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BI-STEP-UP N-PORTS*

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Abstract

The report studies the characterization of a class of nonlossless scattering matrices by means of biorthonormal step-up vectors. It is shown that if any two scattering matrices behave in such a way that one of them is the inverse of the adjoint of the other and such that they preserve inner products in a Hilbert Space, then one can find two sequences of vectors which are biorthonormal and step-up with respect to these scattering matrices.

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1. Introduction

In a recent paper [1] we investigated the problem of characterizing n-port networks by means of orthonormal vectors. It was shown that for a class of lossless n-ports, one can find sequences of orthonormal vectors which, when used as incident voltages, yield reflected voltages consisting of the same sequences with order increased by one. Such sequences have been called orthonormal step-up vectors, and the networks step-up n-ports.

In this report, the step-up concept [2] [3] [4] [5] [6] is generalized to a class of nonlossless n-ports which will be defined as bi-step-up networks. A pair of linear, completely solvable [7] n-ports $\underset{\sim}{M}$ and $\underset{\sim}{N}$ is said to be bi-step-up if there exists a pair of sequences of vectors $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ such that $\underset{\sim}{M}$ maps any $\underline{\varphi}_j$ into a $\underline{\varphi}_{j+1}$ and $\underset{\sim}{N}$ maps any $\underline{\psi}_j$ into a $\underline{\psi}_{j+1}$, and such that the two sequences form a biorthonormal system as is defined below. Such a system of vectors, if it exists, will be called a system of bi-step-up vectors for the pair of n-ports $\underset{\sim}{M}$ and $\underset{\sim}{N}$. Furthermore if the system $\{\underline{\varphi}_j, \underline{\psi}_j\}$ is also complete in the set of square-integrable n-vectors, then $\underset{\sim}{M}$ and $\underset{\sim}{N}$ will be called complete bi-step-up networks.

Given a complete bi-step-up network $\underset{\sim}{M}$ and assuming that the complete bi-step-up sequences $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ have been found, then within the theory, an arbitrary incident voltage $\underline{v}^i$ applied to $\underset{\sim}{M}$ can be represented by the expansion

$$\underline{v}^i = \sum_{j=o}^{\infty} (\underline{\psi}_j, \underline{v}^i)\, \underline{\varphi}_j \qquad (1\text{-}1)$$

The corresponding reflected voltage is then

$$\underline{v}^r = \sum_{j=o}^{\infty} (\underline{\psi}_j, \underline{v}^i)\, \underline{\psi}_{j+1} \qquad (1\text{-}2)$$

In other words, the $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ span the input and output spaces. Equations (1-1) and (1-2) show a useful and simple method of computing signals through the network by the use of biorthonormal expansions.

It is noted that the step-up theory as discussed in [1] [2] [3] [4] applies only to lossless n-ports; to overcome this constraint, the bi-step-up theory in this report is developed with the previous lossless theory resulting as a special case when $\underset{\sim}{M} = \underset{\sim}{N}$.

A brief review of some important results of biorthonormal systems and scattering matrices is given in section 2. Section 3 deals with bi-step-up theory while section 4 gives examples of the above concepts.

2. Review and Background Concepts

Square Integrable Vectors

In what follows, unless otherwise stated, the real variable t will be taken as time; an n-vector $[x_j(t)]$ will be written as $\underline{x}$, a matrix will be denoted by $\underset{\sim}{a}$. Matrix transposition will be indicated by a superscript tilde.

The inner product of two n-vectors $\underline{x}_1 = [x_j^{(1)}(t)]$ and $\underline{x}_2 = [x_j^{(2)}(t)]$ will be written as

$$\langle \underline{x}_1, \underline{x}_2 \rangle = \int_{-\infty}^{\infty} \tilde{\underline{x}}_1(t) \cdot \underline{x}_2(t) \cdot dt \qquad (2\text{-}1)$$

whenever the integral exists; as stated above the tilde, $\tilde{}$, denotes matrix transposition.

