and

$$ab - db - fb - fc - dc - ac -,$$

respectively. Again, since each edge of $T^{2,3}(G)$ is contained either in H_3 or H_4, it follows that any edge of $T^{2,3}(G)$ can be made part of a Hamilton circuit of $T^{2,3}(G)$.

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Constant Resistance, Wide-Sense Solvability, and Self-Duality

Using his concept of system function, Zadeh has shown that every self-dual one-port made of linear time-varying elements is a constant resistance one-port.[1] Recently we gave instances of constant resistance one-ports that have nonlinear, time-varying elements. Some of these one-ports are self-dual networks.[2]–[4] Here we give a precise condition for the truth of the statement, "Every self-dual one-port is constant resistance and conversely every constant resistance one-port is self-dual." That some additional condition is required for this statement to be true is easily shown by an example [see Remark e)]. This statement has recently acquired more importance since wide classes of self-dual one-ports can easily be generated.[4]–[5] This correspondence is an extension of a previous paper[6] in that we adopt exclusively a black-box point of view, and it proves the equivalence completely.

We assume throughout that all one-ports under consideration have been created at $t = -\infty$ and that at the time of their creation they are in their zero state. Similarly, any interconnection of one-ports is assumed to be done at $-\infty$. As a consequence, *all* waveforms under consideration are defined on $(-\infty, \infty)$.

By definition, a one-port $\mathfrak{N}$ is specified as the set of all voltage current pairs $[v(\cdot), i(\cdot)]$ it allows. A one-port $\mathfrak{N}^*$ is said to be the *dual* of $\mathfrak{N}$ whenever the following condition holds: $[f, g] \in \mathfrak{N}^*$ if, and only if, $[g, f] \in \mathfrak{N}$. A one-port $\mathfrak{N}$ is said to be *self-dual* whenever $[f, g] \in \mathfrak{N}$ implies $[g, f] \in \mathfrak{N}$. This point of view amounts to thinking of a one-port as a binary relation on some function space (see Harrison, p. 9[7]) the converse relation is the dual one-port; a one-port is self-dual if, and only if, its defining relation is symmetric. Given a one-port $\mathfrak{N}$, we define the *augmented one-port* $\mathfrak{N}_a$ by the pairs: $[v + i, i] \in \mathfrak{N}_a$ when, and only when, $[v, i] \in \mathfrak{N}$; $\mathfrak{N}_a$ has an obvious interpretation given in Fig. 1(a). Any voltage $e(\cdot)$ such that $e = v + i$ for some $[v, i] \in \mathfrak{N}$ is called an *allowed* voltage of $\mathfrak{N}_a$. We now slightly extend the concept of solvability (see Youla et al., p. 113,[8] and Newcomb, p. 9[9]) by considering only a restricted class of $e(\cdot)$'s, namely those allowed by $\mathfrak{N}_a$. $\mathfrak{N}$ is said to be *wide-sense solvable* (abbreviated as w.s. solvable) if for all allowed $e(\cdot)$, the equation $i(\cdot) + v(\cdot) = e(\cdot)$ has a unique solution $[v(\cdot), i(\cdot)] \in \mathfrak{N}$. Physically, w.s. solvability means that if a voltage source (whose voltage $e(\cdot)$ is an *allowed* voltage of $\mathfrak{N}_a$) is connected to $\mathfrak{N}_a$, then the port voltage and port current of $\mathfrak{N}$ are uniquely determined. Note that the nullator is not solvable in the sense of Youla et al.,[8] and Newcomb[9] but is w.s. solvable.

Manuscript received October 10, 1966; revised March 28, 1967. The research reported here was supported in part by the National Science Foundation under Grant GK-716, in part by the U. S. Army Research Office—Durham under Grant DA-ARO-D-31-124-G576 (both grants to the Electronics Research Laboratory, University of California, Berkeley) and in part by the Air Force Office of Scientific Research under Grant AF-AFOSR-337-63 to Stanford University, Stanford, Calif.

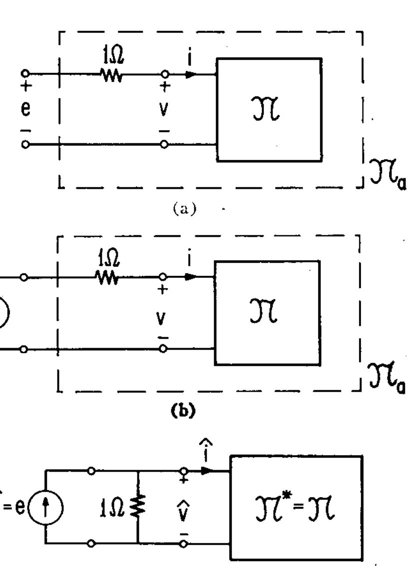

Fig. 1. (a) Physical relation between e, i, and v. (b) Circuit required for testing solvability. (c) The dual of (b). $\mathfrak{N}^*$, the dual of $\mathfrak{N}$, is identical to $\mathfrak{N}$ since $\mathfrak{N}$ is self-dual.

