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STATE-VARIABLE SYNTHESIS FOR INSENSITIVE INTEGRATED CIRCUIT TRANSFER FUNCTIONS

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Abstract

Using state-variable flow graphs and simple operational configurations suitable for integration a theory for insensitive transfer function realization in terms of integrated circuits is discussed. Second order systems are developed with special reference to their sensitivity which is shown to be very low for high gains.

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I. INTRODUCTION

For linear integrated circuits the availability of operational amplifiers seems well-recognized [1] [2] [3] [4]. But, though their use in obtaining low sensitivity designs has been advantageously considered [5], and RC operational amplifier synthesis methods exist [6] [7], the availability of simple but highly theoretical state-variable results for practical design seems to have been overlooked until recently [8]. Here we apply, with slight modifications, some standard systems theory techniques [9, p. 177] to obtain insensitive designs for voltage transfer functions. The resulting structures use a minimum number of grounded capacitors and relatively few resistors and transistors, all of which are in an easily integrated form.

II. Flow-Graph

We begin the theory by considering as given a transfer function of the form

$$T(p) = \frac{a_{n+1}p^n + a_n p^{n-1} + \ldots + a_1}{p^n + b_n p^{n-1} + \ldots + b_1} \qquad (1)$$

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That is, $T(p)$ is the transfer function of a single input, $u(t)$, single output, $y(t)$, differential system with no pole at infinity; $T(p) = \mathcal{L}[y]/\mathcal{L}[u]$ with $\mathcal{L}[\ ]$ the Laplace transform and all a_i and b_i real constants. The introduction of state variables x_i leads to the signal flow graph of Fig. 1a), which is a standard one in the theory of systems [9, p. 177]. Figure 1a) uses positive gain integrators, but negative gain integrators can be introduced by appropriate changes in sign, as shown in Fig. 1b). For generality we have inserted the constant a_{n+1} at infinity. The transfer function for the graph can easily be checked by applying Mason's rule [10, p. 104]. Each element of the graph can be realized by an integrator or a summer (with appropriate scalings) both of which can be obtained in integrated form using operational amplifiers, resistors, and capacitors. In actual fact the unit transmittances of Fig. 1 can be omitted in any physical realization since they are only present in the graph for purposes of illustration. Likewise nonunity coefficients for physical integrators can be incorporated by scaling outgoing branches.

III. Operational Amplifier

A very convenient and economical differential input operational amplifier is as shown in Fig. 2, where, if we subscript voltages according to their point of measurement (with respect to ground)

$$e_C = K(e_B - e_A) \tag{2}$$

The amplifier of Fig. 2 can be conveniently considered as the cascade of a differential input section and a grounded emitter stage of current gain β. Typical values for the circuit are $R_a = R_b = R_c = 10K\Omega$, $\beta = 60$, $2V_o = 15v$, and (measured) $K \approx 1{,}500$.

The gain K of the amplifier can be roughly determined as follows. First let point A be grounded, in which case the emitter-base resistance [11, p. 82] $r_{\epsilon_1} \approx .026/I_{e_1}$ of T_1 is effectively in parallel with R_b. The emitter current I_{e_2} of T_2 is much smaller than I_{e_1} of T_1 since the former is determined by the emitter-base drop of T_3;

$I_{e_2} \approx .5/R_a$, $I_{e_1} \approx V_o/R_b$. Thus the emitter-base resistance of T_2, $r_{\epsilon_2} \approx .026/I_{e_2}$, determines the signal current in the collector of T_2 since r_{ϵ_2} is larger than R_b in parallel with r_{ϵ_1} (with which it is in series). Likewise the effective signal resistance in the collector of T_2 is R_a in parallel with the equivalent input base-emitter resistance βr_{ϵ_3} of T_3, where β is the beta of T_3 and $r_{\epsilon_3} \approx .026/I_{e_3}$ [where $I_{e_3} \approx V_o/R_c$]; this parallel combination is about βr_{ϵ_3}. Thus the voltage gain of the first (differential input) stage is about $-\beta r_{\epsilon_3}/r_{\epsilon_2}$. To obtain the gain of the output stage we note that the emitter signal current is determined by dividing the base emitter voltage by r_{ϵ_3}, and, hence, the gain is $-R_c/r_{\epsilon_3}$. Combining, we have

$$K \approx \beta R_c/r_{\epsilon_2} = \frac{\beta}{.052}\,\frac{R_c}{R_a} \qquad \text{(3a)}$$

Grounding point B in place of A yields the same K within about a .9 multiplier. Note, however, that if a feedback resistor R_f is applied between points A and C, as for the adders and integrators described below, then R_c must be replaced in (3) by the parallel combination of R_c and the effective R_f.