The set of square-integrable n-vectors, $\underline{L}_2$, is defined by $[\varphi_j] = \underline{\varphi} \epsilon \underline{L}_2$ if the φ_j are measurable and

$$\langle \underline{\varphi}, \underline{\varphi} \rangle < \infty \qquad (2\text{-}2)$$

Biorthonormal System

In the Hilbert space, also denoted by $\underline{L}_2$, of $\underline{L}_2$ vectors, two sequences of vectors $\{\underline{\varphi}_n\}$ and $\{\underline{\psi}_n\}$ are said to form a biorthonormal system $\{\underline{\varphi}_n, \underline{\psi}_n\}$ if [8, p. 36]

$$\langle \underline{\varphi}_i, \underline{\psi}_j \rangle = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \qquad (2\text{-}3)$$

The system $\{\underline{\varphi}_j, \underline{\psi}_j\}$ is complete if $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ are fundamental sets of $\underline{L}_2$, i.e., if any $\underline{x} \in \underline{L}_2$ can be represented by

$$\underline{x} = \sum_{i=o}^{\infty} \langle \underline{\psi}_i, \underline{x} \rangle \underline{\varphi}_i = \sum_{i=o}^{\infty} \langle \underline{\varphi}_i, \underline{x} \rangle \underline{\psi}_i \tag{2-4}$$

where the series $\sum_{i=o}^{\infty} \langle \underline{\psi}_i, x \rangle^2$ and $\sum_{i=o}^{\infty} \langle \underline{\varphi}_i, x \rangle^2$ are convergent.

Scattering Matrices

Consider a linear, completely solvable n-port $\underset{\sim}{N}$ [7]. Let $\underline{v}$ and $\underline{i}$ be the allowed n-vector port voltages and currents respectively, then one can define

$$\underline{v}^i = \frac{1}{2}(\underline{v} + \underline{i}) \tag{2-5a}$$

$$\underline{v}^r = \frac{1}{2}(\underline{v} - \underline{i}) \tag{2-5b}$$

as the incident and reflected voltages of the network. Associated with these latter variables there always exists an $n \times n$ matrix of distributions in two variables, called the scattering matrix $\underset{\sim}{s}(t,\tau)$ such that

$$\underline{v}^r = \int_{-\infty}^{\infty} \underset{\sim}{s}(t,\tau)\underline{v}^i(\tau)d\tau \tag{2-6a}$$

$$= \underset{\sim}{s} \bullet \underline{v}^i \tag{2-6b}$$

where the operation of (2-6b) is rigorously defined in [9, p. 221]. For an $\underset{\sim}{s}$ mapping $\underline{L}_2$ into $\underline{L}_2$ we define the norm $\|\underset{\sim}{s}\|$ through

$$\|\underset{\sim}{s}\| = \sup_{\langle \underline{\varphi}, \underline{\varphi} \rangle = 1} \langle s \bullet \underline{\varphi}, s \bullet \underline{\varphi} \rangle^{1/2} \tag{2-7}$$

The adjoint $\underset{\sim}{s}^a(t,\tau)$ of $\underset{\sim}{s}$ is defined by $\underset{\sim}{s}$ through

$$\underset{\sim}{s}^a(t,\tau) = \tilde{\underset{\sim}{s}}(\tau,t) \tag{2-8}$$

For passive networks, we have the following properties and definitions for $\underset{\sim}{s}$ [10]:

$\underset{\sim}{s}$ is antecedal, that is,

$$\underset{\sim}{s}(t,\tau) = \underset{\sim}{0} \quad \text{for all} \quad t < \tau \tag{2-9}$$

$\underset{\sim}{s}$ is a bounded linear continuous transformation of $\underline{L}_2$ vectors into $\underline{L}_2$ vectors.

The norm $\|\underset{\sim}{s}\|$ of $\underset{\sim}{s}$ is bounded by unity

$$\|\underset{\sim}{s}\| = \|\underset{\sim}{s}^a\| \leq 1 \tag{2-10}$$

A passive $\underset{\sim}{s}$ is said to be lossless if the energy at infinity is zero,

$$\mathcal{E}(\infty) = \langle \underline{v}^i, \underline{v}^i \rangle - \langle \underline{v}^r, \underline{v}^r \rangle = 0 \tag{2-11}$$

for all $\underline{v}^i \epsilon \underline{L}_2$, or equivalently

$$\underset{\sim}{s}^a \circ \underset{\sim}{s} = \delta \underset{\sim}{1}_n \tag{2-12}$$

where o denotes the Volterra composition operation [9, p. 229]

$$\underset{\sim}{a} \circ \underset{\sim}{b} = \int_{-\infty}^{\infty} \underset{\sim}{a}(t,\lambda)\underset{\sim}{b}(\lambda,\tau)d\lambda \tag{2-13}$$

δ is the unit impulse $\delta(t-\tau)$ and $\underset{\sim}{1}_n$ is the constant $n \times n$ unit matrix.