When we consider the one-port $\mathfrak{N}$ as a constant resistance one-port, we only allow $\mathfrak{N}$ to be connected to one-ports $\mathfrak{N}'$ such that the connection $\mathfrak{N} - \mathfrak{N}'$ is determinate, i.e., the port voltage $v(\cdot)$ and the port current $i(\cdot)$ of $\mathfrak{N}$ are uniquely determined. Such one-ports $\mathfrak{N}'$ are said to be *compatible* with $\mathfrak{N}$. If $\mathfrak{N}$ is w.s. solvable, then the series connection of a one-ohm resistor and a voltage source e where e is an allowed voltage of $\mathfrak{N}_a$ is a one-port compatible with $\mathfrak{N}$. If all connections $\mathfrak{N} - \mathfrak{N}'$ where $\mathfrak{N}'$ is compatible with $\mathfrak{N}$ have the property that the port voltage $v(\cdot)$ (of $\mathfrak{N}$) is equal to the port current $i(\cdot)$ (of $\mathfrak{N}$), we say that $\mathfrak{N}$ is *constant resistance*. By including a scale factor, this definition can be extended to include the case where for all such connections, $v(\cdot) = ki(\cdot)$, where k is a fixed nonzero real number independent of $i(\cdot)$, $v(\cdot)$, and t. We want now to prove the following theorem.

Theorem: A one-port $\mathfrak{N}$ is constant resistance if, and only if, $\mathfrak{N}$ is w.s. solvable and self-dual.

Proof: 1) Wide-sense solvability and self-duality imply constant resistance. Let $\mathcal{K}(v)$ denote any member of $\{\imath : [v, \imath] \in \mathfrak{N}\}$; $\mathcal{K}$ is not necessarily a function but describes the relation defining $\mathfrak{N}$. From Fig. 1(b), and the w.s. solvability assumption, the equation

$$e = v + \mathcal{K}(v) \tag{1}$$

has a unique solution for all allowed e. Figure 1(c) shows the dual of Fig. 1(b); then, with the notations shown in Fig. 1(c), $\hat{\imath} = v$ and $\hat{v} = i$, by duality. By self-duality, $i = \mathcal{K}(v)$ implies $v = \mathcal{K}(i)$, or what is the same $\hat{\imath} = \mathcal{K}(\hat{v})$. From Fig. 1(c), KCL gives $\hat{j} = e = \hat{v} + \hat{\imath}$, hence

$$e = \hat{v} + \mathcal{K}(\hat{v}). \tag{2}$$

Since for all allowed e, this equation has a unique solution, (1) and (2) imply that $v = \hat{v}$. Hence, $v = i$ and the one-port $\mathfrak{N}$ is equivalent to a one-ohm resistor when it is driven by any allowed voltage source in series with a one-ohm resistor. That it is equivalent to a one-ohm resistor under all *compatible* connections is obvious by contradiction; suppose it were not true, then there would exist a compatible one-port $\mathfrak{N}'$ such that the connection $\mathfrak{N} - \mathfrak{N}'$ has a solution $[\bar{v}, \bar{\imath}]$ with $\bar{v} \neq \bar{\imath}$. Now consider $\mathfrak{N}_a$ driven by the allowed voltage source $\bar{e} \triangleq \bar{v} + \bar{\imath}$: by the w.s. solvability assumption and the definition of $\bar{v}$, $\bar{\imath}$ there is only one possible port voltage and port current, namely, $\bar{v}$ and $\bar{\imath}$. But the previous proof requires $\bar{\imath} = \bar{v}$. This is a contradiction, hence $\mathfrak{N}$ is equivalent to a one-ohm resistor under all compatible connections, i.e., $\mathfrak{N}$ is constant resistance.

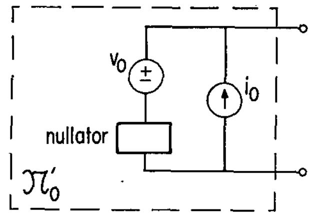

Fig. 2. $\mathfrak{N}'_0$ is compatible with $\mathfrak{N}$, and $\mathfrak{N}'_0$ has only one admissible voltage current pair, namely $[v_0, -i_0]$.

