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STANFORD ELECTRONICS LABORATORIES

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ABSTRACT

The theory of Hilbert transforms of distributions is presented as a justification for calculations made by engineers. The use of distributional Fourier transforms in evaluating Hilbert transforms is illustrated.

Hilbert Transforms - Distributional Theory

In the engineering literature the Hilbert transforms are frequently used to obtain properties of physical systems. Examples can be found in passive circuit theory (real versus imaginary parts of driving-point impedances[1]), systems theory (gain versus phase of realizable minimum phase transfer functions[2]), information theory (sampling theorems[3]), and amplifier theory (optimum gain bandwidth restrictions[4]). Because of its use in these various situations, it is important to have the theory available in the most general context of interest. Unfortunately this has not been the case, and many misleading and false statements can be found in the literature. For instance one finds treatments limited to functions analytic in Re p ≥ 0 applied when singularities occur on Re p = 0[5]. Also, theories can be found which are based upon the real part of a simple pole being an impulse, when, as we show here, the real part is not defined[6]; the user of such a theory is warned to apply it only to functions analytic in Re p ≥ 0[7]. In this report we modify the ideas of this latter theory and use the theory of distributions to correctly obtain results valid when singularities are present on Re p = 0.

The theory proceeds by convoluting the real and imaginary parts, evaluated as $0 < \text{Re } p \to 0$, of a function analytic in Re p > 0 with $1/p$, the latter also being evaluated as $0 < \text{Re } p \to 0$.

Let $p = \sigma + j\omega$ be the complex variable with complex conjugation denoted by a superscript asterisk, *. Further let

$$\frac{\partial}{\partial p} = \tfrac{1}{2}\left[\frac{\partial}{\partial \sigma} - j \frac{\partial}{\partial \omega} \right]$$

$$\frac{\partial}{\partial p^*} = \tfrac{1}{2}\left[\frac{\partial}{\partial \sigma} + j \frac{\partial}{\partial \omega} \right]$$

If $f(p)$ is a function of p then the Cauchy-Riemann conditions become $\frac{\partial f}{\partial p^*} = 0$ and in this case $\frac{\partial f}{\partial p}$ is the derivative. Considering $f(p) = 1/p$, as a distribution, we see that it is analytic everywhere except at $p = 0$ since $\frac{\partial(1/p)}{\partial p^*} = \pi\delta(\sigma,\omega)$; in particular $\frac{\partial(1/p)}{\partial p} = \frac{-1}{p^2}$ [8].

We have

(1) $$\frac{1}{p} = \frac{\sigma}{\sigma^2+\omega^2} - j\,\frac{\omega}{\sigma^2+\omega^2}$$

Consequently

(2a) $$\lim_{\substack{\sigma\to 0\\ \sigma>0}} \frac{1}{p} = \pi\,\delta(\omega) - j\,\frac{1}{\omega}$$

(2b) $$\lim_{\substack{\sigma\to 0\\ \sigma<0}} \frac{1}{p} = -\pi\,\delta(\omega) - j\,\frac{1}{\omega}$$

where $1/\omega$ is to be taken in the distributional sense. (2) results from (1) by using the fact that $\delta(\omega) = \lim_{n\to\infty} \frac{n}{(n^2\omega^2+1)}$ for positive n[9]. Note that if $\sigma\to 0$ in (1) through both positive and negative values no distributional limit exists.One can only conclude that the real part at a pole is not defined, even in the distributional sense. In spite of this (2) is valid and can be used to obtain the Hilbert transform.

Consider now any function $N(p)$ which is analytic in Re $p > 0$ and write

(3) $$N(p) = U(\sigma,\omega) + j\,V(\sigma,\omega)$$

We further assume that

(4) $$\lim_{\substack{\sigma\to 0\\ \sigma>0}} N(p) \triangleq U(\omega) + j\,V(\omega)$$

exists in the distributional sense and that (4) can be (distributionally) convoluted with $1/\omega$. Denoting the convolution by a normal level asterisk,*, the convolution of U and then V with (2a) yields

$$(5a)\qquad \int_{-\infty}^{\infty} U(\lambda) \lim_{\substack{\sigma\to 0\\ \sigma>0}} \frac{1}{p-j\lambda}\, d\lambda = \pi U(\omega) - j\, U(\lambda)*(1/\lambda)$$

$$(5b)\qquad \int_{-\infty}^{\infty} V(\lambda) \lim_{\substack{\sigma\to 0\\ \sigma>0}} \frac{1}{p-j\lambda}\, d\lambda = \pi V(\omega) - j\, V(\lambda)*(1/\lambda)$$

