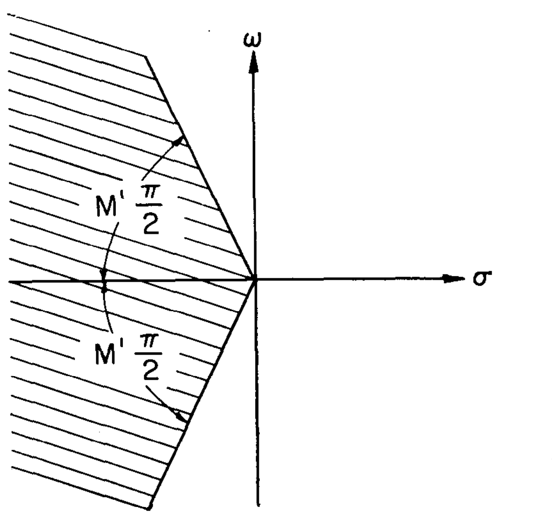

Fig. 1.

that the vectors cannot possibly add to zero (as they must for a natural oscillation) unless s is confined to the shaded region shown in Fig. 1.

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The Noncompleteness of Continuously Equivalent Networks

Schoeffler [1], [2] has recently introduced an interesting extension of the Howitt theory [3] of circuit equivalence to allow for matrices which vary continuously in a parameter. The theory, called that of continuously equivalent networks, appears to be of considerable practical importance. However, in his discussion of the method Calahan [4], page 93, raises the theoretically important question of whether all equivalents can be found by the method. Here we show, by example, that not all equivalents can be found. The example follows that of Brune [5], page 235, and seems reasonable since the Howitt theory is known to be incomplete and the theory of continuously equivalent networks is closely related to that of Howitt. Still, the theories are somewhat different and slightly different techniques must be applied.

Consider a 2-loop RLC 1-port for which we wish to keep the input impedance constant by the use of the theory of continuous equivalence. Then there is a real parameter x such that the symmetric loop resistance matrix $\mathbf{R}(x) = [r_{ij}(x)]$ satisfies [4], page 84,

$$\frac{d\mathbf{R}(x)}{dx} = \tilde{\mathbf{A}}\mathbf{R}(x) + \mathbf{R}(x)\mathbf{A} \tag{1a}$$

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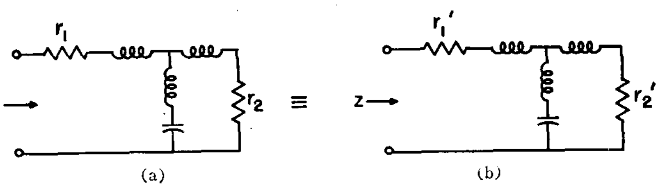

Fig. 1. Equivalent Brune realizations $r_1' \neq r_1$.

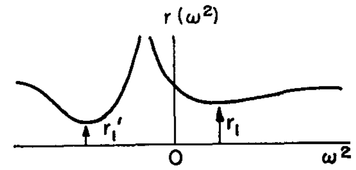

Fig. 2. Determination of r_1' and r_1.

for any real constant $\mathbf{A}$ satisfying

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}. \tag{1b}$$

Here the tilde denotes matrix transposition. The initial conditions $\mathbf{R}(0)$ for (1a) correspond to an initially given configuration, and the object of the theory is to find $\mathbf{R}(x)$, as well as inductor and capacitor parameter matrices, satisfying (1a) but with some desired property. $\mathbf{R}(x)$ for $x \neq 0$ then corresponds to some other circuit configuration, having the same input impedance because of the form of $\mathbf{A}$, as in the Howitt theory. Expanding (1) in this 2-loop case gives

$$\frac{d}{dx}\begin{bmatrix} r_{11} \\ r_{12} \\ r_{22} \end{bmatrix} = \begin{bmatrix} 0 & 2a_{21} & 0 \\ 0 & a_{22} & a_{21} \\ 0 & 0 & 2a_{22} \end{bmatrix}\begin{bmatrix} r_{11} \\ r_{12} \\ r_{22} \end{bmatrix} \tag{2}$$

subject to initial conditions $r_{11}(0)$, $r_{12}(0)$, and $r_{22}(0)$.