2) Constant resistance implies self-duality and w.s. solvability. Let $[v_0, i_0]$ be an arbitrary pair of $\mathfrak{N}$. Consider the one-port $\mathfrak{N}'_0$ shown in Fig. 2; the current source i_0 and the voltage v_0 of $\mathfrak{N}'_0$ are independent sources; the nullator admits only the pair [0, 0]. By KCL, KVL, and the defining relations of the elements of $\mathfrak{N}'_0$, the one-port $\mathfrak{N}'_0$ admits only one pair $[v_0, -i_0]$. The connection $\mathfrak{N} - \mathfrak{N}'_0$ has a unique solution: $[v_0, i_0]$, i.e., $\mathfrak{N}'_0$ is compatible with $\mathfrak{N}$. By the constant resistance assumption, $v_0 = i_0$. Thus we have shown that, for all $[v, i] \varepsilon \mathfrak{N}$, $v = i$. This implies that $\mathfrak{N}$ is self-dual. Given any allowed voltage e, the only solution of $e = v + i$, with $[v, i] \varepsilon \mathfrak{N}$, is $v = i = (e/2)$, i.e., $\mathfrak{N}$ is w.s. solvable. Q.E.D.

Thus, the proof of the theorem shows that our definition of constant resistance one-port is equivalent to the following: "$\mathfrak{N}$ is said to be constant resistance if, and only if, $v = i$ for all $[v, i] \varepsilon \mathfrak{N}$." We prefer our definition since it allows a priori more possibilities; therefore, it is a nontrivial fact that it is equivalent to the one stated previously.

Remarks

a) By interpreting all voltages and all currents as n-vectors one sees that all definitions and derivations are still valid, consequently the theorem holds for n-ports.

b) It should be stressed that the point of view adopted in this paper is strictly black box; only the port voltage and the port current are observable and the set of all pairs $[v, i]$ constitute the complete description of the one-port. An immediate consequence is that the theorem applies to any one-port: its elements may be lumped or distributed, active or passive, linear or nonlinear, time-varying or time-invariant. On the other hand one should keep in mind that the black-box self-duality defined here does not imply, for example, that the graph of the network inside the box is a self-dual graph. For example, the linear time-invariant network of Fig. 3 of a previous paper [3] is self dual in the present (black-box) sense but its graph is not a self-dual graph.

c) Given an arbitrary one-port $\mathfrak{N}$ and its dual $\mathfrak{N}^*$ (as defined in this correspondence), it is possible to use $\mathfrak{N}$ and $\mathfrak{N}^*$ as elements to obtain constant resistance one-ports (see Desoer and Wong, Section III, Examples 1 and 2[4]).

d) Let a be a fixed real number. If, in the one-port shown in Fig. 2, we set $v_0(t) = -i_0(t) = a$ for all t, we then obtain a constant resistance one-port; indeed, its only pair is $[a, a]$. With $a = 0$, we see that the nullator is a constant resistance one-port.

e) The following one-port $\mathfrak{N}_1$ shows that self-duality implies neither constant resistance nor w.s. solvability. Let $\mathfrak{N}_1$ admit only constant voltages and currents and let its admissible pairs be $[V, I]$ where either $V = 2I$ or $V = 2^{-1} I$. $\mathfrak{N}_1$ is clearly self-dual but neither constant resistance nor w.s. solvable.

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A Simple Calculation of the Determinant Polynomial of General Networks

Introduction

In the analysis of lumped element networks with controlled sources the polynomials of the determinant or minors of the network have to be calculated. If the stability of the network has to be checked, special attenuation has to be given to the accuracy of the coefficients of these polynomials. For large networks with a high degree the matrix describing the network may contain elements which are rational functions in the frequency parameter s, thus the evaluation of the determinant becomes unwieldy. In the following, two distinct methods are described which avoid complicated manipulations of the matrix elements. In both cases the coefficients of the pertinent polynomials are calculated by comparatively simple numerical operations.

First Method

A network is described by the equation[2],[6],[7]

$$A(s)\mathbf{X}(s) = \mathbf{E}(s) \tag{1}$$

where

$A(s)$ is an nxn matrix
$\mathbf{X}(s)$ is an n dimensional vector with the unknown node voltages and/or currents.
$\mathbf{E}(s)$ is an n dimensional vector of the exciting voltages and/or currents.

If we restrict this discussion to excitations $e_i(t)$ having rational functions as their Laplace transforms $E_i(s)$, then $\mathbf{E}(s)$ is a vector of rational functions. If only one source $E_r(s)$ is applied to the network, all the unknown voltages or currents $X_i(s)$ can be expressed by Cramer's rule in terms of the determinant of the system and by the minor D_i which corresponds to $E_r(s)$.

$$X_i(s) = \frac{E_r(s) D_i}{D} \tag{2}$$

where $D = \text{Det } A(s)$. The transmission function is

$$T_i(s) = \frac{X_i(s)}{E_r(s)} = \frac{D_i}{D}. \tag{3}$$

D_i can also be expressed in terms of the elements of the matrix $A(s)$.

It is assumed that between two nodes only the parallel connection of a resistor, an inductor, and a capacitor (R, L, C) or a controlled source can appear.[2] Thus the frequency function of each element

Manuscript received November 22, 1966; revised February 2, 1967.