacement in two-variable synthesis to yield Z(p)*

number, the smallest (p. 48 of Reference 5) being the $s$degree of $W(p,s)$ (p. 298 of Reference 8). Next, we replace the unit $s$plane inductors by unit $p$plane resistors. This replacement corresponds to the mathematical operation of setting $s = 1$ in $W(p,s)$. The resulting network is shown in Fig. 1*b* and has the impedance $Z(p)$ when observed at its first $n$ports.

Before proceeding, we define, by analogy with the one-variable case, a matrix $W(p, s)$ to be positive real if it satisfies the following three conditions:

(*a*) W is real when p and s are real in $\text{Re}\, p > 0$ and $\text{Re}\, s > 0$
(*b*) W is holomorphic in $\text{Re}\, p > 0$ and $\text{Re}\, s > 0$
(*c*) $W + \tilde{W}^*$ is positive semidefinite in $\text{Re}\, p > 0$ and $\text{Re}\, s > 0$

Here a superscript tilde (~) denotes matrix transposition and a superscript asterisk (*) denotes complex conjugation.

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If we let a subscript asterisk ($_*$) denote replacement of p and s by $-p$ and $-s$, respectively, then a positive-real matrix $W(p,s)$ is called a lossless matrix if it is meromorphic and satisfies

$$W = -\tilde{W}_*$$

We note that these definitions easily, and meaningfully, extend to an arbitrary number of variables, in which case the concepts needed for the definitions are readily available.[12]

The method of converting the rational one-variable positive-real matrix $Z(p)$ into a two-variable lossless matrix with the property required in eqn. 1 is as follows. The given $n \times n$, $Z(p)$, is written in irreducible form as

$$Z(p) = \frac{\psi(p)}{g(p)} = \frac{M(p) + N(p)}{f(p) + h(p)} \qquad (2)$$

where $\psi(p)$ is a polynomial $n \times n$ matrix, $g(p)$ is the least common denominator of the elements of $Z(p)$ and

$$\left.\begin{array}{ll} 2M = \psi + \tilde{\psi}_* & 2N = \psi - \tilde{\psi}_* \\ 2f = g + g_* & 2h = g - g^* \end{array}\right\} \qquad (3)$$

If we now insert another variable s by defining

$$W(p, s) = \frac{sM(p) + N(p)}{f(p) + sh(p)} \qquad (4)$$

then it is directly seen that eqn. 1 is satisfied. By the following reasoning we can show that $W(p, s)$ is a two-variable lossless matrix.

Since $Z(p)$ is a positive-real matrix, we see that, for any arbitrary complex-constant $n$vector X,

$$\text{Re}\{\tilde{X}^* Z(j\omega) X\} = \frac{f(j\omega)\tilde{X}^* M(j\omega) X - h(j\omega)\tilde{X}^* N(j\omega) X}{|f(j\omega) + h(j\omega)|^2} \geqslant 0 \qquad (5)$$

For $p = j\omega$ and $s = r + jq$, we obtain from eqn. 4

$$\text{Re}\{\tilde{X}^* W(j\omega, r + jq) X\} = \frac{r\{f(j\omega)\tilde{X}^* M(j\omega) X - h(j\omega)\tilde{X}^* N(j\omega) X\}}{|f(j\omega) + rh(j\omega) + jqh(j\omega)|^2} \qquad (6)$$

and hence by eqn. 5

$$\text{Re}\{\tilde{X}^* W(j\omega, r + jq) X\} \geqslant 0 \text{ for } \text{Re}\, s = r > 0 \qquad (7)$$

In eqn. 6 it is to be noticed that

$$|f(j\omega) + rh(j\omega) + jqh(j\omega)| = |f + sh|\Big|_{\substack{p = j\omega \\ s = r + jq}} \qquad (8)$$

Since g is a Hurwitz polynomial, $f(p)/h(p)$ is a reactance function, and hence, for $\text{Re}\, p > 0$,

$$\text{Re}\left\{\frac{f(p)}{h(p)}\right\} \geqslant 0 \qquad (9)$$

and thus $\dfrac{f(p)}{h(p)} \neq -s$ for $\text{Re}\, s > 0$ and $\text{Re}\, p > 0$

In other words, $f + sh$ has no zeros in $\text{Re}\, p > 0$ and $\text{Re}\, s > 0$. Now, for each fixed $s = r + jq$ with $r > 0$, the function $\tilde{X}^* W(p, s) X$ is holomorphic in $\text{Re}\, p > 0$ and

$\mathrm{Re}\{\tilde{X}^* W(j\omega, s)X\} \geqslant 0$; hence by applying the minimum-real-value theorem we come to the conclusion that

$$\mathrm{Re}\{\tilde{X}^* W(p,s)X\} \geqslant 0 \quad \text{for } \mathrm{Re}\, p > 0 \text{ and } \mathrm{Re}\, s > 0$$

Hence, by the definition of a two-variable positive-real matrix, $W(p,s)$ is seen to be a positive-real matrix, and by construction $W \equiv -\tilde{W}_*$. $W(p,s)$ is thus seen to be a two-variable lossless matrix such that $W(p,1) = Z(p)$.

