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ON THE REALIZATION OF MULTIVARIABLE TRANSFER FUNCTIONS

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FOREWORD

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Robert D. Larson
Chief, Electronic Research Branch
Electronic Technology Division

ABSTRACT

By extending the methods of Youla and Ho it is shown how any rational matrix in ν variables, but finite at infinity, has a state-variable type realization.

CONTENTS

ON THE REALIZATION OF
MULTIVARIABLE TRANSFER FUNCTIONS

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SUMMARY

The realized hope is for man
A fulfillment of all that's desired.
Though partial this goal can succeed,
Hopeless concludes end the learned,
For the variables knows he not.
As with man must a system become
From its variables fixed and prepared;
Here do we show it's the case
When a matrix is rational-real
In its variables more than of two,
The method extends Youla's - Ho's,
Though the open concludes learn the end.

I. INTRODUCTION

Realizations of single variable transfer functions, with emphasis on minimal realizations, are treated quite extensively in the literature[1-5]. As an important extension, Youla[6, p. 12] has given a realization of two-variable transfer functions, these latter being useful for the synthesis of lossless lumped-distributed networks where all transmission-line lengths are rationally related. A similar philosophy has been used by Koga[7] for synthesis of two-variable reactance matrices, this method, or that of Youla[6], also allowing the synthesis of lumped-distributed RC networks[8] of interest for integrated circuit design. Since the theory of general (i.e., more than two variable) multivariable transfer functions should have some importance[9], for example in lumped-distributed synthesis with incommensurate line lengths, a realization theory for such transfer functions seems in order. This paper gives a realization of any real-rational multivariable transfer function, finite at infinity, that is in a certain sense minimal. The method is based upon that of Ho[5], which seems simplest for calculations, with, however, heavy reliance on the ideas of Youla[6].

II. PRELIMINARIES

We consider as given an m x n matrix $W(p_1, \ldots, p_{\nu-1}, p_\nu)$ of ν complex variables $p_1, \ldots, p_\nu$ that is rational in the ν variables with real coefficients; desired W is called real-rational. All capital letters, unless otherwise stated, designate matrices. For simplicity we le

$$\underline{p} = (p_1, \ldots, p_{\nu-1}) \quad , \quad s = p_\nu \quad , \tag{2.1a}$$

and thus

$$W(\underline{p}, s) = W(p_1, \ldots, p_{\nu-1}, p_\nu) \quad . \tag{2.1b}$$

As in previous theories, we assume that $W(\underline{p}, s)$ has no pole at $s = \infty$.

What is desired is to find an expansion

$$W(\underline{p}, s) = J(\underline{p}) + H(\underline{p}) \left[s1_k - F(\underline{p})\right]^{-1} G(\underline{p}) \quad , \tag{2.2a}$$

where 1_k is the $k \times k$ identity, such that the set of matrices

$$R = \{F(\underline{p}), \ G(\underline{p}), \ H(\underline{p}), \ J(\underline{p})\} \tag{2.2b}$$

has all entries real-rational in $\underline{p}$; R is called a realization of W. We might call k the size of the "s-state," but, in any event, when k is the smallest possible the realization is called minimal (in s).

Following Youla[6, p. 13], we write

$$W(\underline{p}, s) = \frac{B_0(\underline{p})s^t + B_1(\underline{p})s^{r-1} + \dots + B_r(\underline{p})}{a_0(\underline{p})s^r + a_1(\underline{p})s^{r-1} + \dots + a_r(\underline{p})} , \tag{2.3}$$

where the denominator polynomial $g(\underline{p}, s) = a_0(\underline{p})s^r + \dots + a_r(\underline{p})$ is the least common multiple of all denominators of W; hence the B's are all polynomial in $\underline{p}$. For a general fixed value of $\underline{p}$ we can also expand $W(\underline{p}, s)$ about $s = \infty$ as

$$W(\underline{p}, s) = A_{-1}(\underline{p}) + \frac{A_0(\underline{p})}{s} + \frac{A_1(\underline{p})}{s^2} + \dots = A_{-1}(\underline{p}) + \sum_{l=0}^{\infty} \frac{A_l(\underline{p})}{s^{l+1}} . \tag{2.4a}$$

Further, if a realization is to exist, we can also expand Equation (2.2a) about $s = \infty$ as

$$W(\underline{p}, s) = J(\underline{p}) + \sum_{l=0}^{\infty} H(\underline{p}) \frac{F^l(\underline{p})}{s^{l+1}} G(\underline{p}) . \tag{2.4b}$$

The program is then to identify the two expansions of (2.4) and from these determine F, G, and H; note that already we have $J(\underline{p}) = A_{-1}(\underline{p})$.

