SEL-65-020

THE CHARACTERIZATION BY STEP-UP VECTORS OF n-PORT NETWORKS

by

N. Levan (Monash University, Melbourne)

R.W. Newcomb

February 1965

Technical Report No. 6558-1

Prepared under

National Science Foundation Grant NSF-GP 520

Systems Theory Laboratory

Stanford Electronics Laboratories

Stanford University Stanford, California

THE CHARACTERIZATION BY STEP-UP VECTORS
OF n-PORTS NETWORKS

by
N. Levan (Monash University, Melbourne)
R. Newcomb

ABSTRACT

The causes of time here to spend
Are networks which incident send
Base sequences unto themselves
Indexed one higher; our delves
Necessity show you they carry
Matrices claimed unitary
With para- the prefix attached.
Some problems we barely have scratched

I. INTRODUCTION

The concept of an orthonormal step-up set of functions associated with a given fixed linear single input-output system was first introduced by D. G. Lampard and the theory was developed in a 1962 thesis [1] by the first author. Such a system is one for which there exists an ordered basis sequence of square integrable orthonormal functions which when used as inputs to the given system yield outputs consisting of the same sequence with order increased by one. It was shown that such step-up sequences existed when the given system was of the so called "all-pass" type. Further work on these systems was done recently by Lampard and Levan [2], in which the completeness of the step-up set as well as step-up sets associated with higher order all pass systems were discussed.

During a visit to Monash University in early 1964 by the second author, the connection between step-up transfer functions and lossless reflection coefficients was discussed and it was shown that appropriate sequences of orthonormal step-up functions may be found for lossless 1-port systems. Because lossless n-ports have para-unitary scattering matrices, a property which generalizes that of being all pass, extensions to multidimensional systems appeared to be possible and an appropriate theory was investigated.

Although there are presently several open problems of considerable interest, we here present material so far developed with emphasis on multidimensional systems. One-dimensional and scattering matrix results are reviewed in Section II while the real theory begins in Section III where multidimensional definitions are made (primarily in terms of scattering parameters)- a familiarity with functional analysis concepts is helpful in this and later sections. The necessity of the para-unitary constraint is developed in Section IV where two methods of development are given, one of which makes heavy use of scalar products in a time-domain approach. Section V, which is in part based upon elementary transformations and the Smith canonical form for rational matrices, contains methods of sometimes choosing an appropriate input sequence. Section VI gives an extension to nonpara-unitary matrices.

II. REVIEW AND BACKGROUND CONCEPTS

Consider the system of Fig. 1 which is defined by the real-valued impulse response $h(t)$ which maps the (one-dimensional) input $x(t)$ into the (one-dimensional) output $y(t)$ through the convolution relation

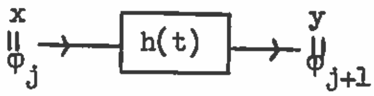

Figure 1
Step-Up System

$$y = h*x \tag{2-1}$$

for any $x\epsilon L_2(0,\infty)$ with $y\epsilon L_2(0,\infty)$, where $L_2(0,\infty)$ is the set of complex-valued square-integrable functions which are zero for $t < 0$. If there exists a sequence $\{\varphi_j\}$, $\varphi_j \epsilon L_2(0,\infty)$, such that

$$\text{if} \quad x = \varphi_j \quad \text{then} \quad y = \varphi_{j+1} = h*\varphi_j \tag{2-2}$$

for all $j = 0,1,2,\ldots$, and such that $\{\varphi_j\}$ is a complete orthonormal sequence in $L_2(0,\infty)$, then the system has been called step-up, and $\{\varphi_j\}$ has been called a natural orthonormal set of step-up functions for the given system. Although we will later slightly change this definition of step-up, by $\{\varphi_j\}$ being orthonormal is meant

$$\int_0^\infty \varphi_j^*(t)\varphi_k(t)dt = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j \neq k \end{cases} \tag{2-3a}$$

where the superscript asterisk denotes complex conjugation, and by completeness is meant that any $\varphi \epsilon L_2(0,\infty)$ has the representation

$$\varphi = \sum_{j=0}^\infty a_j\varphi_j \quad \text{with} \quad \sum_{j=0}^\infty |a_j|^2 < \infty \tag{2-3b}$$

with constant a_j and $| \; |$ denoting the absolute value.

Since a step-up system maps $L_2(0,\infty)$ into $L_2(0,\infty)$

$$H(p) = \mathcal{L}[h] \tag{2-4a}$$

exists in $\operatorname{Re} p > 0$, where $\mathcal{L}[\ ]$ is the Laplace transform. This with the fact [1, p. 14] that H must be all-pass, that is

$$H(p)H(-p) = 1 \tag{2-4b}$$

gives necessary conditions for a system to be step-up. If we let $\Phi_j = \mathcal{L}[\varphi_j]$, an orthonormal set of $L_2(0,\infty)$ functions satisfying Eq. (2-2), but not necessarily complete, can be found from [1, pp. 14-16]

$$\Phi_o(p)\Phi_o(-p) = -\frac{H(p)}{p}\left.\frac{dH(\gamma p)}{d\gamma}\right|_{\gamma = -1} \tag{2-5a}$$

$$\Phi_j(p) = H^j(p)\Phi_o(p) \tag{2-5b}$$

It is worth observing that any two successive members of the sequence $\{\Phi_j\}$ have the transfer function as their ratio.

