SEL-66-100

STATE VARIABLE RESULTS FOR MINIMAL CAPACITOR INTEGRATED CIRCUITS

by

R. W. Newcomb
and
B. D. O. Anderson

October 1966

Technical Report No. 6558-14

Prepared under
National Science Foundation Grant GK-237

Systems Theory Laboratory
Stanford Electronics Laboratories
Stanford University Stanford, California

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It is well recognized that a very practical means of obtaining desirable transmission characteristics with integrated circuits is through the use of operational amplifiers and associated circuitry [1] [2] [3] [4]. In actual fact operational amplifiers are readily available in integrated circuit form [5, p. 67] [6, p. 65] and often have characteristics which are considerably improved over their lumped counterparts. By way of comparison operational amplifiers have long been used in the design of control systems and in analogue computation where their applications are familiar to systems engineers [7]. Such extensive reliance on operational techniques has recently led to rather deep but meaningful mathematical theories of systems some of which we feel should be very applicable to integrated circuit design. In particular we call attention here to the minimal realization procedures which allow synthesis of rational transfer function matrices with resistor-capacitor-operational amplifier circuits using the minimum number of capacitors.

We consider as given an mxn rational transfer function matrix $\underset{\sim}{T}(p)$ of the complex frequency variable p, which, for simplicity, is assumed to have no pole at infinity. This transfer function matrix is to be considered as relating the Laplace transform $\underset{\sim}{U}(p)$ of the input time-domain n-vector $\underset{\sim}{u}(t)$ to the Laplace transform $\underset{\sim}{Y}(p)$ of the output time-domain m-vector $\underset{\sim}{y}(t)$ through

$$\underset{\sim}{Y}(p) = \underset{\sim}{T}(p)\underset{\sim}{U}(p) \qquad (1)$$

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*This work was supported by the Air Force Office of Scientific Research under Contract F44620-67-C-0001 and the National Science Foundation under Grant NSF GK-237.

The techniques of modern system theory then allow one to introduce a k-vector $\underset{\sim}{x}(t)$, called the state, for which there exist four constant matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, $\underset{\sim}{D}$ such that the following equations are satisfied ($\underset{\sim}{1}_k$ is the kxk identity matrix)

$$d\underset{\sim}{x}/dt = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}\underset{\sim}{u} \tag{2a}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\underset{\sim}{x} + \underset{\sim}{D}\underset{\sim}{u} \tag{2b}$$

$$\underset{\sim}{T}(p) = \underset{\sim}{D} + \underset{\sim}{C}[p\underset{\sim}{1}_k - \underset{\sim}{A}]^{-1}\underset{\sim}{B} \tag{2c}$$

The terminology of the theory calls the set of matrices $R = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$ a <u>realization</u> and those realizations for which k assumes its smallest possible value are called <u>minimal</u>. For a minimal realization, the dimension of the state is given by the McMillan degree $\delta[T(p)]$, that is [8, pp. 580-595] [9]

$$k = \delta = \delta[\underset{\sim}{T}(p)] \qquad \text{(for minimal R)} \tag{3}$$

Since, as we show below, Eqs. (2) lead to a physical structure for $\underset{\sim}{T}$ using operational amplifiers and k capacitors (when $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are voltages or currents) it is of interest to find minimal realizations; then the fewest number of possible capacitors will be used. But the procedures for finding minimal realizations, though not simply described, are legion [9] [10, p. 411] [11] [12, p. 547] [13] [14]. Perhaps the simplest method of obtaining a minimal realization follows the ideas almost simultaneously developed by Ho & Kalman [15] and Youla & Tissi [16] [17, pp. 13-21]. Explaining the notation in the constructive procedure which follows, these theories show that a minimal realization is explicitely calculated as