We would like to multiply (5b) by j/π and equate with (5a), however, one trouble must be surmounted; we therefore proceed as follows. The left of (5a) can be analytically continued into Re $p > 0$ by

$$(6a)\qquad N_u(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{U(\lambda)}{p-j\lambda}\, d\lambda \quad , \text{ Re } p > 0$$

and thus, in obvious notation

$$(6b)\qquad U_u(\omega) = U(\omega) \;,\quad V_u(\omega) = -\frac{1}{\pi}\, U(\lambda)*(1/\lambda)$$

Now $U_u(\sigma,\omega) - U(\sigma,\omega) = 0$ in the limit as $\sigma \to 0$ and since this difference is continuously differentiable in Re $p>0$, it vanishes identically in Re $p>0$ by a theorem of potential theory[10]. Consequently, since the real parts coincide, an integration of $\frac{\partial N_u}{\partial p^*} = \frac{\partial N}{\partial p^*} = 0$ shows that $V(\sigma,\omega) = V_u(\sigma,\omega) + V_\infty$ where V_∞ is a constant. Proceeding in a similar manner from (5b) gives $U(\sigma,\omega) = U_v(\sigma,\omega) + U_\infty$. At this point we can multiply (5b) by j/π to get

$$(7a)\qquad U(\omega) = U_\infty + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{V(\lambda)}{\omega-\lambda}\, d\lambda$$

$$(7b)\qquad V(\omega) = V_\infty - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{U(\lambda)}{\omega-\lambda}\, d\lambda$$

$$(7c)\qquad N(p) = jV_\infty + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{U(\lambda)}{p-j\lambda}\, d\lambda \quad , \text{ Re } p>0$$

(7d) $$N(p) = U_\infty + \frac{j}{\pi} \int_{-\infty}^{\infty} \frac{V(\lambda)}{p-j\lambda} d\lambda \quad , \quad \text{Re } p > 0$$

Here it should be recalled that the integrals are in the distributional sense and, hence, when continuous functions U and V are involved they become the Cauchy principal value[8]. c and d of (7) show how a function analytic in Re $p > 0$ can be obtained from a knowledge of only one of its conjugate functions on the boundary. a and b of (7) show how the conjugate functions on the boundary can be calculated, one from the other; when $U_\infty = V_\infty = 0$ these are known as Hilbert transforms. The results are valid under very relaxed conditions, but reduce to the familiar results when U and V are infinitely differentiable functions (analytic N(p) on Re p = 0). In particular all that is required is that $U(\omega)$ and $V(\omega)$ be convolutable with $1/\omega$. Although the "best" necessary properties of U and V for this to be the case don't appear to be known, it is sufficient that they be the sum of a constant (as $U_\infty * (1/\omega) \equiv 0$), a distribution of compact support[11], and a function in $L^q(-\infty,\infty)$, $q > 1$[12]. It is, however, necessary that U and V be bounded at infinity.

Beginning with these formulas, many interesting and equivalent forms can be obtained. Since the manipulations in the distributional and functional cases are identical, reference is made to standard textbooks[13]. However, we do point out that if $U(\omega) = U(-\omega)$ then $V(\omega) = -V(-\omega)$ if we assume $V_\infty = 0$. Also, in this case, by direct calculation from (7c)

(7e) $$N(p) = \frac{2}{\pi} \int_0^{\infty} U(\lambda) \left[\frac{p}{p^2+\lambda^2}\right] d\lambda \quad ; \quad \text{Re } p > 0$$

(7f) $$N^*(p) = N(p^*) \quad ; \quad \text{Re } p > 0$$

Conversely (7f) implies $U(\omega) = U(-\omega)$ as an easy calculation shows. (7e) is a form quoted by Bayard whose ideas, although somewhat imprecise, led to this work[14]. It is often implied that (7f) is the only case of interest in engineering practice. However, this is not the case as is seen by the forms $\tilde{x}^{*}Z(p)x$ which must be considered for positive real matrices, x = complex constant vector, tilde = transpose; e.g. the Z for a gyrator.