A comparison of the step-up condition of Eq. (2-4b) with the conditions on lossless scattering matrices calls forth a multidimensional extension. For this consider a linear, solvable, time-invariant n-port $\underset{\sim}{N}$ [3]. One method of describing $\underset{\sim}{N}$, which is essentially the definition of a network, is to list all the allowed n-vector port currents $\underset{\sim}{i}(t)$ with their accompanying voltages $\underset{\sim}{v}(t)$. By writing

$$2\underset{\sim}{v}^i = \underset{\sim}{v} + \underset{\sim}{i} \tag{2-6a}$$

$$2\underset{\sim}{v}^r = \underset{\sim}{v} - \underset{\sim}{i} \tag{2-6b}$$

one can use a similar description in terms of incident and reflected voltages $\underset{\sim}{v}^i$ and $\underset{\sim}{v}^r$. Using these latter variables there always exists a real $n \times n$ time-domain scattering matrix $\underset{\sim}{s}(t)$ such that $\underset{\sim}{N}$ can be described by

$$\underset{\sim}{v}^r = \underset{\sim}{s} * \underset{\sim}{v}^i \tag{2-7}$$

In terms of incident and reflected voltages the lossless constraint is expressed as [2, p. 9]

$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} [\tilde{v}^{i*}(t)v^{i}(t) - \tilde{v}^{r*}(t)v^{r}(t)]dt = 0 \qquad (2\text{-}8)$$

where we introduce the complex conjugate to handle complex excitations. If the (frequency-domain) scattering matrix

$$S(p) = \mathcal{L}[s] \qquad (2\text{-}9)$$

exists then properties of N are easily transferred to properties of S; for instance if N is finite (that is, has a construction using only a finite number of R's, L's, C's, transformers and gyrators) then S is rational in p. Of particular importance is the fact that S(p) exists for all p in Re $p > 0$ for a passive N [3, p. 11] and that a finite (passive) lossless N has S para-unitary, that is satisfying [4, p. 113]

$$\tilde{S}(-p)S(p) = 1_n \qquad (2\text{-}10)$$

Here the superscript tilde denotes matrix transposition and 1_n denotes the $n \times n$ identity matrix.

The analogy between the all-pass constraint for one-dimensional step-up systems and the para-unitary constraint of n-port networks is striking. This analogy will be used in the following sections to develop a theory of step-up multidimensional systems. Since general system results are obtained from network results by replacing S by a transfer function matrix, we will develop the theory for n-ports without any loss of generality.

III. STEP-UP n-PORTS

In this section we precisely define the concepts appropriate to step-up n-ports.

Unless otherwise stated we will assume all time functions to be zero for $t < 0$. To simplify expressions we also define the scalar product $\langle f, g \rangle$ for two n-vectors $f(t) = [f_i]$, $g(t) = [g_i]$ by

$$\langle f, g \rangle = \int_{0}^{\infty} \tilde{f}^{*}(t)g(t)dt \qquad (3\text{-}1)$$

whenever the integral exists; here, as before, the tilde represents the transpose and the superscript asterisk the complex conjugate. If $\underline{f}$ is a vector of distributions and $\underline{g}$ a vector of testing functions then $< \underline{f}, \underline{g} >$ is also well defined and can be conveniently considered [5, p. 6]. The set of square-integrable n-vectors $\underline{L}_2$ is important and defined by $[\varphi_i] = \underline{\varphi}\epsilon\underline{L}_2$ if the φ_i are measurable and

$$< \underline{\varphi}, \underline{\varphi} > < \infty \qquad (3\text{-}2a)$$

A sequence $\{\underline{\varphi}_j\}$, $\underline{\varphi}_j \epsilon \underline{L}_2$, is orthonormal if

$$< \underline{\varphi}_j, \underline{\varphi}_k > \; = \; \delta_{jk} \qquad (3\text{-}2b)$$

and is complete if any $\underline{\varphi}\epsilon\underline{L}_2$ has the representation

$$\underline{\varphi} = \sum_{j=0}^{\infty} a_j \underline{\varphi}_j \quad ; \quad \sum_{j=0}^{\infty} |a_j|^2 < \infty \qquad (3\text{-}2c)$$

where the constants a_j are found by applying Eq. (3-2b) to the scalar product of $\underline{\varphi}$ and $\underline{\varphi}_j$

$$a_j \; = \; < \underline{\varphi}_j, \underline{\varphi} > \qquad (3\text{-}2d)$$

With these preliminaries consider any linear, solvable, time-invariant n-port $\underline{N}$, then the time-domain scattering matrix $\underline{s}(t)$ exists and maps incident voltages $\underline{v}^i(t)$ into reflected ones $\underline{v}^r(t)$ by Eq. (2-7). We further assume that $\underline{s}$ maps any $\underline{v}^i \epsilon \underline{L}_2$ into a $\underline{v}^r \epsilon \underline{L}_2$; in particular this will be the case if $\underline{N}$ is passive [3, p. 11]. As a consequence we can consider an orthonormal sequence $\{\underline{\varphi}_j\}$, $\underline{\varphi}_j \epsilon \underline{L}_2$, with the $\underline{\varphi}_j$ taken as successive incident voltages. If then

$$\boxed{\underline{\varphi}_{j+1} \; = \; \underline{s} * \underline{\varphi}_j} \qquad (3\text{-}3)$$

for all $j=0,1,2,..$, we call $\underline{N}$ a <u>step-up n-port</u> and $\{\underline{\varphi}_j\}$ a sequence of <u>step-up vectors</u> for $\underline{N}$. If there further exists a complete sequence of step-up vectors, $\underline{N}$ is called a <u>complete step-up n-port</u> and these vectors are called a sequence of <u>natural step-up vectors</u>.

Although $\underline{s}$ is an $n \times n$ matrix of distributions, the fact that it maps $\underline{L}_2$ into itself guarantees that the Laplace transform of Eq. (3-3) can be taken to give

$$\underline{\Phi}_{j+1} = S\underline{\Phi}_j \tag{3-4}$$

for $\operatorname{Re} p > 0$ with $\underline{\Phi}_j = \mathcal{L}[\underline{\varphi}_j]$.

IV. PARA-UNITARY CONSTRAINT

In this section we show that each complete step-up n-port necessarily has a para-unitary scattering matrix which, by the completeness, also has a corresponding impedance and admittance matrix.