A calculation using transistor parameters in more detail gives

$$K \approx \frac{\beta}{r_{\epsilon_1}+\frac{1}{2}r_{\epsilon_2}} \cdot \frac{R_a R_c}{R_a+\beta r_{\epsilon_3}} \qquad \text{(3b)}$$

Equation (3b) agrees with Eq. (3a) when $R_a > \beta r_{\epsilon_3}$ and $r_{\epsilon_1} \approx \frac{1}{2}r_{\epsilon_2}$.

IV. INTEGRATORS AND SUMMERS

The operational amplifier of Fig. 2 can then be used to form the integrators, summers, and gain blocks needed for the signal flow graph of Fig. 1. For an integrator one can use the somewhat standard [12, p. 229] circuit of Fig. 3a) which, for large K is described by

$$e_i \approx -RCpe_o\ , \quad p = d/dt \qquad \text{(4a)}$$

Because of the negative sign, this integrator is most convenient for implementing the flow graph of Fig. 1b). However, for integrated circuits it is often more convenient to have capacitors grounded, in which case Fig. 3b) can be used [13]. For large K Fig. 3b) is described by

$$e_i \approx \frac{R_f C}{2} p e_o \tag{4b}$$

in which case the flow graph of Fig. 1a) is the most appropriate.

The summer is shown in Fig. 4. If we consider admittances $G_{i\pm} = 1/R_{i\pm}$ then we find, using mathematical induction on straightforward but detailed calculations,

$$e_o \approx \sum_{i=1}^{m+} \frac{G_{i+}}{G_+}[1+R_f G_-] e_{i+} - \sum_{i=1}^{m-} R_f G_{i-} e_{i-} \tag{4c}$$

where G_+ and G_- are the parallel combinations of the conductances at the positive and negative inputs

$$G_+ = \sum_{i=1}^{m+} G_{i+} , \quad G_- = \sum_{i=1}^{m-} G_{i-} \tag{4d}$$

Using these integrators and summers the flow graph of Fig. 1 is easily implemented, and in fact uses at most n capacitors, and 3(n+2) transistors (from n+2 operational amplifiers). Depending upon whether Fig. 3a) or 3b) is used either [n+3(n+2)+2(n+2)] = 6n+10 or [n(3+1)+3(n+2)+2(n+2)] = 9n+10 resistors are used. These numbers are for a general nth order system. However, many practical functions of degree n do not require the complete system (that is, some coefficients may be zero). For example, for a low pass (all pole) function no output summer is necessary, similarly for a high pass function (all zeros at zero).

V. SECOND ORDER REALIZATIONS

Since any polynomial with real coefficients can be factored into first or second degree terms which also have real coefficients, T(p) can be realized as a cascade of first or second order systems each of which can be realized as described above. In actual fact first order sections can be realized to within a gain constant by standard passive RC lead-lag networks as shown in Fig. 5 [14, p. 345]. The lead network of Fig. 5a) is described by

$$T(p) = \frac{R_1R_2Cp+R_2}{R_1R_2Cp+R_1+R_2} \tag{5a}$$

and the lag networks of Fig. 5b) has

$$T(p) = \frac{R_2Cp+1}{(R_1+R_2)Cp+1} \tag{5b}$$

Note however that only the lag structure has a grounded capacitor.

As a consequence we need only concern ourselves here with the realization of the general second order transfer function. Using, for simplicity of drawing, the single resistor integrator of Fig. 3a), the circuit configuration for this is shown in Fig. 6. The resulting transfer functions of interest are

$$\frac{\mathcal{L}[e_1]}{\mathcal{L}[e_i]} = T_1(p) = \frac{R_2(R+R_3)R_5(R_6+R_7)}{(R_1+R_2)R_3(R_4+R_5)R_7}\left[\frac{R_8C_1R_9C_2p^2+\frac{(R_4+R_5)R_6}{R_5(R_6+R_7)}R_9C_2p+\frac{R_4}{R_5}}{R_8C_1R_9C_2p^2+\frac{R_1(R+R_3)}{(R_1+R_2)R_3}R_9C_2p+\frac{R}{R_3}}\right] \tag{6a}$$

$$\frac{\mathcal{L}[e_2]}{\mathcal{L}[e_i]} = T_2(p) = -\frac{R_2(R+R_3)}{(R_1+R_2)R_3}\left[\frac{R_9C_2p}{R_8C_1R_9C_2p^2+\frac{R_1(R+R_3)}{(R_1+R_2)R_3}R_9C_2p+\frac{R}{R_3}}\right] \tag{6b}$$

and

$$\frac{\mathcal{L}[e_3]}{\mathcal{L}[e_i]} = T_3(p) = \frac{R_2(R+R_3)}{(R_1+R_2)R_3}\left[\frac{1}{R_8C_1R_9C_2p^2+\frac{R_1(R+R_3)}{(R_1+R_2)R_3}R_9C_2p+\frac{R}{R_3}}\right] \tag{6c}$$

