Step-Up n-Port Networks

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Abstract—This paper presents a method of characterizing n-port networks. It is shown that a large class of lossless n-ports can be characterized by sequences of step-up vectors, i.e., vectors that form successive incident voltages to the networks and are orthonormal at the same time. As a result, lossless networks can be classified into several classes. The most interesting one is, perhaps, the step-up class, which is shown to behave jointly like a unitary operator and a complete step-up operator on separate subspaces of the space of $\underline{L}_2$ vectors.

I. Introduction

TO ANALYZE signals in linear systems, one often represents the input signals as an expansion in orthogonal functions [1]. The corresponding output signal then has the potentially useful property that it can be obtained by superposing responses of the system to these functions. The choice of such a sequence of functions is in general arbitrary, provided the sequence is complete. Such a choice may often depend either on the nature of the class of possible input signals or the nature of the system.

In looking for sequences of functions that are suitable for analyzing input signals and, at the same time, have simple and useful properties with respect to their passage through a given system, the theory of orthonormal step-up functions for a class of linear time-invariant single-input single-output systems, which have been defined as step-up systems, was developed [2]–[5]. A step-up system is one where there exists an ordered sequence of orthonormal functions, which, when used as inputs, yields outputs consisting of the same sequence with order increased by one. If the sequence is also complete, then the system has been called complete step-up.

The step-up systems were shown to have the interesting property of being "all-pass" in a sense similar to the network theory sense. This property, when expressed in the frequency domain, can be regarded as a particular case of the para-unitary property enjoyed by the scattering matrix of a lossless n-port. This was pointed out by Newcomb, and consequently the step-up theory for n-ports was investigated, using mainly a frequency-domain approach [6].

A general theory of time-domain scattering matrices in connection with time-varying n-ports was recently investigated [7] using the theory of distributions, distributional kernels, and linear transformations on Hilbert space. It has been shown in this latter work, among other results, that a time-domain scattering matrix is a bounded linear continuous transformation of square-integrable L_2 vectors into L_2 vectors, and that the lossless property of both time-varying and time-invariant n-ports can be suitably exhibited in the time domain. It was then pointed out that the step-up theory for n-ports can also be investigated by means of the theory of linear transformations on Hilbert space. Such an investigation is presented in this paper.

Section II contains review material as preparation for the subsequent sections. The true step-up theory begins in Section III. Section IV discusses finite step-up n-ports, in particular finite differential systems. These constitute a good example of the preceding concepts.

II. Review and Background Concepts

In what follows, unless otherwise mentioned, the real independent variable will be taken as time t, an n vector $[x_i(t)]$ will be denoted by $\underline{x}(t)$, a matrix $[a_{ij}]$ will be written as A, complex conjugation will be denoted by a superscript asterisk, and matrix transposition will be indicated by a superscript tilde.

To simplify expressions, we also define the inner product $\langle \underline{f}, \underline{g}\rangle$ for two n vectors $\underline{f} = [f_i(t)]$, $\underline{g} = [g_i(t)]$ by

$$\langle \underline{f}, \underline{g}\rangle = \int_{-\infty}^{\infty} \tilde{\underline{f}}^*(t)\underline{g}(t)\, dt \tag{1}$$

whenever the integral exists.

The set of square-integrable n vectors $\underline{L}_2(-\infty, \infty)$ is important and defined by $[\phi_i] = \underline{\phi} \in \underline{L}_2$ if the ϕ_i are measurable and

$$\langle \underline{\phi}, \underline{\phi}\rangle < \infty. \tag{2}$$

Orthonormal Vectors

A sequence $\{\underline{\phi}_j(t);\ j = 0, 1, 2, \cdots, \underline{\phi}_j \in \underline{L}_2\}$ is orthonormal if

$$\langle \underline{\phi}_m, \underline{\phi}_n\rangle = \delta_{m,n} = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n) \end{cases} \tag{3}$$

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and is complete if any $\phi \in L_2$ has the representation

$$\phi = \sum_{i=0}^{\infty} a_i \phi_i \tag{4}$$

provided the constants a_i satisfy the Bessel inequality

$$\sum_{i=0}^{\infty} |a_i|^2 < \infty \tag{5}$$

where $|\ |$ denotes the absolute value. The a_j are found by applying (3) to the inner product of ϕ and ϕ_i:

$$a_j = \langle \phi_j, \phi \rangle, \qquad \text{for all} \quad j. \tag{6}$$

A complete sequence is closed [8, p. 255] if there exists a vector $\psi \in L_2$ such that

$$\langle \psi, \phi_j \rangle = 0, \qquad \text{for all} \quad j. \tag{7}$$

Then this implies

$$\psi \equiv 0. \tag{8}$$

The Time-Domain Scattering Matrix

In this paper we will be mainly concerned with time-variable networks and, hence, review at this point some important properties of the time-varying scattering matrix.