Consider as given a complete step-up n-port $\underline{N}$. Then $\underline{s}$ and S necessarily exist. By the completeness any $\underline{v}^i \epsilon \underline{L}_2$ and any resulting $\underline{v}^r$ can be written as

$$\underline{v}^i = \sum_{j=0}^{\infty} a_j \underline{\varphi}_j \tag{4-1a}$$

$$\underline{v}^r = \sum_{j=0}^{\infty} b_j \underline{\varphi}_j \tag{4-1b}$$

But as a consequence of

$$\underline{v}^r = \underline{s} * \underline{v}^i, \quad \underline{\varphi}_{j+1} = \underline{s} * \underline{\varphi}_j \tag{4-1c}$$

we immediately conclude (see Eq. (3-2d))

$$< \underline{\varphi}_k, \underline{v}^i > = a_k = b_{k+1} = < \underline{\varphi}_{k+1}, \underline{v}^r > ; \; b_o = 0 \tag{4-2}$$

For $\underline{v}^i, \underline{v}^r \epsilon \underline{L}_2$ we have

$$\mathcal{E}(\infty) = < \underline{v}^i, \underline{v}^i > - < \underline{v}^r, \underline{v}^r > \tag{4-3a}$$

$$= \sum_{j=0}^{\infty} |a_j|^2 - \sum_{j=0}^{\infty} |b_j|^2 \equiv 0 \tag{4-3b}$$

Therefore, by Eq. (2-8), $\underline{N}$ is lossless and the para-unitary constraint of Eq. (2-10) follows

$$\tilde{S}(-p)S(p) = 1_n \tag{4-4}$$

By manipulating inside the scalar product of Eq. (4-2) we can obtain an alternate proof of the para-unitary constraint useful for a later section.

$$< \underset{\sim}{\varphi}_k, \underset{\sim}{v}^i > = < \underset{\sim}{\varphi}_{k+1}, \underset{\sim}{v}^r > \tag{4-5a}$$

$$= < \underset{\sim}{s} * \underset{\sim}{\varphi}_k, \underset{\sim}{s} * \underset{\sim}{v}^i > \tag{4-5b}$$

$$= < (\overset{\vee}{\tilde{\underset{\sim}{s}}}{}^* * \underset{\sim}{s}) * \underset{\sim}{\varphi}_k, \underset{\sim}{v}^i > \tag{4-5c}$$

where

$$\overset{\vee}{\underset{\sim}{s}}(t) = \underset{\sim}{s}(-t) \tag{4-5d}$$

which is of course not zero for $t < 0$. Comparing Eqs. (4-5a) and (4-5c), which hold for all $\underset{\sim}{v}^i \epsilon \underset{\sim}{L}_2$ and all $\underset{\sim}{\varphi}_k$, we conclude

$$\overset{\vee}{\tilde{\underset{\sim}{s}}}{}^* * \underset{\sim}{s} = \delta 1_n \tag{4-6}$$

where the complex conjugation can actually be dropped since $\underset{\sim}{s}$ is real; δ is the unit impulse. Eq. (4-6) is of the given form since we require a distributional relation because of the denseness of testing functions (which are contained in $\underset{\sim}{L}_2$) in the set of distributions (which contain $\underset{\sim}{L}_2$). Taking Laplace transforms again gives the para-unitary constraint of Eq. (4-4). This points out that $\mathfrak{L}[\ ]$ is the bilateral Laplace transform and shows that S for a complete step-up n-port is meromorphic if it is continuous for $p = j\omega$ [6, p. 123].

Besides the fact that S is analytic in $\mathrm{Re}\, p > 0$, as $\underset{\sim}{s}$ maps $\underset{\sim}{L}_2$ into $\underset{\sim}{L}_2$, the para-unitary constraint of Eq. (4-4) is a basic necessary condition for $\underset{\sim}{N}$ to be a complete step-up n-port. However, although Eq. (4-4) is necessary it is not sufficient as is seen by the simple example of $\underset{\sim}{s} = \delta C$ with C a constant orthogonal matrix. Eq. (4-4) does hold for more than finite networks, as is seen by the unit delay $s(t) = \delta(t-\tau)$. In spite of the fact that every para-unitary matrix need not correspond to a complete step-up n-port, every para-unitary matrix, analytic in $\text{Re}\ p > 0$, does map any orthonormal sequence $\{\psi_j\}$, $\psi_j \epsilon L_2$, into another orthonormal sequence $\{s*\psi_j\}$. To see this we form

$$< s*\psi_j, s*\psi_k > = < \psi_j, \psi_k > = \delta_{jk}$$

where the argument of Eqs. (4-5) have been used.

A further necessary constraint can be developed on S for a complete step-up N. This is that an impedance matrix $Z = (1_n + S)(1_n - S)^{-1}$ and an admittance matrix $Y = (1_n - S)(1_n + S)^{-1}$ exist. To see this we show that $\{v_j\}$ and $\{i_j\}$, Eq. (2-6), are dense in L_2 when $\{v_j^i\}$ is a dense sequence in L_2 [6, p. 122]. Thus let, for any $K > 1$,

$$v_j^i = \sum_{k=0}^{K-1} a_k \varphi_{j+k} \tag{4-7}$$

where the a_k are arbitrary constants. Forming

$$2z_a = \delta 1_n + s \tag{4-8a}$$

$$2y_a = \delta 1_n - s \tag{4-8b}$$

we have using the step-up property,

$$v_j = 2z_a * v_j^i = (\delta 1_n + s) * v_j^i \tag{4-9a}$$

$$= \sum_{k=0}^{K-1} a_k \varphi_{j+k} + \sum_{k=0}^{K-1} a_k \varphi_{j+k+1} \tag{4-9b}$$

$$= a_o \varphi_j + \sum_{k=1}^{K} (a_k + a_{k-1}) \varphi_{j+k} - a_K \varphi_{j+K} \tag{4-9c}$$