In what follows a scattering matrix $\underset{\sim}{s}$ which does not satisfy equation (2-12) will be generally called nonlossless.

3. Bi-Step-Up n-Ports

We now develop the theory appropriate to bi-step-up n-ports.

Let $\underset{\sim}{M}$ and $\underset{\sim}{N}$ be two linear, time-invariant, completely solvable n-ports whose time-domain scattering matrices $\underset{\sim}{s}_M(t,\tau)$ and $\underset{\sim}{s}_N(t,\tau)$ exist and map $\underline{L}_2$ incident voltages into $\underline{L}_2$ reflected voltages.

Thus we can define a pair of (not necessarily distinct) sequences $\{\underline{\varphi}_j, j = 0, 1, 2, \ldots; \underline{\varphi}_j \in \underline{L}_2\}$ and $\{\underline{\psi}_j, j = 0, 1, 2, \ldots; \underline{\psi}_j \in \underline{L}_2\}$ one associated with each network. $\underline{\varphi}_j$ and $\underline{\psi}_j$ are taken as successive incident voltages of $\underset{\sim}{M}$ and $\underset{\sim}{N}$ respectively and satisfy

$$\underline{\varphi}_{j+1} = \underset{\sim}{s}_M \bullet \underline{\varphi}_j \tag{3-1}$$

$$\underline{\psi}_{j+1} = \underset{\sim}{s}_N \bullet \underline{\psi}_j \tag{3-2}$$

In what follows, we shall investigate those $\underset{\sim}{M}$ and $\underset{\sim}{N}$ for which there exists $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ as defined above and such that the two sequences form a biorthonormal system

$$\langle \underline{\varphi}_j, \underline{\psi}_k \rangle = \delta_{jk} \tag{3-3}$$

The system $\{\underline{\varphi}_j, \underline{\psi}_j\}$ is then defined to be a <u>bi-step-up system</u> for $\underset{\sim}{M}$ and $\underset{\sim}{N}$, and the networks a pair of <u>bi-step-up n-ports</u>. If the sequences $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ are also individually complete in $\underline{L}_2$, then the networks are said to be complete bi-step-up n-ports.

Now let us show that complete bi-step-up n-ports cannot be lossless unless $\underline{\varphi}_j = \underline{\psi}_j$ for all j. If the domain of $\underset{\sim}{s}_M$ and $\underset{\sim}{s}_N$ is the Hilbert space of $\underline{L}_2$ vectors and if $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$, $\underline{\varphi}_j \neq \underline{\psi}_j$, form a bi-step-up system for $\underset{\sim}{M}$ and $\underset{\sim}{N}$, then any $\underline{v}^i$ in $\underline{L}_2$ can be represented by the expansions

$$\underline{v}^i = \sum_{j=o}^{\infty} \langle \underline{\psi}_j, \underline{v}^i \rangle \underline{\varphi}_j = \sum_{j=o}^{\infty} \langle \underline{\varphi}_j, \underline{v}^i \rangle \underline{\psi}_j \tag{3-4}$$

The corresponding reflected voltage $\underline{v}_M^r$ of $\underset{\sim}{M}$ is then

$$\underline{v}_M^r = \underset{\sim}{s}_M \bullet \underline{v}^i = \sum_{j=o}^{\infty} \langle \underline{\psi}_j, \underline{v}^i \rangle \underline{\varphi}_{j+1} = \sum_{j=o}^{\infty} \langle \underline{\varphi}_j, \underline{v}^i \rangle s_M \bullet \underline{\psi}_j \tag{3-5}$$