Consider a linear, completely solvable n-port N [9]. One method of describing N, which is essentially the definition of a network, is to list all the allowed n-vector port currents $i(t)$ and voltages $v(t)$. But a more useful description occurs by writing

$$v^i = \tfrac{1}{2}(v + i) \tag{9a}$$

$$v^r = \tfrac{1}{2}(v - i), \tag{9b}$$

which allows for a characterization in terms of incident and reflected voltages v^i and v^r. Using these latter variables, there always exists an $n \times n$ matrix of distributions in two variables, called the scattering matrix $s(t, \tau)$, such that

$$v^r = \int_{-\infty}^{\infty} s(t, \tau) v^i(\tau)\, d\tau \tag{10a}$$

$$\equiv s \cdot v^i \tag{10b}$$

where the operation of (10b) is defined rigorously by Schwartz [10, p. 221].

The adjoint $s^a(t, \tau)$ of s is defined by s through

$$s^a(t, \tau) = \tilde{s}(\tau, t). \tag{11}$$

For passive N, the following rigorous definitions and properties of s can be found in [11] and [7].

1) s is antecedal, that is,

$$s(t, \tau) = 0, \qquad \text{for all} \quad t < \tau. \tag{12}$$

2) s is a bounded linear continuous transformation of $x \in L_2$ into $s \cdot x \in L_2$ and has a norm [12, p. 149]

$$||s|| = \sup_{||x||=1} \langle s \cdot x, s \cdot x \rangle^{1/2} \tag{13a}$$

bounded by unity

$$||s|| = ||s^a|| \leq 1. \tag{13b}$$

In terms of v^i and v^r, the *lossless* property of passive n-ports is expressed as

$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} [\tilde{v}^i(t) v^i(t) - \tilde{v}^r(t) v^r(t)]\, dt = 0 \qquad \text{for all} \quad v^i \in L_2. \tag{14}$$

Equivalently, a passive lossless s has been shown [7] to satisfy

$$s^a \circ s = \delta \mathbf{1}_n \tag{15}$$

where $\circ$ denotes the (Volterra) composition operation on two distributional kernels [10, p. 229]

$$a \circ b = \int_{-\infty}^{\infty} a(t, \lambda) b(\lambda, \tau)\, d\lambda. \tag{16}$$

Here δ is the unit impulse $\delta(t - \tau)$, while $\mathbf{1}_n$ denotes the constant unit matrix.

In what follows, we shall generally define a scattering matrix s to be lossless for an $x \in L_2$ if, for the specified x,

$$\langle s \cdot x, s \cdot x \rangle = \langle x, x \rangle. \tag{17}$$

Since (17) can be rewritten as $\langle (s^a \circ s) \cdot x - x, x \rangle = 0$, this implies that

$$s^a \circ s \cdot x = x + \alpha \tag{18}$$

where α is any L_2 vector satisfying $\langle \alpha, x \rangle = 0$.

To distinguish between the concepts of lossless and lossless for some x, we will call the former *totally lossless.*

Finally, it is noted that when $s(t, \tau) = s(t - \tau, 0)$ in the above equations, we obtain the known results for time-invariant networks. The operation in (10) then becomes the convolution

$$v^r = \int_{-\infty}^{\infty} s(t - \tau, 0) v^i(\tau)\, d\tau \tag{19a}$$

$$\equiv s * v^i. \tag{19b}$$

Also, for a time-invariant passive network the frequency-domain scattering matrix

$$S(p) = \mathcal{L}[s(t, 0)] \tag{20}$$

exists, where $\mathcal{L}[\ ]$ is the Laplace transform, and properties of s are easily transferred to those of S. For example. the lossless constraint (14) corresponds in this case to the para-unitary relationship in Re $p > 0$:

$$\tilde{S}(-p) S(p) = \mathbf{1}_n \tag{21}$$

where by passivity $S(p)$ is bounded real [13, p. 116]. We will assume that $S(p)$ is meromorphic, and, thus, by the bounded real constraint, holomorphic in Re $p \geq 0$.

III. Step-Up n-Ports

In this section, we precisely define concepts appropriate to step-up n-ports.

Consider any linear, completely solvable, time-invariant

n-port N. Then the time-domain scattering matrix $s(t - \tau, 0)$ exists and maps $\underline{v}^i$ into $\underline{v}^r$ by the convolution relation (19).

We further assume that s is a bounded linear continuous transformation mapping any $\underline{v}^i \in L_2$ into a $\underline{v}^r \in L_2$; in particular, this will be the case if N is passive. Thus, we can consider a system of r sequences $\{\phi_j^{(k)}(t); k = 1, 2, \cdots, r; j = 0, 1, 2, \cdots; \phi_j^{(k)} \in L_2\}$ with the $\phi_j^{(k)}$ taken as successive incident voltages, for each k. Therefore, an arbitrary $\phi_0^{(k)} \in L_2$ defines $\phi_1^{(k)}, \phi_2^{(k)}, \cdots$, through

$$\phi_{j+1}^{(k)} = s * \phi_j^{(k)}, \qquad \text{for all } k, j. \tag{22}$$

In what follows, we shall investigate the class of n-ports for which the r sequences $\{\phi_j^{(k)}\}$ defined in (22) form an orthonormal system, i.e.,

$$\langle \phi_i^{(k)}, \phi_j^{(l)} \rangle = \delta_{ij}\, \delta_{kl}. \tag{23}$$

If (23) holds subject to (22), then N is called a *step-up n-port class* r, written $N \in$ S.U. N_r, and $\{\phi_j^{(k)}\}$ an r *system of step-up vectors*. If the vectors $\{\phi_j^{(k)}\}$ are also complete, then we have *natural step-up vectors* for a *complete step-up n-port class* r, $N \in$ C.S.U. N_r, or simply *a complete step-up* N. It will be seen that r is the dimension of the null space of s^a, and thus for a given s there corresponds one and only one r.