$$A = 1_{\delta,rm} P \Omega S_r Q \tilde{1}_{\delta,rn} \tag{4a}$$

$$B = 1_{\delta,rm} P S_r \tilde{1}_{n,rn} \tag{4b}$$

$$C = 1_{m,rm} S_r Q \tilde{1}_{\delta,rn} \tag{4c}$$

$$D = T(\infty) \tag{4d}$$

The right hand terms are defined as follows: Given an $m \times n$ rational $T(p)$, with $T(\infty)$ finite and well-defined (yielding D), form the least common multiple of all denominators

$$g(p) = a_o p^r + a_1 p^{r-1} + \ldots + a_r \tag{5a}$$

which serves to define r and the constant coefficients $a_o, \ldots, a_r$. Then

$$\Omega = \left[\begin{array}{c|c|c|c} & 1_m & 0 & \\ \cline{2-3} & & 1_m & 0 \\ 0 & 0 & & \\ \cline{3-4} & & \ddots\ 0 & 1_m \\ \hline -\frac{a_r}{a_o} 1_m & -\frac{a_{r-1}}{a_o} 1_m & \cdots & -\frac{a_1}{a_o} 1_m \end{array}\right] \tag{5b}$$

which is a generalized companion matrix. By definition $1_{m,rm}$ is the $m \times rm$ matrix whose first m columns are the $m \times m$ identity matrix 1_m and whose last $(r-1)m$ columns are zero; $\tilde{1}_{n,rn}$ is the transpose after replacing m by n and similarly for $1_{\delta,rm}$ and $\tilde{1}_{\delta,rn}$ where δ, the degree of $T(p)$ is further defined below. To determine S_r, $T(p)$ is expanded about $p = \infty$,

$$T(p) = T(\infty) + \frac{T_o}{p} + \frac{T_1}{p^2} + \ldots \sum_{i=-1}^{\infty} \frac{T_i}{p^{i+1}} \tag{5c}$$

Then $\underset{\sim}{S}_r$ is the generalized Hankel rmxrn matrix

$$\underset{\sim}{S}_r = \begin{bmatrix} \underset{\sim}{T}_o & \underset{\sim}{T}_1 & \cdots & \underset{\sim}{T}_{r-1} \\ \underset{\sim}{T}_1 & \underset{\sim}{T}_2 & \cdots & \underset{\sim}{T}_r \\ \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{T}_{r-1} & \underset{\sim}{T}_r & \cdots & \underset{\sim}{T}_{2r-2} \end{bmatrix} \tag{5d}$$

Finally $\underset{\sim}{P}$, $\underset{\sim}{Q}$ and δ are determined by finding matrices $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ to diagonalize $\underset{\sim}{S}_r$ to $\underset{\sim}{1}_\delta$ and zeros (i.e. δ=rank $\underset{\sim}{S}_r$),

$$\underset{\sim}{P}\underset{\sim}{S}_r\underset{\sim}{Q} = \tilde{\underset{\sim}{1}}_{\delta,rn}\underset{\sim}{1}_{\delta,rm} \tag{5e}$$

Given one minimal realization $R = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$, for example as calculated in Eqs. (4), then all other minimal realizations take the form

$$R_\delta = \{\underset{\sim}{T}_\delta^{-1}\underset{\sim}{A}\underset{\sim}{T}_\delta, \underset{\sim}{T}_\delta^{-1}\underset{\sim}{B}, \underset{\sim}{C}\underset{\sim}{T}_\delta, \underset{\sim}{D}\} \tag{6}$$

where $\underset{\sim}{T}_\delta$ is an arbitrary $\delta x \delta$ nonsingular matrix [11, p. 15] [12, p. 544]. Of course other than minimal realizations exist [i.e. $k > \delta$ is possible for Eq. (2)] but the method of finding all equivalents in the nonminimal case is not as simple as it is for minimal realizations [18].

