If we assume that $\bar{J}_u = \bar{J}_w$, $\bar{F}_x = \bar{F}_v$ and $Q_x = Q_v$, eqn. 11 can be written as

$$Z_{in} = -\frac{\langle \tilde{F}(\bar{J}_u), \bar{J}_u \rangle}{I_u^2} + \frac{\langle F(\bar{J}_u), \bar{F}_v + Q_v \times \tilde{F}(\bar{J}_u), \bar{F}_v + Q_v \rangle}{I_u^2 \langle F(\bar{F}_v + Q_v), \bar{F}_v + Q_v \rangle} \quad . \quad (11)$$

Eqn. 11 is then a variational expression for antenna impedance in a warm magnetoplasma in terms of the induced current, force and fluid-flux distributions along the conductor surfaces, with the assumption that these distributions do not change when $\bar{B}_0$ is reversed. The first term is the one commonly used for computing impedance, while the second is a correction term which accounts for the effect of the induced 'acoustic sources' ($\bar{F}$ and Q).

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FORMULATION OF NETWORK STATE–SPACE EQUATIONS SUITABLE FOR COMPUTER INVESTIGATIONS†

Using a reactive element extraction and the general description of the resulting resistive multiport, the state-variable equations are derived in a manner that is readily implemented on a computer.

The need for state–space equations for a network, or a system, can arise in several situations. For example, the equivalence[1] and synthesis[2,3] of $n$ports, and the stability[4,5] of unexcited networks, can generally be formulated and solved in terms of the state $x(t)$. Likewise, the most promising means of computer analysis of a network seems to be through the state variables.[6] However, although general formulation techniques can be set up in terms of the topological structure of the network,[7] the resulting programs[8] have proven unwieldy. Here we show in a more convenient computational form how to set up, when they exist, the state equations

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$y(t) = H(t)x(t) + J(t)u(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

The theory holds for linear but perhaps time-variable and active networks.

We will consider an $n$port of terminal voltage and current $n$vectors $\boldsymbol{v}(t)$ and $\boldsymbol{i}(t)$. For discussion purposes we will assume that the input $u(t)$ and the output $y(t)$ are incident and reflected $n$vectors:

$$2\boldsymbol{u} = 2\boldsymbol{v}^i = \boldsymbol{v} + \boldsymbol{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

$$2\boldsymbol{y} = 2\boldsymbol{v}^r = \boldsymbol{v} - \boldsymbol{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

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Thus the port description treated in our discussion here is the time-variable scattering (impulse response) matrix $s(t, \tau)$,[9,10] though any other network function can be found by identical techniques. For later reference, we note that

$$\boldsymbol{v} = \boldsymbol{v}^i + \boldsymbol{v}^r, \quad \boldsymbol{i} = \boldsymbol{v}^i - \boldsymbol{v}^r \quad . \quad . \quad . \quad . \quad . \quad . \quad (2c)$$

Consider a finite circuit realisation (Reference 11, p. 11) of a finite $n$port; we will assume the realisation possesses a state-variable description (eqns. 1). For convenience, we apply the equivalence of Fig. 1[12], which guarantees that all dynamical elements are capacitors; all capacitors can then be extracted as a cascade load[13] on a coupling network as shown in Fig. 2.

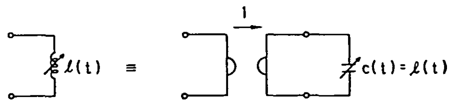

Fig. 1 *Inductor replacement*

The resulting coupling network is completely described by algebraic constraints which may be time-variable, but which we assume can be put into the $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{B}\boldsymbol{I}$ general description.[14] For our purposes, it is most convenient to write this general description in the partitioned form

$$[A_1(t) \vdots A_2(t) \vdots -B_1(t) \vdots -B_2(t)] \begin{bmatrix} \boldsymbol{v}(t) \\ \boldsymbol{v}_c(t) \\ \boldsymbol{i}(t) \\ \boldsymbol{i}_c(t) \end{bmatrix} = \boldsymbol{0} \quad . \quad . \quad (3)$$

Here the A_i and B_i all have $n + c$ rows, where we assume that the capacitors form a $c$port which has a nonsingular capacitance matrix $C(t)$. When it can be applied, a convenient method of interest for machine computation of eqn. 3 results by reducing the indefinite admittance matrix of the $(n + c)$port coupling network, this matrix, and its

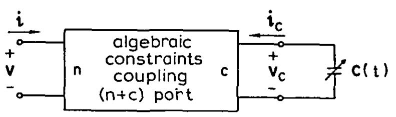

Fig. 2 *Extraction of dynamic elements*

reduction, being easily implemented on a computer. At any rate, a general description can normally be found with comparative ease. In fact, by the arguments of Reference 15, any linear network with a graph is guaranteed to have the description of eqn. 3.

From Fig. 2 we see that

$$\boldsymbol{i}_c = -\frac{dC\boldsymbol{v}_c}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

so that it becomes convenient to choose capacitor charges as pseudostate variables;[5] that is, we define the $c$vector $x_c(t)$ through

$$x_c = C\boldsymbol{v}_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

Substituting eqns. 4 and 2c into 3 gives

$$-B_2\dot{x}_c = [A_1 + B_1]\boldsymbol{v}^r + [A_1 - B_1]\boldsymbol{v}^i + A_2C^{-1}x_c \quad . \quad (5)$$

Eqns. 5 contain all the information of the network, and, by a sequence of transformations to be described, can be brought into the form of eqns. 1.