Since we wish to span the input space by step-up vectors, we are required to consider r sequences, $r \geq 1$, to fully characterize complete step-up N as will become clear in the following developments, where, however, it will often suffice to choose $r = 1$ to illustrate the concept under discussion.

We note that if $N \in$ C.S.U. N_r, then an arbitrary $\underline{v}^i \in L_2$ can be expanded in terms of the natural vectors

$$\underline{v}^i = \sum_{k=1}^{r} \sum_{j=0}^{\infty} a_j^{(k)} \phi_j^{(k)}. \tag{24}$$

The resulting $\underline{v}^r$ is therefore

$$\underline{v}^r = s \cdot \underline{v}^i = \sum_{k=1}^{r} \sum_{j=0}^{\infty} a_j^{(k)} \phi_{j+1}^{(k)}. \tag{25}$$

These last two equations show the utility of the step-up vectors for computations associated with signal processing by the network. Thus, knowing the coefficients $a_j^{(k)}$ for the input completely determines the output by (25).

In what follows, sufficient constraints for a network N to be step-up, as well as necessary and sufficient constraints for N to be completely step-up, will be found.

Theorem 1

N is a step-up n-port if 1) $s^a \cdot \underline{y} = \underline{0}$ has at least one nontrivial L_2 solution, ϕ_0, say, and 2) s is lossless for all vectors of the form $(s)^j * \phi_0$, $j \geq 0$.

Proof: Clearly, if the results hold for $r = 1$, they hold for $r \geq 1$, and, hence, we consider only the case $r = 1$. Let the domain of s be the Hilbert space $\underline{\mathfrak{H}}$ of L_2 vectors. We have [8, p. 207]

$$\underline{\mathfrak{H}} = \underline{\overline{\mathfrak{R}}}(s) \overset{\perp}{\oplus} \underline{\mathfrak{N}}. \tag{26}$$

Here $\overset{\perp}{\oplus}$ denotes both the direct sum and the fact that $\underline{\mathfrak{N}}$ is the orthogonal complement of the closure $\underline{\overline{\mathfrak{R}}}(s)$ of the range of s. We note that $\underline{\mathfrak{N}}$ is also the null space of s^a [8, p. 207]. Thus, if $\underline{y} \in \underline{\mathfrak{N}}(s^a)$, then

$$s^a \cdot \underline{y} = \underline{0} = \int_{-\infty}^{\infty} \tilde{s}(\tau - t) \underline{y}(\tau)\, d\tau. \tag{27}$$

We shall use this equation to calculate step-up vectors for N.

Let ϕ_0 be an L_2 solution of (27), assumed to be initially normalized. We then form

$$\phi_i = (s)^i \bullet \phi_0. \tag{28}$$

For $i > j \geq 1$, we have

$$\langle \phi_i, \phi_j \rangle = \langle (s)^i \cdot \phi_0, (s)^j \cdot \phi_0 \rangle \tag{29a}$$

$$= \langle s^a \circ s \cdot (\phi_{i-1}), \phi_{j-1} \rangle. \tag{29b}$$

If s is lossless for $(s)^{i-1} \cdot \phi_0 = \phi_{i-1}$ and supposing that $\{\phi_j\}$ is not complete, i.e., $\exists\, \underline{\alpha} \in \underline{\mathfrak{H}}$ with $\langle \underline{\alpha}, \phi_j \rangle = 0$ for all j, then by (18) we have

$$\langle \phi_i, \phi_j \rangle = \langle \underline{\alpha} + \phi_{i-1}, \phi_{j-1} \rangle \tag{30a}$$

$$= \langle \phi_{i-1}, \phi_{j-1} \rangle. \tag{30b}$$

Consequently

$$\langle \phi_i, \phi_j \rangle = \langle (s)^{i-j} \cdot \phi_0, \phi_0 \rangle, \qquad i > j \tag{31a}$$

$$= 0 \tag{31b}$$

since $\phi_0 \in \underline{\mathfrak{N}}$ and $(s)^{i-j} \cdot \phi_0 \in \underline{\overline{\mathfrak{R}}}(s)$, which is orthogonal to $\underline{\mathfrak{N}}$. Thus N is a step-up network.

We note that the number of linearly independent L_2 solutions of (27) is the dimension of $\underline{\mathfrak{N}}(s^a)$, which is also the number (r) of the sequences $\{\phi_j^{(k)}\}$.