We now turn to some interesting but simple examples.

E-1 : Let $U(\omega) = 1$ be given with $V(\omega)$ and $N(p)$ desired. As $U(\omega)$ is even, $V(\omega)$ can be chosen odd and (7b) gives $V(\omega) = 0$. Thus $U(\omega) = U_\infty$ and (7d) then gives $N(p) = 1$ which is clearly analytic in Re $p > 0$ as desired. This can also be obtained from (7c) by the use of the residue calculus[15]. This also shows that the constant term of (7a) should not be omitted, as is often the case[16].

E-2 : Let $U(\omega) = \delta(\omega)$ then $V(\omega) = -\frac{1}{\pi}\,\delta(\lambda)*(1/\lambda) = \frac{-1}{\pi\omega}$ since $\delta * f = f$ is valid and defined for any distribution f. Here $V_\infty = 0$ has been assumed. (7c) then gives $N(p) = \frac{1}{\pi p}$. Note that N(p) is analytic in Re $p > 0$ but not on Re $p = 0$. The result is as it should be, since the theory was based upon $1/p$. However, in contrast to statements to the contrary[7], we feel completely safe in applying the theory to this N(p). Recall also that $V(\omega)$ is found by $0 < \sigma \to 0$. We have the acceptable result $N(p) + N(-p) \equiv 0$ from which we can conclude $R(\omega) = \mathrm{Re}\, N(j\omega) \equiv 0 \neq U(\omega)$. This corresponds to approaching the pole along the ω axis in contrast to along the σ axis. To use an $R(\omega)$ found in this manner to obtain $V(\omega)$ we must have $R(\omega) = U(\omega)$, since we must approach the ω axis from the right to apply the theory developed here.

E-3 : As a check on E-2 let $V(\omega) = -\frac{1}{\pi\omega}$ then (7a) gives $U(\omega) = \frac{1}{\pi^2}\,(\frac{1}{\lambda})*(\frac{1}{\lambda})$. Taking Fourier transforms $\mathfrak{F}[\ ]$ gives[17]

$$\mathfrak{F}[U(x)] = -\frac{1}{\pi^2}\,\mathfrak{F}[\tfrac{1}{x}]\,.\,\mathfrak{F}[\tfrac{1}{x}]$$

$$= -\frac{1}{\pi^2}\,(-\pi j\ \mathrm{sgn}\ y)^2$$

$$= 1$$

or, upon inverting, $U(\omega) = \delta(\omega)$. Here $U_\infty = 0$ has been assumed.

E-4 : If $U(\omega) = -\frac{1}{\pi\omega}$ then the reasoning of E-3 gives $V(\omega) = -\delta(\omega)$ and $N(p) = \frac{1}{j\pi p}$. Note that $U(\omega) \neq U(-\omega)$ and $N^*(p) \neq N(p^*)$.

E-5 : Let $N(p) = p = \sigma + j\omega$. Then $U(\omega) = 0$ and $V(\omega) = \omega$. However, the theory can't be applied, since $(\lambda)*(1/\lambda)$ doesn't exist (also note that ω is not in L^q for any $q>1$). Applying (7b) and (7c) to $U(\omega) = 0$ yields $V(\omega) = V_\infty$, $N(p) = jV_\infty$ with V_∞ arbitrary. Similar results are obtained for jp, p^2, etc.

E-6 : Consider a passive, infinite R-C transmission line. We have $N(p) = \frac{1}{+\sqrt{p}}$ as the normalized input impedance. Here the limit as $0<\sigma\to 0$ is $\frac{1}{\sqrt{j\omega}}$, no $\delta(\omega)$ is present as a somewhat extensive calculation shows (convert to magnitude and phase and then take the limit; the essential reason is that $\lim_{n\to\infty}\sqrt{\frac{n}{\pi(n^2t^2+1)}} = 0$). Now, $u(\omega)$ = unit step

$$\frac{1}{\sqrt{j\omega}} = \sqrt{\frac{-j}{\omega}} = \sqrt{\frac{-j\,|\omega|}{\omega^2}}\,[u(\omega) + j\,u(-\omega)] = \frac{1}{\sqrt{2}}(1-j1)|\omega|^{-\frac{1}{2}}\,[u(\omega) + j\,u(-\omega)]$$

where the square roots have been chosen to make the real part positive. Thus

$$\frac{1}{\sqrt{j\omega}} = \frac{|\omega|^{-\frac{1}{2}}}{\sqrt{2}} - j\,\frac{|\omega|^{-\frac{1}{2}}}{\sqrt{2}}\,\text{sgn}\,\omega$$