Equating (2.3) with (2.4a), multiplying both sides by $g(\underline{p}, s)$, and equating the resulting powers of $1/s^j$, for $j > 0$, to zero gives[6, p. 14]

$$a_0(\underline{p})\, A_l(\underline{p}) = -\sum_{i=1}^{r} a_i(\underline{p})\, A_{l-i}(\underline{p}) \quad ; l \geq r \quad . \tag{2.5}$$

Equation (2.5) is the starting point for Ho[5, p. 10], following whom we will say that a sequence of matrices $\{A_i(\underline{p})\}$, with $i \geq 0$, has a <u>realization</u> if there exists an $F(\underline{p})$, $G(\underline{p})$, and $H(\underline{p})$ such that

$$A_i(\underline{p}) = H(p)\, F^i(p)\, G(p) \quad ; \quad i \geq 0 \quad . \tag{2.6}$$

Since Equation (2.4a) yields a sequence to be realized and since the existence of F, G, and H yields a realization (Equation (2.2b)), effort will be concentrated on the realization of a sequence. For this, the generalized Hankel matrix $S_r(\underline{p})$, of order r associated with $\{A_i(\underline{p})\}$, has been recognized as important by Youla[10, p. 4] [6, p. 16] and Ho[5, p. 9]:

$$S_r = \begin{bmatrix} A_0 & A_1 & \cdots & A_{r-1} \\ A_1 & A_2 & \cdots & A_r \\ \vdots & \vdots & \ddots & \vdots \\ A_{r-1} & A_r & \cdots & A_{2r-2} \end{bmatrix} \quad . \tag{2.7a}$$

If τ denotes a unit index shift, then

$$\tau S_r = \begin{bmatrix} A_1 & A_2 & \cdots & A_r \\ A_2 & A_3 & \cdots & A_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ A_r & A_{r+1} & \cdots & A_{2r-1} \end{bmatrix} \qquad (2.7b)$$

is also a matrix of importance, as well as $\tau^i S_r$, for which A_i sits in the upper left corner.

III. REALIZATIONS

We follow the outline of Ho[5] and present a preliminary non-minimal realization. For this we introduce the notation

$$1_{\rho,n} = \left[1_\rho \;\vdots\; 0_{\rho,n-\rho} \right] \quad , \qquad (3.1a)$$

where $0_{\rho,n-\rho}$ is the $\rho \times (n-\rho)$ zero matrix and

$$\Omega_m(\underline{p}) = \left[\begin{array}{c|c|c|c|c} 0_m & 1_m & & 0 & \\ & 0_m & 1_m & & 0 \\ \hline 0 & 0 & \ddots & & \\ & & & 0_m & 1_m \\ \hline -\frac{a_r}{a_0}1_m & -\frac{a_{r-1}}{a_0}1_m & \cdots & & -\frac{a_1}{a_0}1_m \end{array}\right] \quad , \qquad (3.1b)$$

$$F(\underline{p}) = 1_{k, rm} B(\underline{p}) [\tau S_r(\underline{p})] C(\underline{p}) \tilde{1}_{k, rn} , \tag{3.6a}$$

$$G(\underline{p}) = 1_{k, rm} B(\underline{p}) S_r(\underline{p}) \tilde{1}_{n, rn} , \tag{3.6b}$$

$$H(\underline{p}) = 1_{m, rm} S_r(\underline{p}) C(\underline{p}) \tilde{1}_{k, rn} , \tag{3.6c}$$

$$J(\underline{p}) = W(\underline{p}, \infty) . \tag{3.6d}$$

(The symbolism is as in Equations (3.5), (3.1a), (2.7) and (2.4).)