Thus we have

$$\langle \underline{v}^i, \underline{v}^i \rangle - \langle \underline{v}_M^r, \underline{v}_M^r \rangle = \sum_{j=o}^{\infty} \langle \underline{\psi}_j, \underline{v}^i \rangle \langle \underline{\varphi}_j, \underline{v}^i \rangle - \sum_{j,k=o}^{\infty} \langle \underline{\psi}_j, \underline{v}^i \rangle \langle \underline{\varphi}_{j+1}, \underset{\sim}{s}_M \bullet \underline{\psi}_k \rangle \langle \underline{\varphi}_k, \underline{v}^i \rangle \tag{3-6a}$$

which is zero for any $\underline{v}^i$ only when

$$\langle \underline{\varphi}_{j+1}, \underset{\sim}{s}_M \bullet \underline{\psi}_k \rangle = \delta_{jk} \tag{3-6b}$$

But (3-6b) cannot be true for a nonlossless $\underset{\sim}{M}$ since

$$\langle \underline{\varphi}_{j+1}, \underset{\sim}{s}_M \bullet \underline{\psi}_k \rangle = \langle \underset{\sim}{s}_M \bullet \underline{\varphi}_j, \underset{\sim}{s}_M \bullet \underline{\psi}_k \rangle = \langle (\underset{\sim}{s}_M^a \circ \underset{\sim}{s}_M) \bullet \underline{\varphi}_j, \underline{\psi}_k \rangle \tag{3-6c}$$

which (by completeness of the $\underline{\varphi}_j$) is δ_{jk} for all j and k if and only if $\underset{\sim}{s}_M^a \circ \underset{\sim}{s}_M$ is the identity transformation, i.e., equal to $\delta \underset{\sim}{1}_n$, which implies that $\underset{\sim}{M}$ would be lossless. If $\underset{\sim}{M}$ is lossless then $\underline{\psi}_k = \underline{\varphi}_k$, from (3-6b,c) [1], and $\underset{\sim}{M} = \underset{\sim}{N}$.

Theorem 1. Complete bi-step-up n-ports with $\underline{\varphi}_j \neq \underline{\psi}_j$ for some j are nonlossless.

Although necessary and sufficient conditions for two networks to be bi-step-up are not available one can give separate sufficient and separate necessary conditions.

Theorem 2. $\underset{\sim}{M}$ and $\underset{\sim}{N}$ form a pair of bi-step-up n-ports if

(i) $$\langle \underset{\sim}{s}_M \bullet \underline{x}, \underset{\sim}{s}_N \bullet \underline{y} \rangle = \langle \underline{x}, \underline{y} \rangle \tag{3-7}$$

for some $\underline{x}$, $\underline{y}$ in $\underline{L}_2$, in particular for those vectors $\underline{\varphi}_j$ and $\underline{\psi}_j$ defined in (iii) below and

(ii) the equations $$\underset{\sim}{s}_M^a \bullet \underline{\psi}_o = \underline{0} \tag{3-8a}$$

$$\underset{\sim}{s}_N^a \bullet \underline{\varphi}_o = \underline{0} \tag{3-8b}$$

have at least one $\underline{L}_2$ solution and

(iii) $$\underline{\varphi}_j = (\underset{\sim}{s}_M)^j \bullet \underline{\varphi}_o, \quad \underline{\psi}_j = (\underset{\sim}{s}_N)^j \bullet \underline{\psi}_o \tag{3-9}$$

Proof: Assuming the validity of the conditions, we first note from (3-7) that there exist $\underline{f}$, $\underline{g} \in \underline{L}_2$ such that

$$(\underset{\sim}{s}_N^a \circ \underset{\sim}{s}_M) \bullet \underline{x} = \underline{x} + \underline{f} \quad \text{with} \quad \langle \underline{f}, \underline{y} \rangle = 0 \tag{3-10a}$$

$$(\underset{\sim}{s}_M^a \circ \underset{\sim}{s}_N) \bullet \underline{y} = \underline{y} + \underline{g} \quad \text{with} \quad \langle \underline{g}, \underline{x} \rangle = 0 \tag{3-10b}$$

for some $\underline{x}$, $\underline{y}$, $\in \underline{L}_2$.