Conversely, a system of step-up vectors $\{\phi_j^{(k)}\}$ defines an s, which is lossless for all $\phi_j^{(k)}$ since

$$\langle \phi_{j+1}^{(k)} = s \cdot \phi_j^{(k)}, \phi_{j+1}^{(k)} \rangle = \langle \phi_j^{(k)}, \phi_j^{(k)} \rangle. \tag{32}$$

Thus s is also lossless for all $\underline{x}$ of the form

$$\sum_{k=1}^{r} \sum_{j=0}^{\infty} a_j^{(k)} \phi_j^{(k)},$$

i.e., $\underline{x} \in \underline{\mathfrak{SU}}$, the closed subspace spanned by $\langle \phi_j^{(k)} \rangle$. However, it is not always true that $\phi_0^{(k)}$ will be solutions of the equation $s^a \cdot \underline{y} = \underline{0}$, as can be seen by the example of the unit delay $\delta(t - 1)$ of Section IV. Consequently, condition 1) is not a necessary condition even though the above shows 2) to be a necessary condition.

Corollary

A step-up network is totally lossless if it is lossless for all vectors in $\underline{\mathfrak{SU}}^{\perp}$, the orthocomplement of the closed subspace spanned by the step-up sequences.

It will be seen in Theorem 3 that for a totally lossless and step-up network, N, $\underline{\mathfrak{SU}}^{\perp}$ is in fact an invariant subspace $\underline{\mathfrak{I}}$ of s, i.e., $s \cdot \underline{\mathfrak{I}} \subseteq \underline{\mathfrak{I}}$ [8, p. 275]. Furthermore, in $\underline{\mathfrak{I}}$, s behaves like an "onto" (unitary) operator, while within $\underline{\mathfrak{SU}}$ it is clear that s is an "into" (isometry) operator.

Theorem 2

A complete step-up time-invariant network is passive and totally lossless.

Proof: We first show that $\boldsymbol{s}$ satisfies the lossless constraint and then, in the time-invariant case, that the network is passive. Since the calculations are the same for any r, we consider as given a time-invariant complete step-up $N \in$ C.S.U. N_1. Then $\boldsymbol{s}$ necessarily exists. By the completeness of the step-up sequence $\{\phi_i\}$, any $\underline{v}^i \in \underline{\mathcal{H}}$, where, as in Theorem 1, $\underline{\mathcal{H}}$ is the Hilbert space of L_2 vectors, can be written as

$$\underline{v}^i = \sum_{i=0}^{\infty} a_i \phi_i. \tag{33}$$

The resulting $\underline{v}^r$ is then

$$\underline{v}^r = \sum_{i=0}^{\infty} a_i \phi_{i+1}. \tag{34}$$

Consequently, we have

$$\langle \underline{v}^i, \underline{v}^i \rangle = \sum_{i=0}^{\infty} a_i^2 = \langle \underline{v}^r = \boldsymbol{s} * \underline{v}^i, \boldsymbol{s} * \underline{v}^i = \underline{v}^r \rangle \qquad \text{for all } \underline{v}^i \in \underline{\mathcal{H}} \tag{35}$$

and therefore $\boldsymbol{s}$ is totally lossless.

Since $\boldsymbol{s}$ is a map of all of L_2 into L_2, it has a Laplace transform $\boldsymbol{S}(p)$, which is holomorphic in Re $p > 0$ and exists for almost all p on Re $p = 0$ [14, p. 21], [15, p. 80]. By the losslessness of $\boldsymbol{s}$ and by (15), $\tilde{\boldsymbol{S}}(-p)\boldsymbol{S}(p) = \mathbf{1}_n$ on Re $p = 0$ and hence everywhere [16, p. 138]. Since $\boldsymbol{S}(p^*) = \boldsymbol{S}^*(p)$ for a real system (i.e., a system giving real responses for real excitations), $\boldsymbol{S}(p)$ is bounded real [13, p. 116]; hence, the network is passive [17, p. 273]. Q.E.D.

Theorem 3

A totally lossless N is a complete step-up n-port if and only if 1) $\boldsymbol{s}^a \bullet \underline{y} = \underline{0}$ has at least one nontrivial L_2 solution, and 2) $||(\boldsymbol{s}^a)^k \bullet \underline{v}^i|| \to 0$ as $k \to \infty$ for all $\underline{v}^i \in \underline{\mathcal{H}}$, $\underline{\mathcal{H}} = L_2$.