Similarly

$$i_j = 2y_a * v_j^i = (\delta 1_n - s) * v_j^i \tag{4-9d}$$

$$= a_o \varphi_j + \sum_{k=1}^{K} (a_k - a_{k-1}) \varphi_{j+k} - a_K \varphi_{j+K} \tag{4-9e}$$

Choosing

$$a_k = (-1)^k(1-\frac{k}{K}) \quad \text{for } \underset{\sim}{v}_j \tag{4-10a}$$

$$a_k = 1 - \frac{k}{K} \qquad \text{for } \underset{\sim}{i}_j \tag{4-10b}$$

yields

$$\underset{\sim}{v}_j - \underset{\sim}{\varphi}_j = \frac{-1}{K}\sum_{k=1}^{K}(-1)^k \underset{\sim}{\varphi}_{j+k} \tag{4-11a}$$

$$\underset{\sim}{i}_j - \underset{\sim}{\varphi}_j = \frac{-1}{K}\sum_{k=1}^{K} \underset{\sim}{\varphi}_{j+k} \tag{4-11b}$$

Consequently

$$< \underset{\sim}{v}_j - \underset{\sim}{\varphi}_j, \underset{\sim}{v}_j - \underset{\sim}{\varphi}_j > = 1/K = < \underset{\sim}{i}_j - \underset{\sim}{\varphi}_j, \underset{\sim}{i}_j - \underset{\sim}{\varphi}_j > \tag{4-12}$$

Letting $K \to \infty$ shows that $\underset{\sim}{v}_j \to \underset{\sim}{\varphi}_j$ and $\underset{\sim}{i}_j \to \underset{\sim}{\varphi}_j$ for each j; thus $\{\underset{\sim}{v}_j\}$ and $\{\underset{\sim}{i}_j\}$ are dense in $\underset{\sim}{L}_2$ when $\{\underset{\sim}{v}^i_j\}$ is dense in $\underset{\sim}{L}_2$. As a result we can solve for $\underset{\sim}{v}^i$ given $\underset{\sim}{i}$ or $\underset{\sim}{v}$ by

$$\underset{\sim}{v}^i = (\delta 1_n + \underset{\sim}{s})^{-1} * \underset{\sim}{v} \tag{4-13a}$$

$$\underset{\sim}{i} = 2\underset{\sim}{y}_a * (\delta 1_n + \underset{\sim}{s})^{-1} * \underset{\sim}{v} \tag{4-13b}$$

and

$$\underset{\sim}{v}^i = (\delta 1_n - \underset{\sim}{s})^{-1} * \underset{\sim}{i} \tag{4-13c}$$

$$\underset{\sim}{v} = 2\underset{\sim}{z}_a * (\delta 1_n - \underset{\sim}{s})^{-1} * \underset{\sim}{i} \tag{4-13d}$$

Therefore

$$\underset{\sim}{y} = 2\underset{\sim}{y}_a * (\delta 1_n + \underset{\sim}{s})^{-1} = (\delta 1_n - \underset{\sim}{s}) * (\delta 1_n + \underset{\sim}{s})^{-1} \tag{4-14a}$$

and

$$\underset{\sim}{z} = 2\underset{\sim}{z}_a * (\delta 1_n - \underset{\sim}{s})^{-1} = (\delta 1_n + \underset{\sim}{s}) * (\delta 1_n - \underset{\sim}{s})^{-1} \tag{4-14b}$$

necessarily exist.

In summary, a complete step-up n-port necessarily has an $S(p)$

holomorphic in Re $p > 0$ for which $1_n + S$ and $1_n - S$ are nonsingular, also in Re $p > 0$. As will be seen, example 3, section VII, these conditions are still not sufficient.

V. POSSIBLE CALCULATION OF STEP-UP VECTORS

Although the sufficient conditions for $\underset{\sim}{N}$ to be a step-up n-port are not known, we show here two methods which are sometimes useful for finding step-up vectors. The first method is based upon invariant factors and therefore limited to rational S. The second is a generalization of Eq. (2-5a) and may be useful for nonrational S.

As a preliminary we note that for a sequence of step-up vectors

$$\underline{\Phi}_j = S^j \underline{\Phi}_o \tag{5-1}$$

Consequently we concentrate on finding $\underline{\Phi}_o$, subject to the orthogonality constraint which can be expressed through the Parseval theorem for Fourier transforms [7, p. 70]

$$<\underset{\sim}{\varphi}_i, \underset{\sim}{\varphi}_k> = \delta_{ik} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \tilde{\underline{\Phi}}_i^*(j\omega)\underline{\Phi}_k(j\omega)dj\omega \tag{5-2a}$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \tilde{\underline{\Phi}}_o^*(j\omega)S^{k-i}(j\omega)\underline{\Phi}_o(j\omega)dj\omega \tag{5-2b}$$

If $\underset{\sim}{\varphi}_o(t)$ is real then this can be replaced by

$$\frac{1}{2\pi j}\oint_C \tilde{\underline{\Phi}}_{o*}(p)S^{k-i}(p)\underline{\Phi}_o(p)dp = \delta_{ik} \tag{5-2c}$$

where C is any closed contour traversing the imaginary axis and a subscript asterisk denotes replacement of p by $-p$. In Eq. (5-2c) the integrand must vanish at infinity but this will always be true if $\underset{\sim}{\varphi}\epsilon \underset{\sim}{L}_2$.

Method 1:

Here we rely heavily on the theory of polynomial matrices [8, pp. 262-278]. First we obtain a canonical form for rational para-unitary S and from this choose a $\underline{\Phi}_o$ satisfying Eq. (5-2c).