We have from (iii)

$$\langle \underline{\varphi}_j, \underline{\psi}_k \rangle = \langle \underset{\sim}{s}_M \bullet \underline{\varphi}_{j-1}, \underset{\sim}{s}_N \bullet \underline{\psi}_{k-1} \rangle \tag{3-11a}$$

$$= \langle (\underset{\sim}{s}_N^a \circ s_M) \bullet \underline{\varphi}_{j-1}, \underline{\psi}_{k-1} \rangle \tag{3-11b}$$

Assume now that the system $\{\underline{\varphi}_j, \underline{\psi}_j\}$ is not complete, i.e., there exist $\underline{\varphi}$ and $\underline{\psi}$ in $\underline{L}_2$ such that $\langle \underline{\varphi}, \underline{\psi}_j \rangle = \langle \underline{\psi}, \underline{\varphi}_j \rangle = 0$ for all j. Using (3-10a) in (3-11a) with the choice $\underline{f} = \underline{\varphi}$, we have

$$\langle \underline{\varphi}_j, \underline{\psi}_k \rangle = \langle \underline{\varphi}_{j-1} + \underline{\varphi}, \underline{\psi}_{k-1} \rangle \tag{3-12a}$$

$$= \langle \underline{\varphi}_{j-1}, \underline{\psi}_{k-1} \rangle \tag{3-12b}$$

Consequently we have, by iteration

$$\langle \underline{\varphi}_j, \underline{\psi}_k \rangle = \langle (\underset{\sim}{s}_N^a)^{k-j} \bullet \underline{\varphi}_o, \underline{\psi}_o \rangle \quad \text{for} \quad j < k \tag{3-12c}$$

since $\underline{\varphi}_o$ satisfies condition (ii). Similarly for the case $j > k$. The system $\{\underline{\varphi}_j, \underline{\psi}_j\}$ is therefore biorthonormal and $\underset{\sim}{M}$ and $\underset{\sim}{N}$ are bi-step-up n-ports. Q. E. D.

We note that, given a bi-step-up system $\{\underline{\varphi}_j = (\underset{\sim}{s}_M)^j \bullet \underline{\varphi}_o, \underline{\psi}_j = (\underset{\sim}{s}_N)^j \bullet \underline{\psi}_o\}$ then for any $\underline{x}$ and $\underline{y}$ of the subspaces spanned by $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ respectively, we have

$$\underline{x} = \sum_{j=o}^{\infty} a_j \underline{\varphi}_j \tag{3-13a}$$

$$\underline{y} = \sum_{j=o}^{\infty} b_j \underline{\psi}_j \tag{3-13b}$$

Thus

$$\langle s_M \bullet \underline{x}, s_N \bullet \underline{y} \rangle = \sum_{i=o}^{\infty} a_j \, b_j = \langle \underline{x}, \underline{y} \rangle \qquad (3\text{-}14)$$

for all $\underline{x}$ and $\underline{y}$ of these two subspaces respectively.

Theorem 3. The sequences $\{\underline{\varphi}_j = (\underset{\sim}{s}_M)^j \bullet \underline{\varphi}_o\}$ and $\{\underline{\psi}_j = (\underset{\sim}{s}_N)^j \bullet \underline{\psi}_o\}$ form a complete bi-step-up system for the pair of networks M, N only if

$$\text{(i)} \qquad \langle \underset{\sim}{s}_M \bullet \underline{x}, \underset{\sim}{s}_N \bullet \underline{y} \rangle = \langle \underline{x}, \underline{y} \rangle \qquad (3\text{-}15)$$

for all $\underline{x}$ and $\underline{y}$ in $\underline{L}_2$, and

(ii) the equations

$$\underset{\sim}{s}_M^a \bullet \underline{\psi}_o = \underline{0} \qquad (3\text{-}16a)$$

$$\underset{\sim}{s}_N^a \bullet \underline{\varphi}_o = \underline{0} \qquad (3\text{-}16b)$$

have at least one $\underline{L}_2$ solution.