Eqns. 5 can be considered as an alternate form of eqn. 3, for which the coefficient matrix can be triangularised by means of elementary row operations (Reference 16, p. 35). Performing this triangularisation yields

$$\begin{matrix} \delta\{ \\ n\{ \\ 0\{ \\ c-\delta\{ \end{matrix} \underbrace{\begin{bmatrix} \alpha_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{n} \dot{x}_c = \underbrace{\begin{bmatrix} \alpha_r \\ -\beta_r \\ 0 \\ 0 \end{bmatrix}}_{n} \boldsymbol{v}^r + \underbrace{\begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ 0 \end{bmatrix}}_{n} \boldsymbol{v}^i + \underbrace{\begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{bmatrix}}_{c} x_c \quad . \quad . \quad . \quad . \quad (6)$$

The sizes of the submatrixes are indicated in the margin. Thus, δ is the rank of B_2, $\delta \leqslant c$. Since eqn. 1b exists, β_r is an $n \times n$ nonsingular matrix; since eqns. 1 show no direct algebraic relationship between $u = v^i$ and x, $\gamma_i = 0$. However, there are $c - \delta$ constraints among the x_c, which allows the pseudostate vector x_c to be reduced to the state vector x; these are expressed in terms of δ_2. In fact, the state x is a δ vector where the number of state variables can be calculated beforehand from the circuit structure by the method of Bryant.[17,4]

Eqns. 6 can now be reduced to eqns. 1 by purely algebraic means. For convenience of exposition, let us assume that x_c is a state vector; that is, with $\delta = c$,

$$x_c = x \quad \ldots \ldots \quad (7)$$

Then the second n equations of eqn. 6 yield

$$v^r = \beta_r^{-1}\beta_2 x + \beta_r^{-1}\beta_i v^i \quad \ldots \quad (8b)$$

Substitution of eqn. 8b into the first $\delta = c$ equations of eqn. 6 yields

$$\dot{x} = \alpha_1^{-1}[\alpha_2 + \alpha_r\beta_r^{-1}\beta_2]x + \alpha_1^{-1}[\alpha_i + \alpha_r\beta_r^{-1}\beta_i]v^i \quad (8a)$$

Eqns. 8 are the state-variable equations (eqns. 1), when the choices of input and output of eqn. 2 are made; eqns. 8 are thus the ones desired.

If $\delta \neq c$, the last $c - \delta$ equations of eqns. 6 allow the last δ components of x_c to be chosen as x, with the remaining components expressed in terms of x, allowing their elimination in a straightforward manner.

In summary, we have described an algebraic technique of setting up the state-variable equations for a network, when they exist. Of course, alternate methods of setting up state-variable equations are known,[4,5,18] but none seem as convenient for computer calculations, nor as general, as that discussed here. The method used rests upon pulling out dynamical elements to obtain a structure described only by algebraic constraints. However, it should be pointed out that inductors can be extracted without the use of the equivalence of Fig. 1, since eqn. 4a holds for all dynamical elements with a proper interpretation of the variables; Fig. 1 was used for convenience of exposition. The method has relied on the existence of the state equations; that is we assume that eqns. 1 were known to exist beforehand. One advantage in considering the scattering matrix in a general discussion is that, for finite passive networks, we are guaranteed there are no derivatives of the input present,[19] such that no added terms for eqn. 1b are ever needed, in contrast to the case when immittances are considered. Further, in the useful time-invariant situation, a finite passive network will always possess a scattering matrix (Reference 11, p. 153), and then eqns. 1 will always exist. Of course, there are many active networks which are also described by eqns. 1. In general, given the network function, as the scattering matrix $s(t, \tau)$, the conditions for the existence of eqn. 1 follow simply from the work of Youla;[20] however we do not believe that the conditions for existence of eqns. 1 are known if the circuit realisation (diagram) is given.

The treatment given appears to limit considerations to an entire matrix. However, any desired submatrixes can be considered (e.g. voltage transfer functions) by simply ignoring all but the output of interest, and by setting all but the input of interest to zero. Further, internal properties of a network can be studied by appropriately attaching ports at internal points of interest.

In the case where $u = v$, $y = i$ and $\delta = c$, and the coupling network possesses an admittance matrix, the theory results in a very simple technique, the implementation of which will be described in a future work.

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FIELD REPRESENTATION IN CYLINDRICAL CO-ORDINATES

Formulas are presented which give the electrostatic field in a half-space when the radial field is prescribed on the plane $z = 0$ to be zero for $r > 1$ and given for $r < 1$. This corresponds to a prescribed field in a circular aperture in a metal plate, and is applied to the case of the fringe field of a flanged cylindrical tube.

The calculation of an electromagnetic field in an extended region, given the field in part of the region, say on a bounding surface, is one of the basic problems in e.m. theory. Many particular solutions, some of them based on applications of Green's theorem, are known.

In cylindrical co-ordinates some particular solutions are known which involve integrations over imaginary points in space. Thus,[1] if $f(z)$ is the potential along the axis of co-ordinates, and ρ is the radial co-ordinate, the potential V in space is

$$V = \frac{1}{\pi}\int_0^\pi f(z + j\rho\cos\Phi)d\Phi \quad \ldots \ldots \quad (1)$$

It is readily shown that this is a solution of Laplace's equation and reduces to $f(z)$ at $\rho = 0$.

The present letter is concerned with a solution of Laplace's equation in the half-space $z < 0$, in which the radial electric field is zero at $z = 0$, except for a prescribed field in the region $0 < r < 1$. The problem arises in the solution of the fringe field around the aperture of a flanged cylindrical tube. A somewhat more general geometry, which yields the same method, concerns the field around the aperture of a flanged cone, as shown in Fig. 1.

If $f(\Phi)$ is any function of Φ, symmetrical around $\Phi = \pi/2$,