Note that though T_3 results from T_1 by letting R_4 and R_7 become infinite, T_2 is not a special case of T_1. Using T_1, T_2, and T_3 any second order pole-zero pattern can be obtained, except that right half plane zeros require sign reversal of the T_1 numerator p coefficient by connecting R_7 to the junction of R_4 and R_5 and replacing the output summer with an emitter follower. It is also of interest to note that at the resonant frequency

$$\omega_o = \sqrt{\frac{R}{R_8C_1R_9C_2R_3}} \tag{7}$$

for which $R_8C_1R_9C_2p^2 + R/R_3 = 0$, we have

$$T_2(j\omega_o) = -R_2/R_1 \tag{6d}$$

If we also normalize by letting $R_1 = R_3 = R_5 = R_6 = R_8C_1 = R_9C_2 = 1$ then we obtain the simple and useful forms

$$T_1(p) = \frac{R_2(1+R)(1+R_7)}{(1+R_2)(1+R_4)R_7}\left[\frac{p^2 + \frac{1+R_4}{1+R_7}p + R_4}{p^2 + \frac{1+R}{1+R_2}p + R}\right] \tag{6e}$$

$$T_2(p) = -\frac{R_2(1+R)}{(1+R_2)}\left[\frac{p}{p^2 + \frac{1+R}{1+R_2}p + R}\right] \tag{6f}$$

$$T_3(p) = \frac{R_2(1+R)}{(1+R_2)}\left[\frac{1}{p^2 + \frac{1+R}{1+R_2}p + R}\right] \tag{6g}$$

with $\omega_o = \sqrt{R}$ and $T_2(j\omega_o) = -R_2$. These expressions for T_1 and T_2 show which parameters are important for obtaining a desired pole-zero pattern.

VI. SENSITIVITY OF SECOND ORDER REALIZATIONS

At this point we will determine how insensitive the above mentioned realizations are to parameter changes. Since changes in the magnitude of transfer functions are of primary importance we define the system sensitivity to changes in the value of any parameter x as [15]

$$S_x^{|T(p)|} = \frac{x}{T(p)} \frac{\partial |T(p)|}{\partial x} \tag{8}$$

Since the sensitivity is a (real) function of frequency it will be investigated at the frequency of most interest, ω_o, the resonant frequency. For most high Q cases ω_o is the frequency of maximum signal output and hence variations at ω_o are usually maximum.

We obtain after lengthy but straightforward calculations

$$S_R^{|T_1(j\omega_o)|} = S_{R_3}^{|T_1(j\omega_o)|} = 0 \tag{9a}$$

$$-S_{R_1}^{|T_1(j\omega_o)|} = S_{R_2}^{|T_1(j\omega_o|} = 1 \tag{9b}$$

In addition to the very desirable lack of sensitivity to changes in R or R_3 the total effect of R_1 and R_2 approximate zero if, as would be the usual case, they have similar temperature coefficients since their effects are equal and opposite. This cancellation is particularly attractive since the individual sensitivities themselves are not large.

As can be seen from Eqs. (6a,b,c) the pole positions are independent of R_4, R_5, R_6, and R_7, and we therefore have zero pole position sensitivity to each of these parameters. The system sensitivity is, of course, not zero since the position of the zeroes is affected by these parameters in the completely general case; however, these are usually minor effects and depend in detail on the specific function synthesized so will not be discussed further here. For the case of the low pass, all pole second order function, Eq. (6c) [for which the output summer, K_o, can be eliminated], we then have zero system sensitivity to R_4, R_5, R_6, and R_7 since they are no longer used. For the high pass case we let $R_4 \to 0$ and $R_6 \to 0$ in Eq. (6a). This is equivalent to taking the output from e_4 and eliminating the output summer, K_o. Again we have zero sensitivity to R_4, R_5, R_6, R_7. If we restrict our consideration of the effects of R_8, R_9, C_1, and C_2 to their effect on the pole positions, for the same reasons as above, we find that the sensitivity of the magnitude of the denominator to each of these parameters is:

$$S_{R_8}^{|D(j\omega_o)|} = S_{C_1}^{|D(j\omega_o)|} = 0 \qquad (10a)$$

$$S_{R_9}^{|D(j\omega_o)|} = S_{C_2}^{|D(j\omega_o)|} = 1 \qquad (10b)$$

Investigating the sensitivity of the resonant frequency of the system to variations in R_8, R_9, C_1 and C_2 we find, from $S_x^{\omega} = \frac{x}{\omega}\frac{\partial\omega}{\partial x}$,

$$S_{R_8}^{\omega_o} = S_{R_9}^{\omega_o} = S_{C_1}^{\omega_o} = S_{C_2}^{\omega_o} = -\frac{1}{2} \qquad (11)$$

In summary, it can be seen that the sensitivity to the passive components is always equal to or less than 1, and most importantly that these sensitivities are <u>independent</u> of the system Q.

If we now consider the sensitivity of the system to amplifier gain change, by using the exact expression for the integrators (Fig. 3a)

$$T(p) = \frac{-K}{1+(1+K)RCp} \qquad (12)$$

we find that, for large K_1 and K_2, the denominators for Eqs. (6) are given to a high degree of approximation by

$$D(p) = R_8C_1R_9C_2p^2 + \left[\frac{1}{K_1} + \frac{1}{K_2}\left(\frac{R_8C_1}{R_9C_2}\right) + \frac{R_1(R+R_3)}{R_3(R_1+R_2)}\right] R_9C_2p + \frac{R}{R_3} \qquad (13a)$$

Identifying coefficients through

$$D = \alpha p^2 + \beta p + \gamma \qquad (13b)$$

we find

$$S_{K_1}^{|D(j\omega_o|} = -\frac{QR_9C_2}{K_1\sqrt{\alpha\gamma}}, \quad Q = \frac{\sqrt{\alpha\gamma}}{\beta} \qquad (14a)$$

where Q is the quality factor for $T = 1/D$. Since $\sqrt{\alpha\gamma} = \sqrt{R_8C_1R_9C_2R/R_3}$ we see that $R_9C_2 \approx \sqrt{\alpha\gamma}$ in the usual cases where $R_8C_1 = R_9C_2$ and $R \approx R_3$, thus

$$S_{K_1}^{|D(j\omega_o)|} \approx -\frac{Q}{K_1} \tag{14b}$$

For the all pole second order system we thus obtain

$$S_{K_1}^{|T_3(j\omega_o)|} \approx +\frac{Q}{K_1} \tag{14c}$$

As there is no difference in D between K_1 and K_2 we immediately also have; when $R_8C_1 = R_9C_2$,

$$S_{K_2}^{|T_3(j\omega_o)|} \approx \frac{Q}{K_2} \approx -S_{K_2}^{|D(j\omega_o)|} \tag{14d}$$

If we consider a typical amplifier gain of $K_1 \approx K_2 \approx 1500$ used to realize a $Q = 500$ system we obtain

$$S_{K_1}^{|T_3(j\omega_o)|} \approx \frac{1}{3} \approx S_{K_2}^{|T_3(j\omega_o)|} \tag{15}$$

This can be compared to a controlled source, [16] [17] or NIC realization which would have a sensitivity to gain change or NIC conversion factor of 1000 or more!

In addition to the extreme improvement in sensitivity, this method of RC active synthesis has absolute stability, since neither a change in the passive elements nor in the amplifier gains can cause the poles to move into the right half plane. It should also be noted that in the case of the very useful bandpass second order function, Eq. (6b) including the effect of K_1 and K_2, we find that the center frequency voltage ratio is still $T_2(j\omega_o) = -R_2/R_1$ and thus we see that the amplifier gains have no effect. In the cases of second order functions which are a) low pass (with all zeros at infinity), b) high pass (with all zeros at zero), or c) band pass (with one zero at zero and one at infinity) only three amplifiers are required and the outputs are e_3, e_4, and e_2 (of Fig. 6) respectively.

VII. CONCLUSIONS

Using state-space systems theory results, Fig. 1, we have shown how any degree n transfer function can be realized by integrated circuit structures. For this a simple differential amplifier and corresponding summers and integrators, which use only grounded capacitors, have been introduced, Fig. 2-4.