Instead of eqn. 4 as the defining equation for $W(p,s)$ we could choose

$$W(p,s) = \frac{M(p) + sN(p)}{sf(p) + h(p)} \quad \ldots \ldots \quad (10)$$

which is also a two-variable reactance matrix with the required property, $W(p1,) = Z(p)$. The proof for this follows exactly the same line of argument as before.

It should also be noted that the same procedure can be used to convert an $n$variable positive-real matrix into an $(n + 1)$variable reactance matrix (treat p as a vector variable $\boldsymbol{p}$). In the $n$variable case the result is of importance, since the procedure allows the synthesis of $n$variable positive-real matrixes once a method of synthesis for $(n + 1)$variable lossless matrixes is established.[11]

In conclusion, a synthesis of rational positive-real matrixes $Z(p)$ results on forming the lossless two-variable matrix $W(p,s)$ of eqn. 4, which can then be synthetised in terms of lossless p and $s$plane elements by recent results, as shown in Fig. 1a. The replacement of $s$plane inductors by $p$plane resistors gives the original $Z(p)$ as shown in Fig. 1b.

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GYRATORS USING OPERATIONAL AMPLIFIERS

An ideal negative-impedance invertor (Lundry type) using a voltage-controlled voltage source is described, and is illustrated by a practical circuit using an operational amplifier. It is used in conjunction with negative-impedance convertors to realise new gyrator circuits.

A negative-impedance invertor (n.i.i.) is an active 2-port which has an input impedance proportional to the negative of the load admittance.[1,2] It has a chain matrix:

$$[a]_{\text{n.i.i.}} = \begin{bmatrix} 0 & \mp R_2 \\ \pm \dfrac{1}{R_1} & 0 \end{bmatrix} \quad \ldots \ldots \ldots \quad (1)$$

On replacing the current-controlled current source by a voltage-controlled voltage source[3] in an n.i.i. circuit given by Lundry,[1] Fig. 1a results. The chain matrix of this circuit is

$$[a]_{\text{n.i.i}} = \begin{bmatrix} \dfrac{1}{G} & \dfrac{R_2(G+1)}{G} \\ -\dfrac{G-1}{GR_1} & \dfrac{R_2}{GR_1} \end{bmatrix} \quad \ldots \ldots \quad (2)$$

When $|G| \gg 1$, eqn. 2 approximates to eqn. 1. Operational amplifiers are, in effect, voltage-controlled voltage sources and can thus be used to realise Fig. 1a, as shown in Fig. 1b.

Theoretical investigation has shown that Fig. 1b has both ports short-circuit stable (s.c.s.) when $G > 0$, and these are open-circuit stable (o.c.s.) when $G < 0$. In Fig. 1b, G is positive, and can be made negative by connecting −ve to node B and +ve to node A, where −ve and +ve denote inverting and noninverting input terminals.

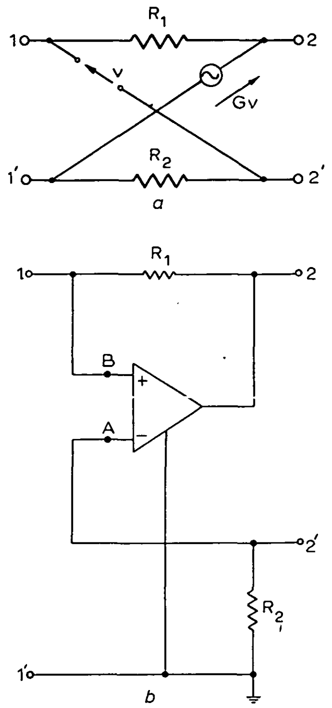

Fig. 1 *Negative-impedance-invertor circuit*
a Theoretical circuit
b Practical circuit

Gyrators: A gyrator[4] is a 2-port having an input impedance proportional to the load admittance, and has a chain matrix:

$$[a]_{\text{gyr}} = \begin{bmatrix} 0 & \pm R_2 \\ \pm \dfrac{1}{R_1} & 0 \end{bmatrix} \quad \ldots \ldots \ldots \quad (3)$$

Several gyrator circuits have been described in the past.[5–15] Most of these[5–9] are realisations of an ideal circuit given by Sharpe,[12] where two voltage-controlled current sources are connected in parallel and back to back. Other realisations[13–15] consist of electronic circuits which exhibit gyrator properties but are, nevertheless, imperfect since a_{11} and a_{22} are large. On using negative resistances, a_{11} and a_{22} are reduced to zero. This Section describes a different type of gyrator using negative-impedance convertors (n.i.c.) and n.i.i.s. Practical circuits, using operational amplifiers, are also described, and these circuits are, to the author's knowledge, novel.*

Eqn. 2 shows that if either R_2 or R_1 is made negative, Fig. 1a becomes a gyrator. In practice, this can be accomplished by using n.i.c.s. A suitable scheme is shown in Fig. 2a. Dotted lines enclose a current-inversion n.i.c. (c.n.i.c.) using an operational amplifier with a differential input.[16] Let G_1 and G_2 (G_1 or $G_2 \gg 1$) be the differential gains of amplifiers A_1 and A_2, respectively (gains of A_1 and A_2 with inverting input terminal earthed).

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