By multiplication of S by its least common multiple $\lambda(p)$ of denominators, with leading coefficient normalized to unity, λS becomes a polynomial matrix. From the theory of elementary transformations there exist polynomial matrices P and Q_o of constant nonzero determinants such that

$$S = P\Lambda Q_o \tag{5-3a}$$

$$\lambda\Lambda = \text{diag.}\ [h_1, h_2, \ldots, h_n] \tag{5-3b}$$

Here the h_i are the (nonzero) invariant factor of λS and have the property that h_i divides h_{i+1}, written $h_i \mapsto h_{i+1}$. Cancelling common factors it is convenient to write

$$\Lambda = \text{diag.}\left[\frac{\Omega_1}{\lambda_1}, \frac{\Omega_2}{\lambda_2}, \ldots, \frac{\Omega_n}{\lambda_n}\right] \tag{5-3c}$$

where $\Omega_i \mapsto \Omega_{i+1}$, $\lambda_{i+1} \mapsto \lambda_i$. We can see that Λ and Λ_*^{-1} are equivalent, that is that there exist constant determinant polynomial matrices T and R such that

$$T\Lambda R = \Lambda_*^{-1} = \text{diag.}\left[\frac{\lambda_{1*}}{\Omega_{1*}}, \frac{\lambda_{2*}}{\Omega_{2*}}, \ldots, \frac{\lambda_n}{\Omega_n}\right] \tag{5-4}$$

This results from the fact that $\tilde{S}_* = S^{-1}$ or

$$S = P\Lambda Q_o = \tilde{P}_*^{-1}\Lambda_*^{-1}\tilde{Q}_{o*}^{-1} \tag{5-5a}$$

that is

$$\tilde{P}_* S Q_o^{-1} = \tilde{P}_* P\Lambda = \Lambda_*^{-1}\tilde{Q}_{o*}^{-1}Q_o^{-1} \tag{5-5b}$$

Equation (5-4), through the divisibility requirements $\Omega_i \mapsto \Omega_{i+1}$, $\lambda_{i+1} \mapsto \lambda_i$, shows that a reversal of order of Eq. (5-4) yields Eq. (5-3c). Equating term by term

$$\lambda_{1*} = \Omega_n,\ \lambda_{2*} = \Omega_{n-1},\ \ldots,\ \lambda_{n*} = \Omega_1 \tag{5-6a}$$

or finally

$$\Lambda = \text{diag.}\left[\frac{\lambda_{n*}}{\lambda_1}, \frac{\lambda_{n-1*}}{\lambda_2}, \ldots, \frac{\lambda_{1*}}{\lambda_n}\right] \tag{5-6b}$$

If we let

$$\Xi = \text{diag.}\left[\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right] \tag{5-7a}$$

then there exists a permutation matrix A such that

$$\Lambda = \Xi A \Xi_*^{-1} \tilde{A} \tag{5-7b}$$

Incorporating $\tilde{A}$ into Q_o, $Q = \tilde{A}Q_o$, gives the canonical form

$$\boxed{S = P \Xi A \Xi_*^{-1} Q} \tag{5-8}$$

For this decomposition Ξ is unique, but unfortunately P and Q are not. We note that since S is analytic in Re $p \geq 0$, the λ_i are Hurwitz polynomials.

At this point we can choose

$$\boxed{\underline{\Phi}_o = \alpha_m S Q^{-1} \Xi_* \underline{E}_m} \tag{5-9a}$$

where α_m is a real constant to be determined and $\underline{E}_m$ is an n-vector of zeros except for +1 in the m th position. $\underline{\Phi}_o$ should really be indexed with m and serves to define a sequence of step-up vectors if $\lim_{p\to\infty} \underline{\Phi}_o = 0$, that is if $\underline{\Phi}_o \epsilon \underline{L}_2$, which need not always be the case as even $S = 1_n$ shows. We choose

$$\boxed{\alpha_m^2 = \frac{2\pi j}{\int_{-j\infty}^{j\infty} \tilde{\underline{E}}_m \Xi \tilde{Q}_*^{-1} Q^{-1} \Xi_* \underline{E}_m \, d j\omega}} \tag{5-9b}$$

When $\underline{\Phi}_o(\infty) = \underline{0}$ the integral defining α_m exists and yields a real α_m. This choice of α_m automatically gives Eq. (5-2c) when i=k. To see that Eq. (5-2c) is satisfied even when $i \neq k$ first consider $k > i$, then by choosing C to enclose the right half plane

$$\frac{\alpha_m^2}{2\pi j} \oint_C \tilde{\underline{E}}_m \Xi \tilde{Q}_*^{-1} S^{k-i-1} P \Xi A \underline{E}_m \, dp = 0 \tag{5-10a}$$

since the integrand has no poles in $\operatorname{Re} p > 0$. If $k < i$ then

$$\frac{\alpha_m^2}{2\pi j} \oint_C \tilde{\underline{E}}_m \tilde{A} \Xi_* \tilde{P}_* \tilde{S}^{i-k-1} Q^{-1} \Xi_* \underline{E}_m \, dp = 0 \qquad (5\text{-}10b)$$

since C can be chosen to enclose the left half plane where no poles of the integrand occur.

Other $\underline{\Phi}_o$ can be found by multiplying Eq. (5-9a) by arbitrary para-unitary matrices which are analytic in $\operatorname{Re} p > 0$. In the one-dimensional case this method always gives a sequence of step-up vectors since Eq. (5-9a) will have all its zeros at infinity.

Method 2:

Here we look for a sequence of natural step-up vectors by considering expansions of exponentials in terms of the step-up vectors to obtain a generalization of Eq. (2-5a).