Because an important consideration in the practical implementation of the method is that the pole positions become increasingly sensitive to the polynomial coefficients as the degree of the polynomial increase, the factorization into degree one and two sections was considered. It was shown that these sections are relatively insensitive to parameter changes. Degree one or two sections can be realized as special cases of Fig. 1 or by the almost identical structures of Figs. 5 and 6. As seen in Section VI, the sensitivity of structures obtained by these techniques is extremely low. In particular it has been shown that the general second order transfer function can be obtained with a minimal number of capacitors in a low sensitivity RC active configuration with nearly unlimited Q capability without increasing the sensitivity to the passive components and only appreciably increasing the sensitivity to the active elements at Q's greater than 1000. If we consider D of Eq. (13a) we see that the maximum Q is given, when $R_1 = 0$, by

$$Q_{max} = \frac{K_1 K_2}{K_1 + K_2} \tag{17}$$

Thus, $Q_{max} = 750$ in a typical case with amplifier gains of 1,500. Preliminary experimental results verify Eq. (6c) with the denominator of Eq. (13a).

The effect on the performance of this integrator due to the finite input and output impedances of the real amplifier is equivalent to the use of an ideal $\begin{pmatrix} R_{in} = \infty \\ R_{out} = 0 \end{pmatrix}$ voltage amplifier having lower gain. The effect is at most a factor of 2 and can be compensated for if necessary by using an amplifier of higher open loop gain.

VIII. ACKNOWLEDGMENT

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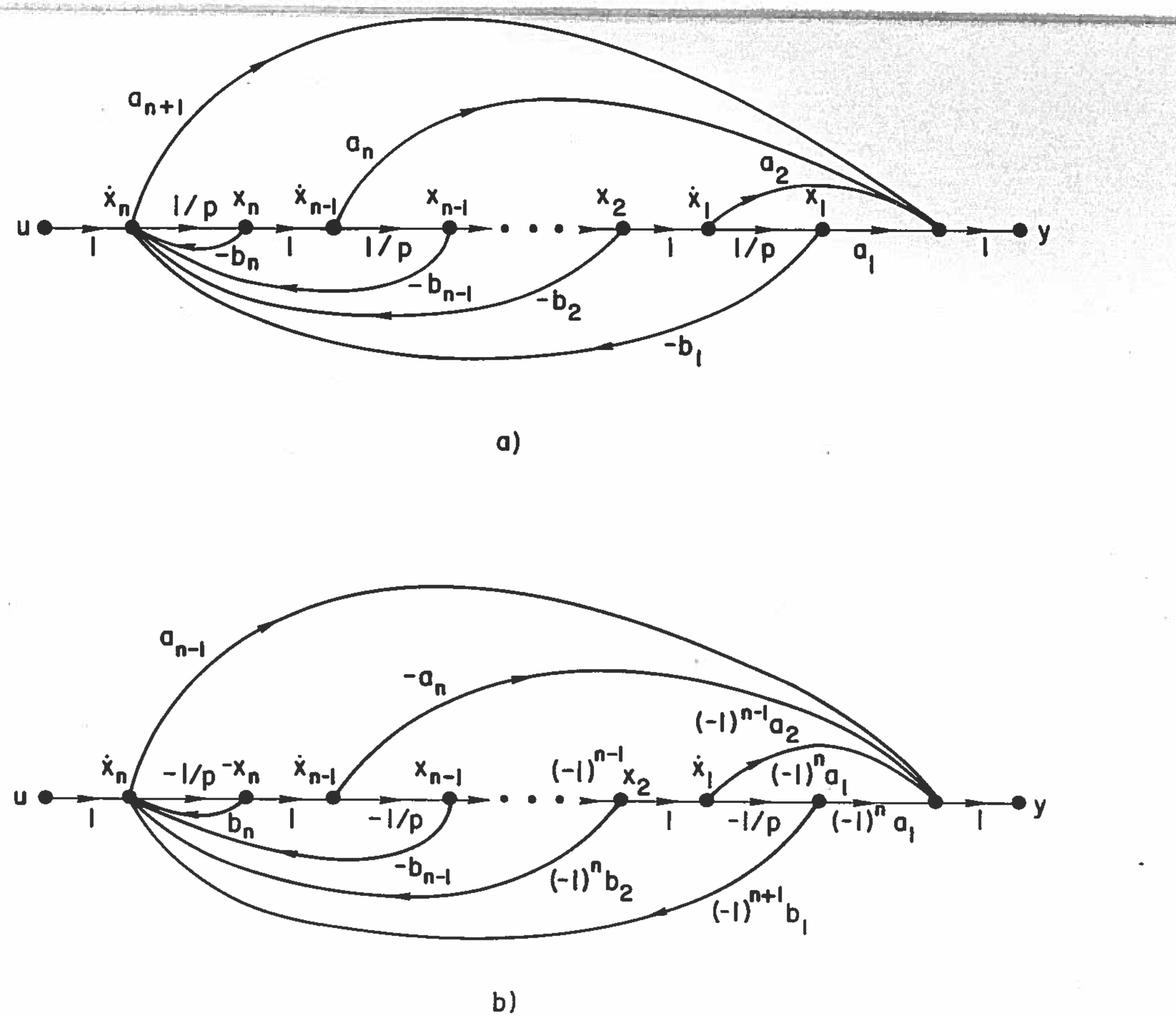