Proof:

The procedure is to take the realization of Theorem 1 and reduce its dimension by manipulation of Equations (3.4). For this it is convenient to introduce

$$S_r^{\#}(\underline{p}) = C(\underline{p}) \tilde{1}_{k, rn} 1_{k, rm} B(\underline{p}) , \tag{3.7}$$

which acts as a pseudo-inverse for S_r since

$$S_r S_r^{\#} S_r = S_r , \quad S_r^{\#} S_r S_r^{\#} = S_r^{\#} . \tag{3.8a}$$

For example, the first expression of (3.8a) is shown as follows:

$$S_r S_r^{\#} S_r = B^{-1} \tilde{1}_{k, rn} 1_{k, rm} C^{-1} C \tilde{1}_{k, rn} 1_{k, rm} B B^{-1} \tilde{1}_{k, rn} 1_{k, rm} C^{-1} = S_r . \tag{3.8b}$$

To simplify expansions we recall Equation (3.3b) and note that a shift in the indices of S_r can be effected by also postmultiplying by $\tilde{\Omega}_n^i$, which acts on the columns in the same manner as premultiplication by Ω_m^i acts on the rows:

$$\Omega_m^i S_r = \tau^i S_r = S_r \tilde{\Omega}_n^i . \tag{3.9}$$

Now consider the realization of Theorem 1. We have by Equation (3.4a) (see Ho[5, p. 13]):

$$A_i = 1_{m,rm}(\Omega_m^i S_r)\tilde{1}_{n,rn} \quad (3.10a)$$

$$= 1_{m,rm}\Omega_m^i S_r S_r^{\#} S_r \tilde{1}_{n,rn} \quad (3.10b)$$

$$= 1_{m,rm} S_r \tilde{\Omega}_n^i S_r^{\#} S_r \tilde{1}_{n,rn} \quad (3.10c)$$

$$= 1_{m,rm} S_r S_r^{\#} S_r \tilde{\Omega}_n^i S_r^{\#} S_r \tilde{1}_{n,rn} \quad (3.10d)$$

$$= 1_{m,rm} S_r C\tilde{1}_{k,rn} 1_{k,rm} B S_r \tilde{\Omega}_n^i C\tilde{1}_{k,rn} 1_{k,rm} B S_r \tilde{1}_{n,rn} \quad (3.10e)$$

$$= (1_{m,rm} S_r C\tilde{1}_{k,rn})(1_{k,rm} B S_r \tilde{\Omega}_n^i C\tilde{1}_{k,rn})(1_{k,rm} B S_r \tilde{1}_{n,rn}) \quad (3.10f)$$

$$= (1_{m,rm} S_r C\tilde{1}_{k,rn})(1_{k,rm} B S_r \tilde{\Omega}_n C\tilde{1}_{k,rn})^i(1_{k,rm} B S_r \tilde{1}_{n,rn}) \quad (3.10g)$$

$$= H F^i G \quad . \quad (2.6)$$

Here the step from Equations (3.10f) to (3.10g) is justified, for example, by noting from Equations (3.8a), (3.7) and (3.9) that

$$1_{k,rm} B S_r \tilde{\Omega}_n^2 C\tilde{1}_{k,rn} = 1_{k,rm} B \Omega_m S_r S_r^{\#} S_r \tilde{\Omega}_n C\tilde{1}_{k,rn} \quad (3.11a)$$

$$= 1_{k,rm} B S_r \tilde{\Omega}_n C\tilde{1}_{k,rn} 1_{k,rm} B S_r \tilde{\Omega}_n C\tilde{1}_{k,rn} \quad . \quad (3.11b)$$

From Equation (3.10g) we see that Equation (3.6) gives a realization; it only remains to show that the realization is minimal.