Let $\{\varphi_j\}$ be a sequence of natural step-up vectors, then for $\operatorname{Re} p > 0$ we can make the expansion

$$e^{-pt}\underline{E}_m u(t) = \sum_{j=0}^{\infty} a_j \underset{\sim}{\varphi}_j(t) \qquad (5\text{-}11)$$

where $\underline{E}_m$ is as before, all zeros but +1 in the m th position, and $u(t)$ is the unit step function. By Eq. (3-2d)

$$a_j = \langle \underset{\sim}{\varphi}_j, e^{-pt}\underline{E}_m u(t) \rangle \qquad (5\text{-}12a)$$

$$= (\mathcal{L}[\underset{\sim}{\varphi}_j^*])_m = (S^j \mathcal{L}[\underset{\sim}{\varphi}_o^*])_m \qquad (5\text{-}12b)$$

$$= (S^j \underline{\Phi}_o^*(p^*))_m \qquad (5\text{-}12c)$$

where $(\quad)_m$ denotes the m th component of the vector and the fact that s is real has been used.

If we now let

$$\hat{\underline{\Phi}}_o(p) = \underline{\Phi}_o^*(p^*) \qquad (5\text{-}13)$$

and form

$$e^{-\gamma pt}\tilde{\underline{E}}_i \cdot e^{-pt}\underline{E}_k u(t) = e^{-p(1+\gamma)t}\delta_{ik}u(t) \qquad (5\text{-}14a)$$

we get from Eqs. (5-11) and (5-12)

$$e^{-p(1+\gamma)t}\delta_{ik}u(t) = \sum_{j=0}^{\infty}(S^j\hat{\underline{\Phi}}_o)_k e^{-\gamma pt}\tilde{\underline{E}}_i\underline{\varphi}_j(t) \qquad (5\text{-}14b)$$

Integrating yields

$$\frac{\delta_{ik}}{p(1+\gamma)} = \sum_{j=0}^{\infty}(S^j\hat{\underline{\Phi}}_o)_k(S^j(\gamma p)\underline{\Phi}_o(\gamma p))_i \qquad (5\text{-}14c)$$

$$= \sum_{j=0}^{\infty}\left\{\sum_{\ell=1}^{n}S^j_{k\ell}\hat{\Phi}_{o\ell}\right\}\left\{\sum_{m=1}^{n}S^j_{im}(\gamma p)\Phi_{om}(\gamma p)\right\} \qquad (5\text{-}14d)$$

where $S^j_{k\ell}$ is the (k,ℓ) entry in S^j. Expanding the sums

$$\frac{\delta_{ik}}{p(1+\gamma)} = \sum_{j=0}^{\infty}\sum_{\ell=1}^{n}\sum_{m=1}^{n}S^j_{k\ell}(p)\hat{\Phi}_{o\ell}(p)\Phi_{om}(\gamma p)S^j_{im} \qquad (5\text{-}14e)$$

Comparing this with the general matrix expansion

$$A = BCD \qquad (5\text{-}15a)$$

$$A_{ki} = \sum_{\ell=1}^{n}\sum_{m=1}^{n}B_{k\ell}C_{\ell m}D_{mi} \qquad (5\text{-}15b)$$

shows that Eq. (5-14e) is

$$\frac{1}{p(1+\gamma)}1_n = \sum_{j=0}^{\infty}S^j(p)\hat{\underline{\Phi}}_o(p)\tilde{\underline{\Phi}}_o(\gamma p)\tilde{S}^j(\gamma p) \qquad (5\text{-}16a)$$

$$= \hat{\underline{\Phi}}_o(p)\underline{\Phi}_o(\gamma p) + \sum_{j=1}^{\infty}S^j(p)\hat{\underline{\Phi}}_o(p)\tilde{\underline{\Phi}}_o(\gamma p)\tilde{S}^j(\gamma p) \qquad (5\text{-}16b)$$

or

$$\underline{\hat{\Phi}}_o(p)\underline{\tilde{\Phi}}_o(\gamma p) = \frac{1}{p(1+\gamma)}1_n - \sum_{j=1}^{\infty} S^j(p)\underline{\hat{\Phi}}_o(p)\underline{\tilde{\Phi}}_o(\gamma p)\tilde{S}^j(\gamma p) \qquad (5\text{-}16c)$$

Premultiplying by $\tilde{S}_*$ and postmultiplying by $\tilde{S}^{-1}(\gamma p)$ yields

$$\tilde{S}_*(p)\underline{\hat{\Phi}}_o(p)\underline{\tilde{\Phi}}_o(\gamma p)\tilde{S}^{-1}(\gamma p) = \frac{\tilde{S}_*(p)\tilde{S}^{-1}(\gamma p)}{p(1+\gamma)} - \sum_{j=0}^{\infty} S^j(p)\underline{\hat{\Phi}}_o(p)\underline{\tilde{\Phi}}_o(\gamma p)\tilde{S}^j(\gamma p) \qquad (5\text{-}16d)$$

$$= \frac{\tilde{S}_*(p)\tilde{S}^{-1}(\gamma p) - 1_n}{p(1+\gamma)} \qquad (5\text{-}16e)$$

where Eq. (5-16a) has been used to obtain the last expression. Cancelling the S terms on the left of Eq. (5-16d) gives

$$\underline{\hat{\Phi}}_o(p)\underline{\tilde{\Phi}}_o(\gamma p) = \frac{1_n - S(p)\tilde{S}(\gamma p)}{p(1+\gamma)} \qquad (5\text{-}16f)$$