Figure 1. Signal Flow Graph Realizing a General Transfer Function with a) Positive and b) Negative Gain Integrators

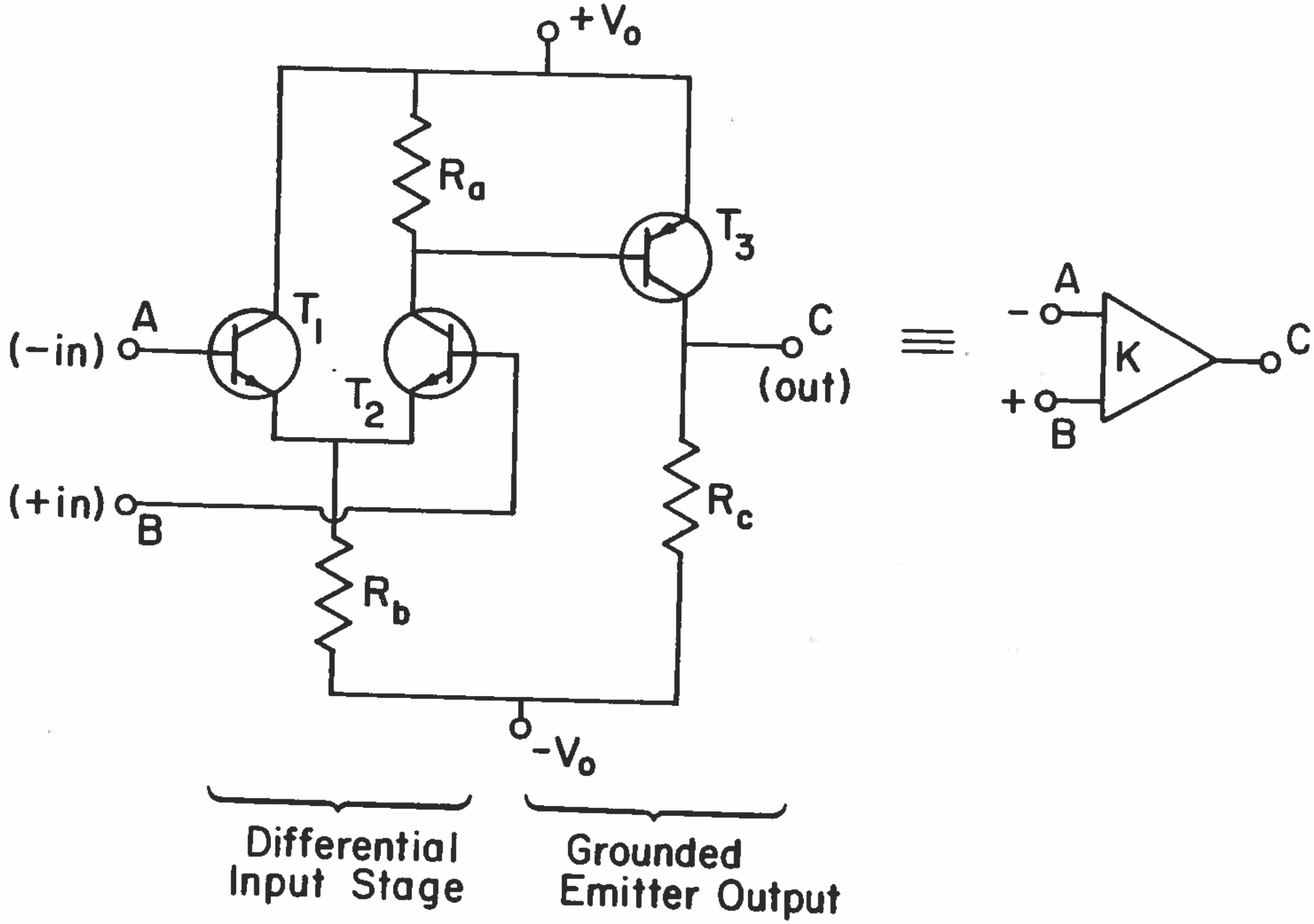

Figure 2. The Basic Operational Amplifier.