Proof: Given a complete bi-step-up system $\{\underline{\varphi}_j = (\underset{\sim}{s}_M)^j \bullet \underline{\varphi}_o, \underline{\psi}_j = (\underset{\sim}{s}_N)^j \bullet \underline{\psi}_o\}$, then for any $\underline{x}, \underline{y} \in \underline{L}_2$

$$\underline{x} = \sum_{j=o}^{\infty} a_j \, \underline{\varphi}_j \qquad (3\text{-}17a)$$

$$\underline{y} = \sum_{j=o}^{\infty} b_j \, \underline{\psi}_j \qquad (3\text{-}17b)$$

Then

$$\langle \underset{\sim}{s}_M \bullet \underline{x}, \underset{\sim}{s}_N \bullet \underline{y} \rangle = \sum_{j=o}^{\infty} a_j \, b_j = \langle \underline{x}, \underline{y} \rangle \text{ for all } \underline{x}, \underline{y} \in \underline{L}_2 \qquad (3\text{-}18)$$

by the completeness property. From (3-17a) we have

$$\langle \underline{\psi}_o, \underset{\sim}{s}_M \bullet \underline{x} \rangle = \langle \underline{\psi}_o, \sum_{j=o}^{\infty} a_j \, \underline{\varphi}_{j+1} \rangle \qquad (3\text{-}19a)$$

$$= 0 \quad \text{for all } \underline{x} \in \underline{L}_2 \qquad (3\text{-}19b)$$

Therefore

$$\underset{\sim}{s}_M^a \bullet \underline{\psi}_o = \underline{0} \qquad (3\text{-}20)$$

Similarly

$$\underset{\sim}{s}_N^a \bullet \underline{\varphi}_o = \underline{0} \qquad (3\text{-}21)$$

and the theorem is proved. Q. E. D.

From equation (3-15) we have by taking $\underline{x} = \underline{y}$

$$\langle [\underset{\sim}{s}_N^a \circ \underset{\sim}{s}_M - \delta \underset{\sim}{1}_n] \bullet \underline{x}, \underline{x} \rangle = 0 \quad \text{for all } \underline{x} \in \underline{L}_2 \qquad (3\text{-}22a)$$

and

$$\langle [x, [\underset{\sim}{s}_M^a \circ \underset{\sim}{s}_N - \delta 1_n] \bullet \underline{x} \rangle = 0 \quad \text{for all } \underline{x} \in \underline{L}_2 \qquad (3\text{-}22b)$$

Hence

$$\underset{\sim}{s}_N^a \circ \underset{\sim}{s}_M = \delta \underset{\sim}{1}_n = \underset{\sim}{s}_M^a \circ \underset{\sim}{s}_N \qquad (3\text{-}23)$$

Thus, given an $\underset{\sim}{s}_M$ one can, in principle, find an $\underset{\sim}{s}_N$ which satisfies (3-23). However, even if $\underset{\sim}{s}_M$ and $\underset{\sim}{s}_N$ satisfy equation (3-23) it does not necessarily mean that they will be bi-step-up, as can be seen from the following result.

Theorem 4. If $\underset{\sim}{s}_M$ and $\underset{\sim}{s}_N$ satisfy equation (3-23) and if they map complete orthonormal sequences of $\underline{L}_2$ into complete biorthonormal systems of $\underline{L}_2$ then $\underset{\sim}{s}_M$ and $\underset{\sim}{s}_N$ cannot be bi-step-up.

Proof: Let $\{\underline{f}_i\}$ be a complete orthonormal sequence of $\underline{L}_2$, then

$$\langle \underset{\sim}{s}_M \bullet \underline{f}_i, \underset{\sim}{s}_N \bullet \underline{f}_j \rangle = \langle \underline{f}_i, \underline{f}_j \rangle = \delta_{ij} \qquad (3\text{-}24)$$

the system $\{\underset{\sim}{s}_M \bullet \underline{f}_i, \underset{\sim}{s}_N \bullet \underline{f}_i\}$ is therefore biorthonormal.

Let $\mathfrak{n}$ be the closed subspace spanned by the sequence $\{\underset{\sim}{s}_M \bullet \underline{f}_i\}$ and $\underline{\mathfrak{n}}^{\perp}$ the orthocomplement of $\underline{\mathfrak{n}}$, then

$$\langle \underline{z}, \underset{\sim}{s}_M \bullet \underline{f}_i \rangle = 0, \quad \text{for all } \underline{z} \in \underline{\mathfrak{n}}^{\perp} \text{ and all } i \qquad (3\text{-}25)$$

or

$$\langle \underset{\sim}{s}_M^a \bullet \underline{z}, \underline{f}_i \rangle = 0, \quad \text{for all } i \qquad (3\text{-}26)$$