Using $U(\omega) = |\omega|^{-\frac{1}{2}}/2$ in the Hilbert transform gives this $V(\omega)$ since[17]

$$\begin{aligned}\mathfrak{F}[V(x)] &= -\frac{1}{\pi}\,\mathfrak{F}[U(x)].\,\mathfrak{F}[1/x]\\ &= -\frac{1}{\pi}\,[\tfrac{2}{\sqrt{2}}\,\cos\tfrac{\pi}{4}\,.\,(-\tfrac{1}{2}!)(2\pi|y|\,)^{-3/2}]\,[-j\pi\,\text{sgn}\,y]\\ &= -\frac{1}{\sqrt{2}}\,[-2j\,\sin\tfrac{\pi}{4}\,.\,(-\tfrac{1}{2}!)(2\pi|y|\,)^{-3/2}\,\text{sgn}\,y]\end{aligned}$$

$V(\omega)$ results directly by inversion. Observe that $N(p)$ has a singularity which is not a pole and that this in fact lies on the ω axis.

E-7 : Consider a positive real symmetric matrix $Z(p)$. If $Z(p)$ is analytic at infinity and the only singularities on $\text{Re}\,p = 0$ are poles, then $\lim_{\substack{\sigma\to 0\\ \sigma>0}} \tilde{x}\,Z(p)x$ has its real part as a positive distribution[18], x = real vector. Since $\frac{1}{p+\lambda^2/p}$ has its angle, $\sphericalangle$, between that of -p and +p we see from (7e) and the positive nature of the above Hermitian form[19]

$$|\sphericalangle \tilde{x} Z(p) x| \leq |\sphericalangle p| \quad , \text{Re } p > 0$$

This proof appears to hold if Z has algebraic singularities on Re p = 0. However, only in the case where these are poles or "half poles" can we as yet show that the limit is a distribution. If $Z \neq \tilde{Z}$, (7e) is no longer valid and we must use the results of Desoer and Kuh to obtain a generalization of this result[20].

E-8 : Let $U(\omega) = \frac{1}{\omega^2+1} = \frac{j}{2}\left[\frac{1}{\omega+j} - \frac{1}{\omega-j}\right]$. $V(\omega) = -\frac{1}{\pi} U(\lambda) * (1/\lambda)$

can be obtained by the Fourier transform[21]

$$\mathfrak{F}[V(x)] = -\frac{1}{\pi}\left(\frac{j}{2}\right)[-2\pi j e^{-2\pi y} u(y) - 2\pi j e^{2\pi y} u(-y)] \cdot [-\pi j \text{ sgn } y]$$

$$= \left(-\tfrac{1}{2}\right) \cdot [-2\pi j e^{-2\pi y} u(y) + 2\pi j e^{2\pi y} u(-y)]$$

Inverting gives

$$V(\omega) = -\tfrac{1}{2}\left[\frac{1}{\omega+j} + \frac{1}{\omega-j}\right] = \frac{-\omega}{\omega^2+1}$$

Since U and V are continuous, this gives $N(j\omega) = \frac{1}{\omega^2+1}[1-j\omega]$, which results from $N(p) = \frac{1}{p+1}$. Using the method of this example (i.e. a partial fraction expansion in conjunction with Fourier transforms of distributions) we can find $V(\omega)$ for any rational $U(\omega)$ which is finite at infinity. For this it should be remembered that for $c = c_1 + jc_2$ and m a positive integer[21]

$$\mathfrak{F}\left[\frac{1}{(x+c)^m}\right] = \frac{2\pi j(-2\pi j y)^{m-1}}{(m-1)!} e^{2\pi j c y} u(-c_2 y) \text{ sgn } c_2, c_2 \neq 0$$

$$\mathfrak{F}\left[\frac{1}{(x+c_1)^m}\right] = \frac{-\pi j(-2\pi j)^{m-1}}{(m-1)!} e^{2\pi j c_1 y} \text{ sgn } y$$

Note that the second of these is not the limit of the first as $c_2 \to 0$.