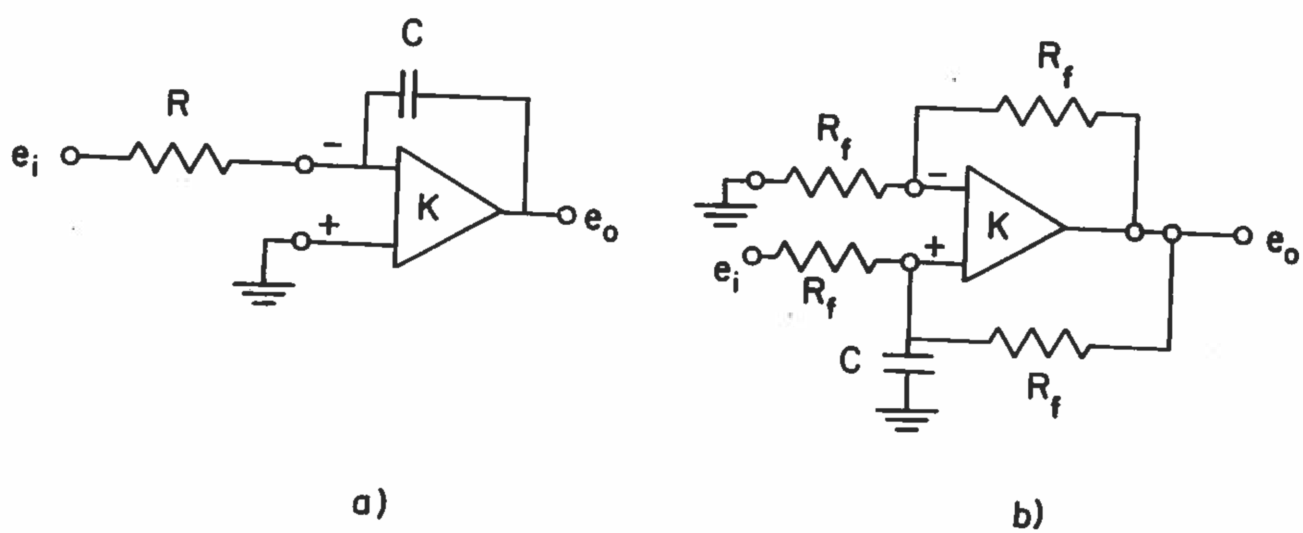

Figure 3. Negative a) and Positive b) Gain Integrators.

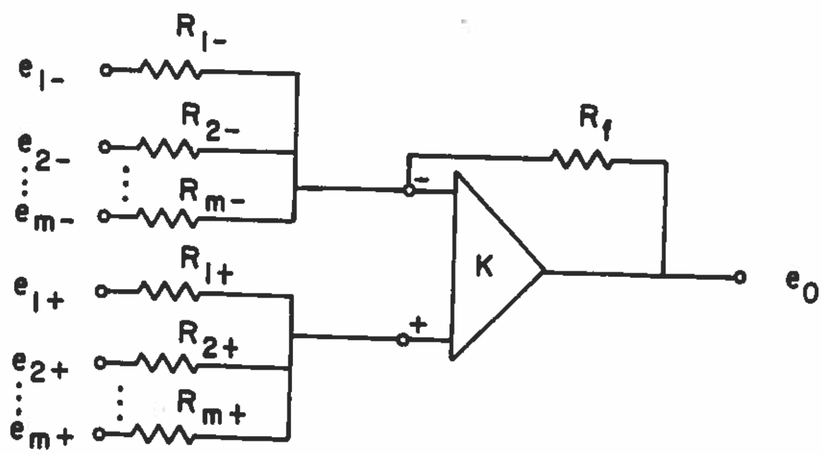

Figure 4. Summer.