Letting $\gamma = -1$ leads to an indeterminate expression which can be evaluated by using L'Hospital's rule

$$\underline{\hat{\Phi}}_o\underline{\tilde{\Phi}}_{o*} = \frac{\frac{\partial}{\partial\gamma}[1_n - S(p)\tilde{S}(\gamma p)]\big|_{\gamma = -1}}{\frac{\partial}{\partial\gamma}[p(1+\gamma)]\big|_{\gamma = -1}} \qquad (5\text{-}16g)$$

$$= -S \cdot \frac{1}{p} \cdot \frac{\partial\tilde{S}(\gamma p)}{\partial\gamma}\Big|_{\gamma = -1} \qquad (5\text{-}16h)$$

$$= -S \cdot \frac{1}{p} \cdot \frac{d\tilde{S}(\gamma p)}{d\gamma p} \cdot \frac{\partial\gamma p}{\partial\gamma}\Big|_{\gamma = -1} \qquad (5\text{-}16i)$$

$$= -S \frac{d\tilde{S}_*}{dp_*} \qquad (5\text{-}16j)$$

When $\underset{\sim}{\varphi}_o$ is real, Eq. (5-16h) corresponds to Eq. (2-5a), but Eq. (5-16j) can be finally simplified to

$$\boxed{\hat{\underline{\Phi}}_o \tilde{\underline{\Phi}}_{o*} = S\frac{d\tilde{S}_*}{dp}} \tag{5-17}$$

At this point we see that

$$S\frac{d\tilde{S}_*}{dp} = \widetilde{[S\frac{d\tilde{S}_*}{dp}]_*} \tag{5-18}$$

by differentiating $1_n = S\tilde{S}_*$. As a consequence we can factor $S\frac{d\tilde{S}_*}{dp}$ by the Gauss factorization [9, p. 89] when S is rational. Only in the case where $S\frac{d\tilde{S}_*}{dp}$ has rank one will this lead to a vector $\underline{\Phi}_o$, but then the completeness comes into question.

VI. NONLOSSLESS n-PORTS

The preceding results can be generalized to nonpara-unitary matrices by considering two sequences $\{\varphi_j\}$ and $\{\varphi_j'\}$ corresponding to a given s and an associated s' for which

$$\varphi_{j+1} = s*\varphi_j \tag{6-1a}$$

$$\varphi_{j+1}' = s'*\varphi_j' \tag{6-1b}$$

The relation constraining s' to s is defined as that of biorthogonality

$$< \varphi_i, \varphi_k' > = \delta_{ik} \tag{6-2}$$

Using the second method previously given in section IV one finds

$$S' = \tilde{S}_*^{-1} \tag{6-3a}$$

and that further

$$\hat{\underline{\Phi}}_o' \tilde{\underline{\Phi}}_{o*} = S\frac{d\tilde{S}_*'}{dp} \tag{6-4a}$$

$$\hat{\underline{\Phi}}_o \tilde{\underline{\Phi}}_{o*}' = S'\frac{d\tilde{S}_*}{dp} \tag{6-4b}$$

VII. INTERESTING EXAMPLES

Several examples illustrate some of the points of the theory while showing some of the problems still to be solved.

Example 1:

Let

$$S = \begin{bmatrix} 0 & \dfrac{a_*}{a} \\ -\dfrac{b_*}{b} & 0 \end{bmatrix}$$

with a and b Hurwitz polynomials of leading coefficient unity. Then

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_* & 0 \\ 0 & b_* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \quad P \qquad \Xi \qquad A \qquad \Xi_*^{-1} \qquad Q$$

is a useful form, which is canonical only if $b \nrightarrow a$. Then two suitable $\underline{\Phi}_o = \alpha_m S Q^{-1} \Xi_* \underline{E}_m$ are

$$\underline{\Phi}_o = \alpha_1 \begin{bmatrix} 0 & a_*/a \\ -b_*/b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1/a_* & 0 \\ 0 & 1/b_* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \dfrac{\alpha_1}{a} \end{bmatrix}$$

$$\underline{\Phi}_o = \alpha_2 \begin{bmatrix} 0 & a_*/a \\ -b_*/b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1/a_* & 0 \\ 0 & 1/b_* \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{\alpha_2}{b} \\ 0 \end{bmatrix}$$

If either a or b is a constant then one at least of these $\underline{\Phi}_o$ is a constant and $\alpha_m = 0$ results from Eq. (5-9b). Note that if the S multiplier in the expression for $\underline{\Phi}_o$ is deleted, that is if $\underline{\Phi}_o$ is replaced by $\tilde{S}_* \underline{\Phi}_o$, then $\underline{\Phi}_o$ would have poles in the right half plane.

If one calculates $Sd\tilde{S}_*/dp$ for Eq. (5-17) one sees that $Sd\tilde{S}_*/dp$ can be of rank two.

$$S\frac{d\tilde{S}_*}{dp} = \begin{bmatrix} \left(\frac{a_*}{a}\right)\frac{d}{dp}\left(\frac{a_*}{a}\right) & 0 \\ 0 & \left(\frac{b_*}{b}\right)\frac{d}{dp}\left(\frac{b_*}{b}\right) \end{bmatrix}$$

Here $Sd\tilde{S}_*/dp = 0_2$ if a and b are constants and $\underline{\Phi}_o = \underline{0}$ results. If

$$a = p + \alpha, \quad \alpha > 0$$

then

$$\frac{a_*}{a}\frac{d}{dp}\left(\frac{a_*}{a}\right) = \frac{2\alpha}{(p+\alpha)(-p+\alpha)}$$

If

$$a = p^2 + \alpha p + \beta \quad , \quad \alpha > 0, \beta > 0$$

then

$$\frac{a_*}{a}\frac{d}{dp}\left(\frac{a_*}{a}\right) = \frac{\sqrt{2\alpha}\,(p+\sqrt{\beta})}{p^2 + \alpha p + \beta} \cdot \frac{\sqrt{2\alpha}\,(-p+\sqrt{\beta})}{p^2 - \alpha p + \beta}$$

Thus if a and b are of degree one or two $Sd\tilde{S}_*/dp$ is easily factored into matrices of rank two.