Had we wished the imaginary part $V_-(\omega)$ for a function analytic in Re $p < 0$ we could realize that, by (2b), (7b) can be used with U replaced by -U. In the example at hand, then $V_-(\omega) = +\frac{\omega}{\omega^2+1}$. As a check, this is the imaginary part of $N(p) = \frac{-1}{p-1}$, whose real part agrees with the given $U(\omega)$.

Conclusions

We have shown how previously unjustified calculations of engineers can be fully justified. In particular the Hilbert transforms have been extended to distributions, and, as a consequence, their use when working with impulses, etc., is fully justified. Using this theory very general results can be obtained, as illustrated by the examples. Further, the use of distributional Fourier transforms offers a convenient way of evaluating Hilbert transforms, a fact which has apparently been previously overlooked.

Finally we wish to mention that the results are a special case of a very general, but abstract, theory contained in Schwartz's second volume on Distribution Theory[22].

Appendix

We here show that if

$$f_n(x) = \frac{n}{\pi(n^2x^2+1)}$$

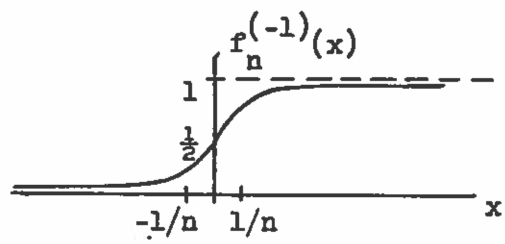
then

$$\lim_{n\to\infty} f_n(x) = \delta(x)$$

We can choose, as the first integral.

$$f_n^{(-1)}(x) = \frac{1}{\pi} \tan^{-1} nx + \frac{1}{\pi}$$

which is graphed as



Clearly $f_n^{(-1)}(x)$ converges to $u(x)$. Applying the definition of distributional limit, this shows that $\lim_{n\to\infty} f_n(x) = \delta(x)$.

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2. S.J. Mason and H.J. Zimmermann, Electronic Circuits, Signals, and Systems, John Wiley & Sons, 1960, p. 360.
3. S. Goldman, Information Theory, Prentice-Hall, 1953, p. 76.
4. Mason and Zimmerman, loc. cit. p. 357.
5. Ibid.,p. 352.
6. E.A. Guillemin, Synthesis of Passive Networks, John Wiley & Sons, 1957, p. 291.
7. Ibid., p. 296.
8. L. Schwartz, Théorie des distributions, Vol. I, Hermann, 1957, p. 49.
9. This is shown in the appendix by applying the theory of J. Mikusiński and R. Sikorski, The elementary theory of distributions (I), Rozprawy Matematyczne XII, Warsaw, 1957. The theory agrees with reference 8. Here n is limited to integer values which presents no problem for (2) by continuity in Re p >0.
10. O.D. Kellogg, Foundations of Potential Theory, Dover, 1953, p. 213.
11. L. Schwartz, Théorie des distributions, Vol. II, Hermann, 1959, p. 11. Recall that the support is essentially the largest set over which the distribution is non-zero.
12. E.C. Tichmarsh, Reciprocal formulae involving series and integrals, Mathematische Zeitschrift, Vol. 25, 1926, pp. 321-347, especially p. 322. Recall that f(x) is in $L^q(-\infty,\infty)$ if it can be integrated and $\int_{-\infty}^{\infty}|f(x)|^q\,dx<\infty$, the integral being in the Lebesque sense.
13. Tuttle, loc. cit., p. 399.
14. M. Bayard, Théorie des réseaux de Kirchhoff, Éditions de la Revue d'Optique, 1954, p. 257.
15. S. Seshu and N. Balabanian, Linear Network Analysis, John Wiley & Sons, 1959, p. 537.

16. Mason and Zimmerman, loc. cit., p. 350. Note that $H(j\omega) = 1$ is realizable since the impulse response is zero for $t < 0$, p. 319.

17. M.J. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge, 1958, p. 43.

18. Schwartz, Vol. I, loc. cit., p. 29.

19. Bayard, loc. cit., p. 259. The result stated here generalizes that in Bayard to matrices.

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21. Lighthill, loc. cit., p. 44.

22. Schwartz, Vol. II, loc. cit., p. 18-20.