Let

$$P(\underline{p}) = [\tilde{H}(\underline{p}) | \tilde{F}(\underline{p})\tilde{H}(\underline{p}) | \ldots | \tilde{F}^{r-1}(\underline{p}) H(\underline{p})] \quad , \quad (3.12a)$$

and

$$Q(\underline{p}) = [G(\underline{p}) | F(\underline{p}) G(\underline{p}) | \ldots | F^{r-1}(\underline{p}) G(\underline{p})] \quad , \quad (3.12b)$$

which are analogous to the ordinary observability and controllability matrices[11, pp. 500, 504]. Then by direct multiplication, using Equations (2.6) and (2.7), we find that

$$S_r(\underline{p}) = \widetilde{P}(\underline{p})\, Q(\underline{p}) \qquad (3.13)$$

Now suppose that there exists a realization with F of size $k_0 \times k_0$ with $k_0 < k = \text{rank } S_r(\underline{p})$. But this supposition gives a contradiction since

$$\text{rank } S_r \leq \min [\text{rank } P, \text{ rank } Q] \leq k_0 < k = \text{rank } S_r \quad , \qquad (3.14)$$

where the middle inequality follows from P being $k_0 \times (mr)$ and Q being $k_0 \times (rn)$.

Q.E.D.

It should be observed, by noting the number of rows and columns of P and Q , that

$$\text{rank } P(\underline{p}) = \text{rank } Q(\underline{p}) = \text{rank } S_r(\underline{p}) = k \quad , \qquad (3.15)$$

which gives an algebraic demonstration of the size of the ranks of the observability and controllability matrices. Using ideas identical to those of Youla[6, pp. 22, 26], one can show that

$$k = \text{rank } S_r = \delta_s [W(\underline{p}, s)] \quad , \qquad (3.16a)$$

where $\delta_s[W]$ is the degree of W in s called the s-degree and defined by

$$\delta_s[W(\underline{p}, s)] = \max_{\underline{p}_0} \delta [W(\underline{p}_0, s)] \quad , \qquad (3.16b)$$

where $\underline{p}_0$ is "fixed" and $\delta\ [W(\underline{p}_0, s)]$ is the degree of McMillan [12, pp. 580-595].

IV. TRANSFORMATION OF MINIMAL REALIZATIONS

As with the scalar theory, we can find all minimal realizations by the use of similarity transformations. Again the procedure follows that of Ho[5, p. 15]

Theorem 3:

Any two minimal realizations $R_1 = \{F_1(\underline{p}), G_1(\underline{p}), H_1(\underline{p}), J(\underline{p})\}$ and $R_2 = \{F_2(\underline{p}), G_2(\underline{p}), H_2(\underline{p}), J(\underline{p})\}$ are related by a nonsingular $T(\underline{p})$ through

$$F_2(p) = T(p)\, F_1(p)\, T^{-1}(p) \;,$$

$$G_2(\underline{p}) = T(\underline{p})\, G_1(\underline{p}) \;,$$

$$H_2(\underline{p}) = H_1(\underline{p})\, T^{-1}(\underline{p}) \;, \tag{4.1}$$

where T is given by

$$T(\underline{p}) = [P_2(\underline{p})\, \widetilde{P}_2(\underline{p})]^{-1}\, P_2(\underline{p})\, \widetilde{P}_1(\underline{p}) \tag{4.2a}$$

$$= Q_2(\underline{p})\, \widetilde{Q}_1(\underline{p})\, [Q_1(\underline{p})\, \widetilde{Q}_1(\underline{p})]^{-1} \;, \tag{4.2b}$$

and P_1, P_2, Q_1, and Q_2 are as in Equations (3.12) with appropriate subscripts.

Proof:

We can rewrite Equations (4.1) as

$$T\, G_1 = G_2 \;,\quad T\, F_1 = F_2\, T \;,\quad H_1 = H_2\, T \;. \tag{4.3a}$$