But the orthonormal sequence $\{\underline{f}_i\}$ is assumed to be complete, therefore

$$\underset{\sim}{s}_M^a \bullet \underline{z} = 0 \quad \text{for all } \underline{z} \in \underline{\mathfrak{n}}^{\perp} \qquad (3\text{-}27)$$

Now, if the biorthonormal system $\{\underset{\sim}{s}_M \bullet \underline{f}_i, \underset{\sim}{s}_N \bullet \underline{f}_i\}$ is also complete then it is clear that $\underline{z} = \underline{0}$ is the only vector of $\underline{L}_2$ which satisfies equation (3-25) or equation (3-27). As a consequence, we have from (3-9)

$$\underline{\psi}_i = \underset{\sim}{s}_N^i \bullet (\underline{\psi}_o = \underline{z} = \underline{0}) = \underline{0} \qquad (3\text{-}28a)$$

By a similar argument it can be seen that

$$\underline{\varphi}_j = \underline{0} \qquad (3\text{-}28b)$$

Thus, the bi-step-up sequences are zero (do not exist) and $\underset{\sim}{s}_M$ and $\underset{\sim}{s}_N$ cannot be bi-step-up scattering matrices.

4. Examples

To illustrate the various theorems and concepts we give four examples. The first one exhibits a complete bi-step-up system while the next two consider noncomplete but bi-step-up systems.

Example 1. Consider the one-ports described by

$$s_M(t-\tau) = \delta(t-\tau) - (\alpha+\beta)e^{-\beta(t-\tau)}u(t-\tau) \qquad (4\text{-}1a)$$

and

$$s_N(t-\tau) = \delta(t-\tau) - (\alpha+\beta)e^{-\alpha(t-\tau)}u(t-\tau) \qquad (4\text{-}1b)$$

where $u(t-\tau)$ is the unit step function.

It can be easily verified that

$$s_M^a \circ s_N = \delta = s_N^a \circ s_M \qquad (4\text{-}2)$$

Direct calculations, using (3-8), give

$$\varphi_o(t) = \sqrt{\alpha+\beta}\, e^{-\alpha t}\, u(t) \qquad (4\text{-}3a)$$

$$\psi_o(t) = \sqrt{\alpha+\beta}\, e^{-\beta t}\, u(t) \qquad (4\text{-}3b)$$

Consequently, for $j \geq 0$,

$$\varphi_j(t) = \sqrt{\alpha+\beta}\, e^{-\alpha t}\, L_j\, [(\alpha+\beta)t]u(t) \qquad (4\text{-}3c)$$

$$\psi_j(t) = \sqrt{\alpha+\beta}\, e^{-\beta t}\, L_j\, [(\alpha+\beta)t]u(t) \qquad (4\text{-}3d)$$

where L_j is a Laguerre polynomial. The bi-step-up system is complete in this case since each of $\{\varphi_j\}$ and $\{\psi_j\}$ are separately complete. We note that if $\beta \geq \alpha > 0$ then (4-1a) describes a passive network while (4-1b) defines an active one. If $\alpha = \beta > 0$ then the two networks are identical and in fact lossless (inductors).

Example 2. Consider the (direct sum) 2-ports of scattering matrices

$$\underset{\sim}{s}_M(t-\tau) = \delta(t-\tau)\underset{\sim}{1}_2 - \begin{bmatrix} 2\alpha e^{-\alpha(t-\tau)} & 0 \\ 0 & (\alpha+\beta)e^{-\beta(t-\tau)} \end{bmatrix} u(t-\tau) \qquad (4\text{-}4a)$$

and

$$\underset{\sim}{s}_N(t-\tau) = \delta(t-\tau)\underset{\sim}{1}_2 - \begin{bmatrix} 2\alpha e^{-\alpha(t-\tau)} & 0 \\ 0 & (\alpha+\beta)e^{-\alpha(t-\tau)} \end{bmatrix} u(t-\tau) \qquad (4\text{-}4b)$$