Example 2:

Consider the special case of Example 1 where $a = 1$, $b = p^2 + 5p + 6$. Using the theory of equivalent matrices gives as a possible factorization,

$$S = \begin{bmatrix} 0 & 1 \\ -\dfrac{p^2 - 5p + 6}{p^2 + 5p + 6} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{p+5}{60} b_* & -1 \\ \frac{p+5}{60} b_* & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{b} & 0 \\ 0 & \frac{bb_*}{b} \end{bmatrix} \begin{bmatrix} -b_* & b \\ \frac{p-5}{60} & -\frac{p+5}{60} \end{bmatrix}$$

$$= P \Lambda Q_o$$

With $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Q_o$ we have

$$Q^{-1} \Xi_* = \begin{bmatrix} -b/b_* & -\frac{p+5}{60} \\ -1 & -\frac{p-5}{60} \end{bmatrix}$$

Since this has a pole at infinity the choice of $\underline{\Phi}_o$ given in Eq. (5-9a) leads to $\varphi_o \notin L_2$. Consequently some factorizations into the canonical form are unacceptable.

Example 3:

One wonders what the conditions for the existence of a sequence of step-up vectors is given a para-unitary S. Although necessary, it is clear that the existence of an impedance and an admittance is not sufficient to guarantee the existence of step-up vectors, as is shown by $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Any set of step-up vectors for this S is generated by any normalized $\tilde{\underset{\sim}{\varphi}}_o = [\varphi_{o_1}, \varphi_{o_2}]$ giving $\tilde{\underset{\sim}{\varphi}}_1 = [\varphi_{o_2}, -\varphi_{o_1}]$, $\tilde{\underset{\sim}{\varphi}}_{2k-2} = (-1)^{k+1}\tilde{\underset{\sim}{\varphi}}_o$, $\tilde{\underset{\sim}{\varphi}}_{2k-1} = (-1)^{k+1}\tilde{\underset{\sim}{\varphi}}_o$.

VIII. CONCLUSIONS

The step-up characterization is an interesting and seemingly important description of a large class of systems. This work has shown that all systems with para-unitary descriptions must so be considered. By a change of variable from voltages and currents to incident and reflected variables all lossless n-ports have been brought into view. Although the treatment here has been primarily concerned with n-port networks the theory is clearly meaningful for all systems having the step-up property.

Several questions remain open. Perhaps the most important is that of completeness of the step-up vectors found. Nowhere has it been shown, even in the one port case, that the general method of finding step-up vectors will lead to a complete sequence. In the multidimensional situation the question of calculating $\underset{-}{\Phi}_o$ is still somewhat open, as are the general necessary and sufficient conditions for a system to be step-up.

It is interesting to note how one can physically generate a step-up sequence given a step-up n-port and the first member of the sequence $\underset{\sim}{\varphi}_o$ Using n-port circulators this can be accomplished by the network of Fig. 2.

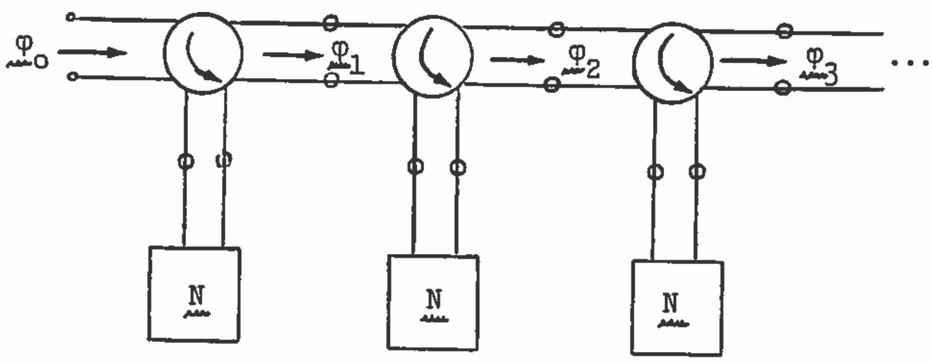

Figure 2

Physical Generation of a Step-Up Sequence

ACKNOWLEDGEMENTS

The authors sincerely wish to acknowledge the kind assistance and enthusiasm of Professor D. G. Lampard (Monash University) in bringing them together. The assistance of Brian Anderson in checking the manuscript and suggesting Fig. 2, as well as the interest of J. Hiller (University of New South Wales) and Professor R. M. Huey (University of New South Wales) in pointing out the first author's work to the second, the support of NSF Grant GP-520 in preparation of the manuscript is also acknowledged. Finally we wish to thank Barbara Serrano for her care in preparation of the manuscript.

REFERENCES

1. N. Levan, "Expansions for the Analysis of Signals and Identification of Networks," The University of New South Wales, School of Electrical Engineering, M.S. thesis, March 1962.
2. N. Levan, "Orthogonal Step-Up Functions in Linear Networks," Monash University, Electrical Engineering Department Report MEE 64-2, Dec. 1964.
3. R. W. Newcomb, "The Foundations of Network Theory," The Inst. of Engineers, Australia, Electrical and Mechanical Engineering Trans., vol. EM6, no. 1, May 1964, pp. 7-12.
4. Y. Oono, "Application of Scattering Matrices to the Synthesis of n-Ports," IRE Trans. on Circuit Theory, vol. CT-3, no. 2, June 1956. pp. 111-120.
5. R. W. Newcomb and R. G. Oliveira, "Laplace Transforms - Distributional Theory," Stanford Electronics Laboratories Technical Report No. 2250-3, June 1963.
6. D. C. Youla, L. J. Castriota, and H. J. Carlin, "Bounded Real Scattering Matrices and the Foundations of Linear Passive Network Theory," IRE Trans. on Circuit Theory, vol. CT-6, no. 1, March 1959, pp. 102-124.
7. N. Wiener, "The Fourier Integral and Certain of Its Applications," Dover, N. Y., 1933.
8. M. Bôcher, "Introduction to Higher Algebra," Macmillan, N. Y., 1907.
9. R. W. Newcomb, "A Bayard-Type Nonreciprocal n-Port Synthesis," IEEE Trans. on Circuit Theory, vol. CT-10, no. 1, March 1963.