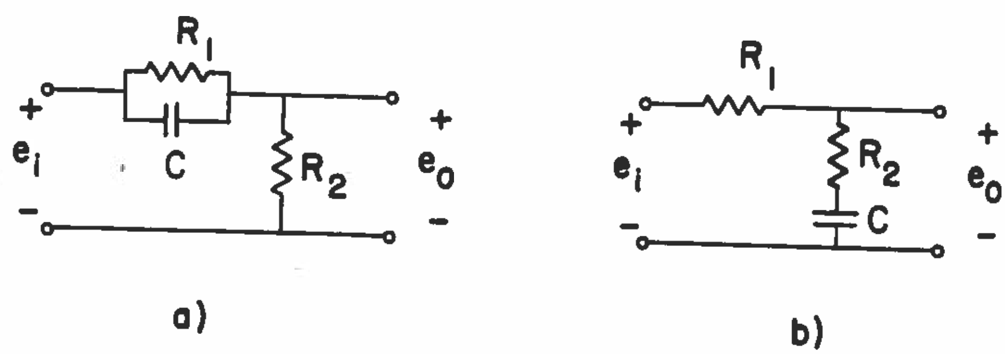

Figure 5. Passive First Order Systems,
a) Lead and b) Lag.

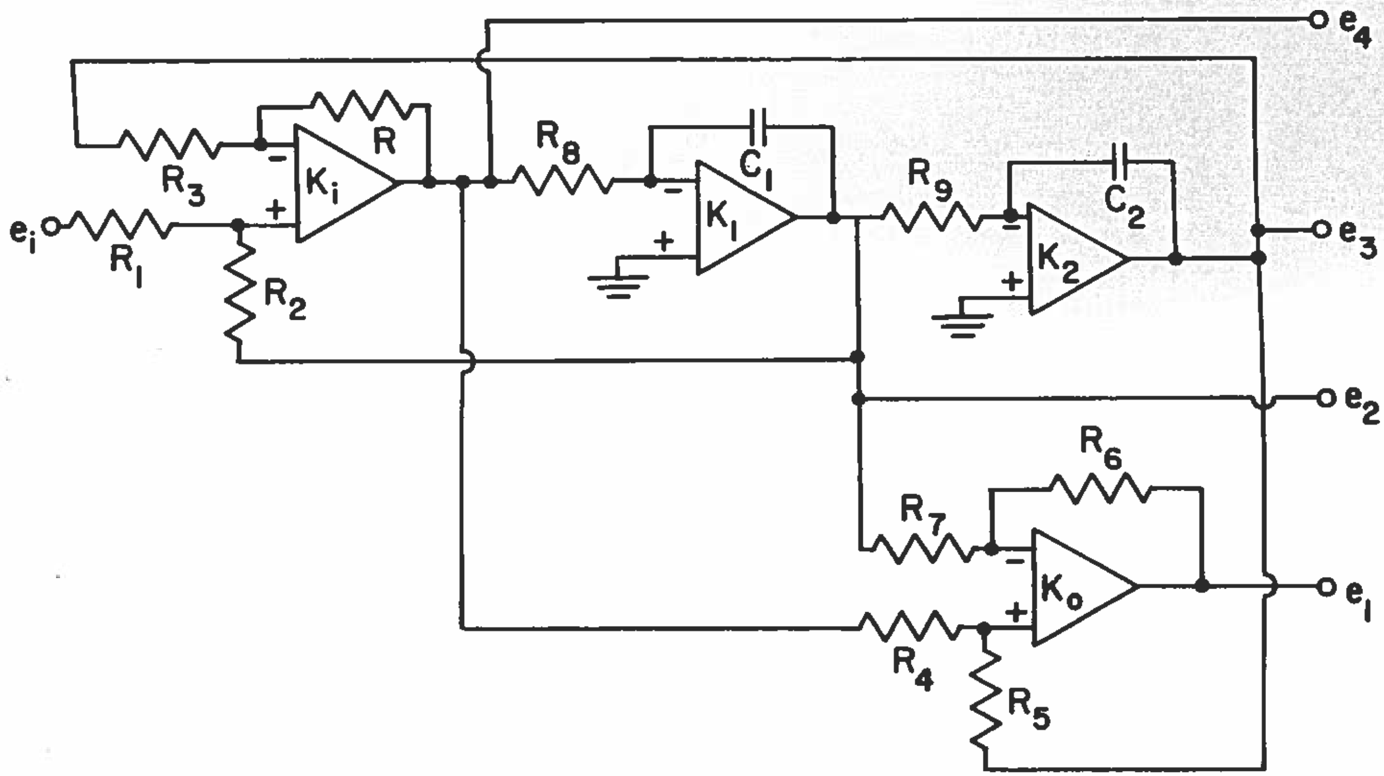

Figure 6. General Second Order Function Schematic.