We find $\underset{\sim}{s}_M^a \circ \underset{\sim}{s}_N = \underset{\sim}{s}_N^a \circ \underset{\sim}{s}_M = \delta\underset{\sim}{1}_2$ and (3-8) applies to give, for $j \geq 0$

$$\underline{\varphi}_j = \begin{bmatrix} \sqrt{\alpha}\, e^{-\alpha t} L_j\ (2\alpha t) \\ \sqrt{\frac{\alpha+\beta}{2}}\, e^{-\alpha t} L_j\ [(\alpha+\beta)t] \end{bmatrix} u(t) \qquad (4\text{-}5a)$$

and

$$\underline{\psi}_j = \begin{bmatrix} \sqrt{\alpha}\, e^{-\alpha t} L_j\ (2\alpha t) \\ \sqrt{\frac{\alpha+\beta}{2}}\, e^{-\beta t} L_j\ [(\alpha+\beta)t] \end{bmatrix} u(t) \qquad (4\text{-}5b)$$

Note again that when $\alpha = \beta$ the two networks are the same and lossless. Nevertheless the $\{\underline{\varphi}_j\}$ and $\{\underline{\psi}_j\}$ are not complete, though, by Theorem 2, the two networks form a bi-step-up pair. Theorem 3, which gives only necessary conditions is of course not violated and this example shows that the conditions of Theorem 3 are not sufficient.

Example 3. For the 2-ports defined by

$$\underset{\sim}{s}_M(t-\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \delta(t-\tau) + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} 2e^{-3(t-\tau)}u(t-\tau) \qquad (4\text{-}6a)$$

$$\underset{\sim}{s}_N(t-\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \delta(t-\tau) + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} 2e^{-(t-\tau)}u(t-\tau) \qquad (4\text{-}6b)$$

we find, again for (3-8) since $\underset{\sim}{s}_M^a \circ \underset{\sim}{s}_N = \delta\underset{\sim}{1}_2$,

$$\tilde{\Phi}_o = \sqrt{2}\,[1\ 1]e^{-3t}u(t), \quad \tilde{\Psi}_o = \sqrt{2}\,[1\ 1]e^{-t}u(t) \qquad (4\text{-}6c)$$

$$\tilde{\Phi}_{2n} = \sqrt{2}\,[1\ 1]e^{-3t}L_{2n}(4t)u(t), \quad \tilde{\Psi}_{2n} = \sqrt{2}\,[1\ 1]e^{-t}L_{2n}(4t)u(t) \qquad (4\text{-}6d)$$

$$\tilde{\Phi}_{2n+1} = \sqrt{2}\,[1\ -1]e^{-3t}L_{2n+1}(4t)u(t), \quad \tilde{\Psi}_{2n+1} = \sqrt{2}\,[1\ -1]e^{-t}L_{2n+1}(4t)u(t) \qquad (4\text{-}6e)$$

Although the bi-step-up sequences of this and the previous example are not complete, we have shown the existence of the bi-step-up sequences for the 2-ports under consideration.

Example 4. The networks described by

$$s_M(t,\tau) = C\,\delta(t-\tau) \qquad (4\text{-}7a)$$

and

$$s_N(t,\tau) = \tilde{C}^{-1}\delta(t-\tau) \qquad (4\text{-}7b)$$

where C is a constant nonsingular $n \times n$ matrix, cannot be bi-step-up, even though $s_M^a \circ s_N = \delta 1_n$, since they map complete orthonormal sequences of L_2 into complete orthonormal sequences of L_2 and Theorem 4 applies. Note also that $s_M^a \bullet \Psi_o = 0$ has no nonzero solution and hence condition (ii) of Theorem 2 is violated.

5. Conclusion

The bi-step-up property is an interesting characteristic of a class of nonlossless networks. This work has shown that if any two distinct nonlossless scattering matrices behave in such a way that one of them is the inverse of the adjoint of the other, and provided that they preserve inner products in a Hilbert space then the two scattering matrices can be characterized by bi-step-up vectors. The theory is also formulated to yield the previous lossless theory when the two networks M and N are identical.

One important question of considerable interest still remains open; this is that of the completeness of the bi-step-up vectors found (sufficiency, analogous to Theorem 3). Finally, we note that the above theory can be suitably extended to operators in a Banach space $\underline{B}$ by replacing $\underline{L}_2$ by $\underline{B}$ everywhere that $\underline{L}_2$ occurs.

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