Proof: As in Theorem 2, we consider as given an $N \in$ C.S.U. N_1. Then (34) holds, from which it follows

$$\langle \phi_0, \underline{v}^r = \boldsymbol{s} * \underline{v}^i \rangle = 0, \qquad \text{for all } \underline{v}^i \in \underline{\mathcal{H}} \tag{36}$$

or

$$\langle \boldsymbol{s}^a \bullet \phi_0, \underline{v}^i \rangle = 0, \qquad \text{for all } \underline{v}^i \in \underline{\mathcal{H}}. \tag{37}$$

Hence, we have shown 1)

$$\boldsymbol{s}^a \bullet \phi_0 = \underline{0}. \tag{38}$$

To show 2), we operate with $(\boldsymbol{s}^a)^k$ for any given $k > 0$ on both sides of (34), which is valid by the completeness of N. Remembering that $\phi_0 \in \underline{\mathfrak{N}}(\boldsymbol{s}^a)$,

$$(\boldsymbol{s}^a)^k \bullet \underline{v}^i = \sum_{i=0}^{\infty} a_{i+k} \phi_i, \qquad \text{for all } \underline{v}^i \in \underline{\mathcal{H}}. \tag{39}$$

Hence

$$\langle (\boldsymbol{s}^a)^k \bullet \underline{v}^i, (\boldsymbol{s}^a)^k \bullet \underline{v}^i \rangle = \sum_{i=0}^{\infty} a_{i+k}^2, \qquad \text{for all } \underline{v}^i \in \underline{\mathcal{H}}. \tag{40}$$

Consequently

$$\lim_{k\to\infty} ||(\boldsymbol{s}^a)^k \cdot \underline{v}^i|| \to \underline{0}, \qquad \text{for all } \underline{v}^i \in \underline{\mathcal{H}} \tag{41}$$

where $||\ ||$ denotes the L_2 norm (13a). Thus, $(\boldsymbol{s}^a)^k \bullet \underline{v}^i$ tends strongly to $\underline{0}$ as n tends to infinity, for all $\underline{v}^i \in \underline{\mathcal{H}}$ [12, p. 58].

Conversely, if 1) holds, then, by Theorem 1, N is a step-up network. Supposing for the moment that the step-up sequence $\{\phi_i\}$ is not complete, then

$$\underline{\mathcal{H}} = \underline{\mathcal{J}} \overset{\perp}{\oplus} \underline{\mathcal{SU}} \tag{42}$$

where $\underline{\mathcal{J}}$ is the orthogonal complement of $\underline{\mathcal{SU}}$, the closed subspace spanned by $\{\phi_i\}$. If $\underline{\alpha} \in \underline{\mathcal{J}}$, then

$$\langle \underline{\alpha}, \phi_j = (\boldsymbol{s})^j \cdot \phi_0 \rangle = 0, \qquad \text{for all } j = 0, 1, \cdots. \tag{43}$$

By the losslessness of $\boldsymbol{s}$, we also have

$$\langle \boldsymbol{s} \cdot \underline{\alpha}, \phi_{j+1} = (\boldsymbol{s})^{j+1} \cdot \phi_0 \rangle = 0, \qquad \text{for all } j = 0, 1, 2, \cdots. \tag{44}$$

Furthermore, $\boldsymbol{s} \cdot \underline{\alpha} \in \underline{\mathfrak{R}}(\boldsymbol{s})$, the range of $\boldsymbol{s}$; therefore, it is also orthogonal to $\underline{\phi}_0 \in \underline{\mathfrak{N}}$. Thus, $\boldsymbol{s} \cdot \underline{\alpha}$ is also in $\underline{\mathcal{J}}$; this shows that $\underline{\mathcal{J}}$ is an invariant subspace of $\boldsymbol{s}$ [8, p. 275].

Some $\underline{v}^i \in \underline{\mathcal{H}}$ can now be written as

$$\underline{v}^i = b\underline{\alpha} + \sum_{i=0}^{\infty} c_i \phi_i. \tag{45}$$

Operating with $(\boldsymbol{s}^a)^k$ on both sides of (45), we have

$$(\boldsymbol{s}^a)^k \cdot \underline{v}^i = b(\boldsymbol{s}^a)^k \cdot \underline{\alpha} + \sum_{i=0}^{\infty} c_{i+k} \phi_i. \tag{46}$$

Then

$$\langle (\boldsymbol{s}^a)^k \cdot \underline{v}^i, (\boldsymbol{s})^k \cdot \underline{v}^i \rangle = b^2 \langle \underline{\alpha}, \underline{\alpha} \rangle + \sum_{i=0}^{\infty} c_{i+k}^2 \tag{47}$$

where, for the first term on the right-hand side, we have made use of the fact that $\boldsymbol{s}$ is an "onto" lossless operator for all $\underline{\alpha} \in \underline{\mathcal{J}}$.

Letting k tend to infinity and if 2) holds, then (47) becomes

$$b^2 \langle \underline{\alpha}, \underline{\alpha} \rangle = 0. \tag{48}$$

This shows that $\underline{\alpha} = \underline{0}$ for all $\underline{\alpha} \in \underline{\mathcal{J}}$; the step-up sequence is therefore complete and N is a complete step-up network.

From the above discussions, we have the following.

Lemma 1

If $\boldsymbol{s}$ is totally lossless and $\underline{y} = \underline{0}$ is the only solution of $\boldsymbol{s}^a \cdot \underline{y} = \underline{0}$, then $\boldsymbol{s}$ maps complete orthonormal sequences into complete orthonormal sequences.