Multiplying the middle equation by G_1 on the right and then by $F_1 G_1$, etc. gives:

$$T[G_1 \mid F_1 G_1 \mid \dots \mid F_1^{r-1} G_1] = [G_2 \mid F_2 G_2 \mid \dots \mid F_2^{r-1} G_2] \quad , \tag{4.3b}$$

or

$$T Q_1 = Q_2 \quad . \tag{4.3c}$$

By Equation (3.15), Q_1 is of rank k and hence has a left inverse; this can be exhibited explicitly. Thus, we multiply Equation (4.3c) on the left by $\widetilde{Q}_1$ and then note that $Q_1 \widetilde{Q}_1$ is nonsingular. The non-singularity of Q_1 is seen by noting that there is a permutation matrix P_0 (which then satisfies $P_0 \widetilde{P}_0 = 1_{rn}$) such that $Q_1(\underline{p}) = [A(\underline{p}) \mid D(\underline{p})] P_0$ with $A(\underline{p})$ nonsingular. Then we find that

$$Q_1 \widetilde{Q}_1 = [A \mid D]\, P_0 \widetilde{P}_0 \begin{bmatrix} \widetilde{A} \\ \widetilde{D} \end{bmatrix} = [A \widetilde{A} + D \widetilde{D}] \quad .$$

But, by observing the behavior for real $\underline{p}$ we see that $A \widetilde{A} + D \widetilde{D}$ is nonsingular, being the sum of a positive definite matrix and a positive semi-definite one. Consequently, $T Q_1 \widetilde{Q}_1 = Q_2 \widetilde{Q}_1$ yields the T of Equation (4.2b); similar arguments based upon left multiplication of Equation (4.3a) by H_2, etc. yield Equation (4.2a).

The T of Equation (4.2b) then satisfies, by its construction, $T G_1 = G_2$ and $T F_1 = F_2 T$; we need to see that $H_1 = H_2 T$ and that T is nonsingular. The nonsingularity of T is established by noting that

$$S_r = \widetilde{P}_1 Q_1 = \widetilde{P}_2 Q_2 = \widetilde{P}_2 T Q_1 \tag{4.4}$$

by Equations (3.13) and (4.3c). But $k = \text{rank } S_r = \text{rank } P_2 = \text{rank } Q_r$ by Equation (3.15), and hence $(\text{rank } T) \geq k$ by Equation (4.4); since

T is k x k , its rank must be k . To see that $H_1 = H_2 T$, we first note that the two expressions for T of Equations (4. 2) are identical as is seen by, from Equation (4. 4), $P_2 \tilde{P}_1 Q_1 \tilde{Q}_1 = P_2 \tilde{P}_2 Q_2 \tilde{Q}_1$ and multiplication by $(P_2 \tilde{P}_2)^{-1}$ and $(Q_1 \tilde{Q}_1)$ on the left and the right. But the first expression for T , Equation (4. 2a), satisfies $H_1 = H_2 T$ by construction.

Q. E. D.

V. DISCUSSION

This work has shown, by a constructive method, that any transfer function m x n matrix $W(\underline{p}, s)$ that is rational in ν variables $\underline{p} = (p_1, \ldots, p_{\nu-1})$ and $s = p_\nu$, with real coefficients, has a realization of minimal size for any one of the variables (chosen as p_ν). That is, given $W(\underline{p}, s)$ there exist four matrices F , G , H , and J such that a new (m+k) x (n+k) matrix

$$M(\underline{p}) = \begin{bmatrix} J(\underline{p}) & -H(\underline{p}) \\ G(\underline{p}) & -F(\underline{p}) \end{bmatrix} \qquad (5.1)$$

is also rational in $\underline{p}$ with real coefficients and there is the possibility of choosing $k = \delta_s[W(\underline{p}, s)]$, the minimum size.

Considering $M(\underline{p})$ as an impedance matrix loaded in the impedance $s1_k$, we see that the input impedance is

$$W(\underline{p}, s) = J(\underline{p}) + H(\underline{p})[s1_k - F(\underline{p})]^{-1} G(\underline{p}) \quad . \qquad (2.2a)$$

This shows the meaning of the theory; that is, a realization gives a method of extracting one type of element, corresponding to $s1_k$, from $W(\underline{p}, s)$ to yield $M(\underline{p})$, on which the procedure can further be applied. Alternatively, if we drop the two minus signs from $M(\underline{p})$ of Equation (5.1), we can interpret $W(\underline{p}, s)$ and $M(\underline{p})$ as scattering matrices. In such physical situations, several other properties need to be obtained, or established, for the result to be of much practical use. For example, one would like to obtain

$$\delta_{p_i}[M(\underline{p})] = \delta_{p_i}[W(\underline{p}, s)] \quad . \tag{5.2}$$