The interest in minimal realizations for integrated circuits rests on the fact that they allow synthesis using readily available resistor-capacitor-operational amplifier structures incorporating a minimum number of capacitors. Thus, immediately from the state variable equations, Eqs. (2), one can set up the block diagram structure of Fig. 1 [19, p. 390]. In the figure each of the gain blocks $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, $\underset{\sim}{D}$ can be constructed as a multidimensional interconnection of operational amplifiers and resistors, similarly for the two multidimensional summers [19, p. 341] [20, pp. 538-548]. The $\frac{1}{p}\mathbf{1}_k$ block represents k uncoupled integrators each of which can be constructed with one capacitor and an operational amplifier [20, p. 541]; thus when k is minimum, k=δ, the minimum number of capacitors is used. By circuitry imminently suitable to integrated structures, such as differential voltage-to-current convertors [21], the variables $\underline{u}$ and $\underline{y}$ of Fig. 1 can be assumed to be voltages, at least in the electrical case.

However, in the case where $\underline{T}(p)$ is an nxn admittance matrix one can proceed in a somewhat different manner, as suggested by Youla [12, p. 548], by forming the constant (n+k)x(n+k) admittance matrix

$$\underline{Y}_c = \begin{bmatrix} D & -B \\ C & -A \end{bmatrix} \tag{7}$$

from a given realization $R = \{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$. By loading an (n+k)-port resistive network synthesizing $\underline{Y}_c$ in its final k ports by unit capacitors, as shown in Fig. 2, $\underline{T}(p)$ results. Again, when the realization is minimal, k=δ, a minimum number of capacitors is used. Resistors and active gain elements can be used to synthesize $\underline{Y}_c$, but if $\underline{T}(p)$ is positive-real, a transformation of the form used in Eq. (6) can be found to guarantee $\underline{Y}_c$ positive-real, when k=δ [22]. A positive-real $\underline{Y}_c$ can be synthesized with passive resistors and gyrators, the latter being obtainable with passive resistors and operational amplifiers [23].

In summary, using techniques of state-variable realization theory methods of synthesizing rational matrices have been outlined which are inherently convenient for integrated circuit structures since only resistors, operational amplifiers and a minimum number of capacitors need be used. The methods should be contrasted with other recent integrated circuit synthesis techniques [24] where comparison is favorable.

R. W. Newcomb
B. D. O. Anderson
Stanford Electronics Laboratories
Stanford, California

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FIGURE CAPTIONS

1. Operational Simulation of State Equations.
2. Admittance Matrix Synthesis.

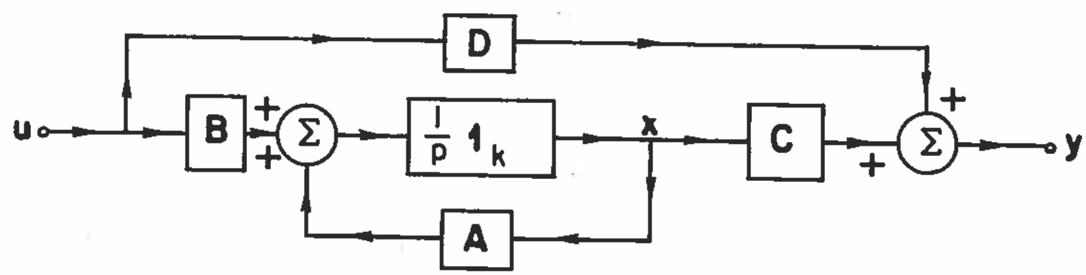

FIGURE I

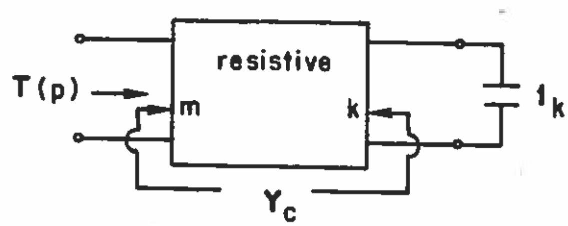

FIGURE 2