Proof: Let $\{\underline{\psi}_i\}$ be a complete orthonormal sequence in $\underline{\mathcal{H}}$, then

$$\langle \boldsymbol{s} \cdot \underline{\psi}_k, \boldsymbol{s} \cdot \underline{\psi}_l \rangle = \langle \underline{\psi}_k, \underline{\psi}_l \rangle = \delta_{kl} \tag{49}$$

since $\boldsymbol{s}$ is totally lossless. Let $\underline{P}$ be the closed subspace spanned by the orthonormal sequence $\{\boldsymbol{s} \bullet \underline{\psi}_i\}$, and $\underline{P}^\perp$ the orthocomplement of $\underline{P}$, then for all $\underline{y} \in \underline{P}^\perp$

$$\langle \underline{y}, \boldsymbol{s} \cdot \underline{\psi}_i \rangle = 0, \qquad \text{for all } j \tag{50a}$$

or

$$\langle s^a \cdot \underline{y}, \underline{\psi}_i \rangle = 0, \qquad \text{for all} \quad j. \tag{50b}$$

But $\{\underline{\psi}_i\}$ is complete; therefore, we must have

$$s^a \cdot \underline{y} = 0, \qquad \text{for all} \quad \underline{y} \in \underline{P}^{\perp}. \tag{51}$$

Thus if $\underline{y} = \underline{0}$, then from (50a), $\{s \cdot \underline{\psi}_i\}$ is complete and consequently [8, p. 273] s is "onto" lossless.

Examples of this case are $s = \delta(t - \tau)c$ with c a constant orthogonal matrix and

$$s = \begin{bmatrix} 0 & \delta(t-\tau) \\ -\delta(t-\tau) & 0 \end{bmatrix}.$$

We note that if $\underline{y} \neq \underline{0}$ and if $s^a \cdot \underline{y} = \underline{0}$, then by Theorem 1, s is step-up; but by (50), $\langle s \cdot \underline{\psi}_n \rangle$ is not complete. Hence, we have the following.

Lemma 2

If a totally lossless s maps complete orthonormal sequences into incomplete orthonormal sequences, then it is step-up.

Finally we note that, in the above, only the passivity result of Theorem 2 rests upon the time-invariance constraint of the network.

IV. Finite Step-Up n-Ports and Examples

In this section, we consider finite networks and from these give some explicit examples of matrices $s(t, \tau)$ and $S(p)$ yielding step-up sequences; we also indicate how to calculate the first member of these sequences.

By a finite network we will mean an n-port described by equations of the form

$$C(p, t)\underline{v}^r(t) = D(p, t)\underline{v}^i(t) \tag{52}$$

where C and D are $n \times n$ matrices, with $p = d/dt$ the differential operator at this point. Passivity of the network then implies that the scattering matrix $s(t, \tau)$ relating $\underline{v}^i$ to $\underline{v}^r$ takes the form [7]

$$s(t, \tau) = A(\tau)\, \delta(t - \tau) + \Phi(t)\tilde{\Psi}(\tau)u(t - \tau) \tag{53}$$

where elements of the $n \times r$ matrices Φ and Ψ are $L_2(T, \infty)$ and $L_2(-\infty, T)$ functions, respectively, for all finite T, and $u(t - \tau)$ is the unit step function. The columns of Φ are assumed to be linearly independent, since they can always be chosen so. The lossless constraint

$$s^a \circ s = \delta 1_n \tag{15}$$

then implies [18, p. 37]

$$s^a \bullet \underline{\Phi}_k = \underline{0}, \qquad k = 1, \cdots, r \tag{54}$$

for all columns $\underline{\Phi}_k$ of the matrix $\Phi(t)$. The number of linearly independent columns r of Φ is the degree δ of s [18, p. 23], [19, pp. 543, 580], which is also the minimum number of reactive elements in any synthesis of s [18], [20], [21]. In this case, a system of step-up sequences is defined by

$$\underline{\varphi}_0^{(k)} = \underline{\Phi}_k, \qquad k = 1, \cdots, r = \delta \tag{55}$$

as is shown by Theorem 1. We comment that a finite lossless n-port N is at least of the class S.U. N_δ.

In the time-invariant case, $\Phi(t)\tilde{\Psi}(\tau)$ is stationary, that is,

$$\Phi(t)\tilde{\Psi}(\tau) = \Phi(t - \tau)\tilde{\Psi}(0) \tag{56}$$

and one knows that each column is of the form $t^k e^{-at} \cos bt\, \underline{w}$ or $t^k e^{-at} \sin bt\, \underline{w}$, where $\underline{w}$ is a constant vector.

As a simple example, consider the degree one one-port lossless scattering matrix

$$s(t - \tau, 0) = \delta(t - \tau) - 2ae^{-at}e^{a\tau}u(t - \tau) \tag{57a}$$

to which corresponds

$$S(p) = \frac{p - a}{p + a}. \tag{57b}$$

Then ϕ_0 can be taken as $e^{-at}u(t)$ (straightforward calculations will verify that $s^a \circ e^{-at}u(t) = 0$). The successive members of the sequence ϕ_1, ϕ_2, etc. will be such that

$$\phi_i(t) = L_i(2at)e^{-at}u(t) \tag{58}$$

where the $L_i(x)$ are the Laguerre polynomials. In this case, the step-up sequence $\phi_0, \phi_1, \cdots, \phi_i \cdots$ is complete [8, p. 259].