But even the determination of the degree of $M(\underline{p})$ seems difficult. Nevertheless, when $\nu = 2$, and when W and M (on deletion of the minus signs) are para-unitary, Youla[6, p. 39] has shown how to obtain Equation (5.2). Likewise, one would like to have $M(\underline{p})$ holomorphic in the same region as $W(\underline{p}, s)$, but generally rational $B(\underline{p})$ and $C(\underline{p})$ are required at condition (3.5), in which case it appears that not much can be said about the singularities of $M(\underline{p})$. Still, when $\nu = 2$, we have the Smith-McMillan canonical form for S_r obtainable by polynomial matrices with polynomial inverses, in which case $M(\underline{p})$ can be guaranteed holomorphic in $\operatorname{Re} p_i \geq 0$ if $W(\underline{p}, s)$ is[6, p. 22].

A practical case of much interest concerns lossless immittance $W(\underline{p}, s)$, that is, those for which $W(\underline{p}, s) = -\widetilde{W}(-\underline{p}, -s)$, or in shorthand notation $W = -\widetilde{W}_*$ (where the sub $*$ denotes replacement of all variables by their negatives). In the lossless case it is easy to verify that if M of Equation (5.1) defines a realization, then so does $-\widetilde{M}_*$. This being the case, Theorem 3 shows that there is a T such that

$$-\widetilde{M}_* = (1_n \dotplus T)\, M\left(1_n \dotplus T^{-1}\right) \quad . \tag{5.3}$$

Now, W being lossless, we would like to find an M_l of the form of Equation (5.1) such that M_l is lossless, i.e. $M_l = -\widetilde{M}_{l_*}$. But again, any such minimal M_l must come from M through

$$M_l = \left(1_n \dotplus L\right) M\left(1_n \dotplus L^{-1}\right) \quad . \tag{5.4}$$

Insertion of Equation (5.3) into (5.4) shows that we wish to obtain the factorization

$$T = \widetilde{L}_* L \quad . \tag{5.5a}$$

From an insertion of Equation (5.3) into itself, we find that

$$T = \widetilde{T}_* \quad . \tag{5.5b}$$

It also appears (but is not yet shown)[6, p. 34] that $T(\underline{p})$ is a non-negative definite matrix for all variables arbitrarily imaginary, i.e. $\underline{p} = j\underline{\Omega}$ (we write $T(j\underline{\Omega}) \geq 0$). Consequently, it is desirable to have a factorization for matrices $\widetilde{T}_* = T$, with $T(j\underline{\Omega}) \geq 0$ to obtain L, which would also be of use for extracting resistors to give non-lossless syntheses in terms of lossless ones[13, p. 8].

In contrast to the one-variable case where several real-rational factorizations are available[14, 15, p. 89], real-rational factorizations for ν-variable $T = \tilde{T}_*$ with $T(j\underline{\Omega}) \geq 0$ are presently unavailable. Using the Gauss method[15, p. 89] we can obtain for such a $T(\underline{p})$:

$$T = \tilde{A}_* D^{-1} A \quad , \tag{5.6}$$

where $A(\underline{p})$ is real-polynomial in $\underline{p}$ and $D = D_*$ is a diagonal matrix with $D(j\underline{\Omega}) \geq 0$. The problem is then reduced to the factorization of D, this being equivalent to the factorization of a scalar $d(\underline{p})$. If we require a scalar factorization of $d(p) = d(-p)$ with $d(j\underline{\Omega}) \geq 0$, as really needed for Equation (5.6), we must generally go outside the domain of real-rational factors, as is seen by $d(p_1, p_2) = -p_1^2 - p_2^2 = (p_1+jp_2)(-p_1+jp_2)$. Consequently, it appears that in some physical situations other constraints must be placed upon matrices T, satisfying $T = \tilde{T}_*$ and $T(j\underline{\Omega}) \geq 0$, to allow the factorization of Equation (5.6).

In summary, although a type of minimal realization for any $W(\underline{p}, s)$ has been obtained, this raises more questions than it solves.

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