More generally, we may consider an n-port degree one lossless $S(p)$ of the form

$$S(p) = 1_n + \frac{2a}{p + a}\underline{w}\tilde{\underline{w}} \tag{59}$$

where $\underline{w}$ satisfies

$$\tilde{\underline{w}}\underline{w} = 1. \tag{60}$$

Forming

$$s(t - \tau, 0) = \delta(t - \tau)1_n + 2ae^{-a\tau}\underline{w}e^{at}\tilde{\underline{w}}u(t - \tau), \tag{61}$$

we observe from (54) that

$$\phi_0 = e^{-at}u(t)\underline{w}. \tag{62}$$

Corresponding to this, we have

$$\phi_i(t) = L_i(2at)e^{-at}u(t)\underline{w}. \tag{63}$$

In this case, the sequence $\phi_0, \phi_1, \cdots, \phi_i$, is not complete since any L_2 vector of the form $\sum a_i\phi_i$ is of the form $\phi(t)\underline{w}$, where $\phi(t)$ is a scalar.

There are, however, multiport lossless scattering matrices that will define complete sequences. An example of such a matrix is that of the two-port of Fig. 1:

$$S(p) = \frac{1}{p + 2}\begin{bmatrix} p & 2 \\ -2 & -p \end{bmatrix}. \tag{64}$$

We find

$$\tilde{\underline{\phi}}_0(t) = [\sqrt{2}e^{-2t}\ \sqrt{2}e^{-2t}]u(t) \tag{65a}$$

and, with the $L_i(4t)$ Laguerre polynomials,

$$\tilde{\underline{\phi}}_{2i}(t) = [\sqrt{2}e^{-2t}L_i(4t)\ \ \sqrt{2}e^{-2t}L_i(4t)]u(t) \tag{65b}$$

$$\tilde{\underline{\phi}}_{2i+1}(t) = [\sqrt{2}e^{-2t}L_i(4t)\ \ -\sqrt{2}e^{-2t}L_i(4t)]u(t). \tag{65c}$$

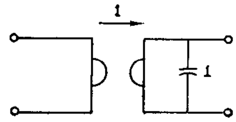

Fig. 1. Complete step-up two-port.

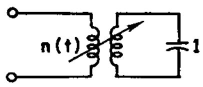

Fig. 2. Complete step-up time-variable one-port.

As a simple example of time-varying step-up networks, we consider the one-port network of Fig. 2 whose scattering matrix is [18, p. 17]

$$s(t, \tau) = \delta(t - \tau) - \varphi(t) \frac{\varphi(\tau)}{\displaystyle\int_\tau^\infty \varphi^2(\lambda)\, d\lambda} u(t - \tau) \tag{66}$$

where $\varphi(t)$ is square-integrable over $[T, \infty)$ for all finite T. It is noted that the time-varying turns ratio of the transformer is

$$n(t) = \varphi(t)\left[2 \int_t^\infty \varphi^2(\lambda)\, d\lambda\right]^{-1/2}.$$

Choosing $\varphi(t) = u(t)/(1 + t)$ and $T = 0$, we find

$$\begin{aligned}
\phi_0(t) &= \frac{u(t)}{1+t}\\
\phi_1(t) &= \frac{u(t)}{1+t}[1 - \log(1+t)]\\
\phi_2(t) &= \frac{u(t)}{1+t}[1 - 2\log(1+t) + \tfrac{1}{2}\log^2(1+t)]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\phi_i(t) &= \frac{u(t)}{1+t}[\text{polynomial of degree } i \text{ in } \log(1+t)].
\end{aligned} \tag{67}$$

It can be easily seen that this sequence can be derived from the Laguerre sequence $\{e^{-t'}L_i(2t')\}$ by the substitution $t' = \log(1 + t)$, and as a consequence it is also complete over $[0, \infty)$.

V. Conclusion

The step-up characterization is an interesting and seemingly important description of a large class of systems. This work has shown that all lossless networks can be classified into "lossless for some" and "totally" lossless classes. Within the latter class, one also has the "onto" (unitary) lossless and the "into" (isometry) lossless or step-up networks. Furthermore, we have shown the interesting property that every step-up network behaves jointly like a unitary operator in the invariant subspace $\underline{\mathcal{I}}$ on the one hand, and a complete step-up operator in the subspace $\underline{\mathcal{S}\mathfrak{U}}$ on the other.

Finally, it is interesting to note how one can physically generate a step-up sequence given a step-up n-port and the first member $\phi_0^{(k)}$, say. Using n-port circulators, this can be accomplished by the network of Fig. 3. Finally, we note that the lossless constraint (15) does hold for more than finite networks, as is seen by the unit delay

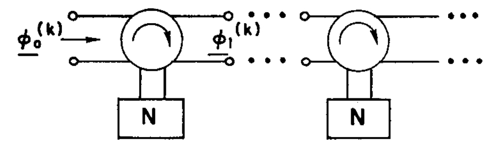

Fig. 3. Physical generation of step-up sequences.

$s(t) = \delta(t - 1)$. For this case, we have

$$s^a \bullet \phi_0 = \phi_0(t + 1) \tag{68}$$

and, as a consequence, we cannot find a function $\phi_0(t)$ with $\phi_0(t + 1) = 0$ for all t that generates a step-up sequence. However, if we choose $\phi_0(t) = \sin \pi t/\pi t$, which is band-limited, then we have the step-up sequence

$$\phi_j(t) = \frac{\sin \pi(t - j)}{\pi(t - j)}, \qquad j = 0, 1, 2, \cdots, \tag{69}$$

which is not complete. Nevertheless, (69), together with the step-down sequence $\sin \pi(t + j)/\pi(t + j)$, does form a complete sequence.

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Equivalent Circuit and Loss Invariant for Helicon Mode Semiconductor Devices

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***Abstract*—A basic nonreciprocal two-port component in the HF range is considered consisting of two spatially orthogonal coils wound on an indium antimonide slab. A dc magnetic field is impressed perpendicular to the axes of the coils. Local helicon mode theory, assuming a single type of carrier, is used to derive the frequency-dependent network matrix of the device. From this matrix a broadband equivalent circuit is synthesized, which contains one gyrator. The resultant network structure is then analyzed to arrive at a loss invariant or figure of merit for the nonreciprocal device and the invariant is found to be frequency independent. This loss invariant is used to calculate the minimum possible insertion loss of an isolator formed by imbedding the helicon mode element in the most general lossless reciprocal circuit. The minimum loss is shown to be**

$$\text{Minimum insertion loss} = 10 \log \left(1 + \frac{1}{u^2}\right) \text{dB},$$

where u is the tangent of the Hall angle in the InSb slab. Two-port experimental measurements were made (using a low-frequency embodiment of a microwave directional coupler) over the frequency band 5–15 MHz to verify the equivalent circuit and the value of the invariant. These data provided reasonable verification of the equivalent circuit, the invariant, and the minimum loss figure. A circuit for a minimum insertion loss isolator was constructed, and at fields of 2600 and 5600 gauss the measured minimum insertion loss at room temperature was found to be 2.8 and 1.5 dB, respectively. This data compares with the theoretically predicted figures of 3.1 and 1.0 dB. These results are all under infinite isolation conditions. A derivation is given to show the optimum benefits to be obtained by trading reduced isolation for improved insertion loss. Finally, the theory of invariance is employed to derive expressions for the minimum insertion loss of a helicon mode gyrator and three-port circulator.

I. Basic Physical Considerations for Helicon Mode Coupled Coils

An isolator is a passive two-port device that allows the flow of power in one direction only. Ideally, it completely absorbs power propagating from port 1 to port 2, but no loss is encountered in the opposite direction. It has been known for some time that room temperature isolators can be built at radio frequencies by utilizing the nonreciprocal helicon effect exhibited by certain semiconductors, which are generally 3–5 compounds and because of high mobility, most effectively indium antimonide [1], [6]. The heart of these isolators is a pair of spatially orthogonal coils wound on a semiconducting core, as in Fig. 1, where InSb is indicated as the core material. Since the axes of the two coils are perpendicular, they are uncoupled in the absence of a dc magnetic field. However, imposing a dc magnetic field $\boldsymbol{B}_0$ normal to the large faces of the semiconductor slab, excites the helicon mode and causes nonreciprocal coupling between the two coils.

We now consider the basic physical mechanism for nonreciprocal coupling between the coils. In 1962, Chambers and Jones [2] published a theory describing this coupling based on a single-type current carrier with isotropic conduction. To approximate the geometry of Fig. 1, consider a thin infinite sheet of helicon semiconductor. A dc magnetic field $\boldsymbol{B}_0$ is impressed perpendicular to the face of the sheet. The RF magnetic field inside the material is calculated when a time-varying linearly polarized magnetic field $\boldsymbol{b} = \boldsymbol{b}_0 e^{j\omega t}$, is imposed parallel to the plane of the sheet. The vector magnetic flux density inside the sample consists of two phasor components, $\boldsymbol{b}_L$ and $\boldsymbol{b}_T$, the first lying in a direction parallel to $\boldsymbol{b}_0$ and the second perpendicular to $\boldsymbol{b}_0$. Two expressions of particular interest are

$$\frac{\mu_L}{\mu_0} = \frac{|\boldsymbol{b}_L|}{|\boldsymbol{b}_0|} = 1 - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \frac{1 + 2jQx_n}{1 + jQ(x_n - x_n^{-1})}, \tag{1}$$

and

$$\frac{\mu_T}{\mu_0} = \frac{|\boldsymbol{b}_T|}{|\boldsymbol{b}_0|} = \frac{4u}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \frac{1}{1 + jQ(x_n - x_n^{-1})}, \tag{2}$$

where[1]

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[1] The properties of the device here depend basically on the macroscopic ac Hall effect. The theory requires a significant Hall field that in turn calls for $\mu = \tan \theta = w_c\tau \geq 1$ where w_c is the electron cyclotron frequency, τ the relaxation time.