At his point let us consider an impedance z which has the two realizations of Fig. 1, both obtained through the Brune process. Here r_1 and $r_1' \neq r_1$ are two stationary values of $r(\omega^2) = Ev\ z(j\omega)$ with r_1' being evaluated for some $\omega^2 < 0$, as shown in Fig. 2 [5], page 230. Using Fig. 1(a) as the initial configuration gives $r_{11}(0) = r_1$, $r_{12}(0) = 0$, and $r_{22}(0) = r_2$. On assuming $r_{22}(x)$ nonconstant, that is, $a_{22} \neq 0$, (2) is solved to obtain

$$r_{22}(x) = r_2 e^{2a_{22}x} \tag{3a}$$

$$r_{12}(x) = r_2 \frac{a_{21}}{a_{22}} e^{a_{22}x}[e^{a_{22}x} - 1] \tag{3b}$$

$$r_{11}(x) = r_1 + r_2 \frac{a_{21}^2}{a_{22}^2}[e^{a_{22}x} - 1]^2. \tag{3c}$$

We wish to find an x_0 such that $r_{11}(x_0) = r_1'$ with $r_{22}(x_0) = r_2'$ and $r_{12}(x_0) = 0$, that is, such that Fig. 1(b) results. But this is impossible since $r_{12}(x_0) = 0$ means $a_{21} = 0$ is chosen to satisfy (3b), as $x_0 \neq 0$, in which case $r_{11}(x_0) = r_1 \neq r_1'$. Consequently, the circuit of Fig. 1(b) can not be derived from that of Fig. 1(a) within the framework of the present theory of continuously equivalent networks.

Some comments are in order. First, one sees that there is really nothing in the theory to preclude using the equation

$$\frac{d\mathbf{R}(x)}{dx} = \mathbf{B}\mathbf{R}(x) + \mathbf{R}(x)\mathbf{A} \tag{4a}$$

with arbitrary constant

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} \\ 0 & b_{22} \end{bmatrix}. \tag{4b}$$

Although (4) is derived by Calahan, he rules it out by placing unnecessary (but nevertheless useful) constraints [4], page 84. Still Fig. 1(b) can not be derived from Fig. 1(a) when **B** is free to be chosen different from $\tilde{\mathbf{A}}$ as the reader can verify using the above arguments. Second, although **A** and **B** must be chosen independent of the frequency variable in the above treatment, this is not necessary if **R** is replaced by the loop impedance matrix in (4a); this probably would yield all equivalents but would probably be hard to verify. The third observation is that there are presently two theories available which will essentially derive all equivalents from a given realization. The first method is due to Oono and Yasuura [6]; this is well developed in the network context but apparently little understood since it seems little used for the content it contains. The second method is due to Kalman [7] and is presently under development [8].

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Nonsymmetric Lattice Sections, II

In a previous note[1] with the same title, a necessary condition for realization of a nonsymmetric inductance capacitance (LC) lattice was derived. This condition was originally derived in[2] by a different method and was proved to be sufficient. A different proof of sufficiency will be given here to complete the previous note.[1] At the same time some new properties of LC functions will be proved.

In order to subtract from a given LC function $Z_1(s)$ a reactance

$$z_{11}(s) = \frac{As^4 + Bs^2 + C}{s^3 + Ds}, \tag{1}$$

which is realizable by a nonsymmetric LC lattice section of the type shown in Fig. 1, it is necessary that $Z_1(s)$ must have numerator degree higher than 3, and the expression

$$Z_1'(j\omega_0)Z_1'(\omega_0) = \frac{Z_1(j\omega_0)}{j\omega_0} \cdot \frac{Z_1(\omega_0)}{\omega_0} \tag{2}$$

in ω_0 must have at least one positive root. We now prove that (2) is not only necessary but also sufficient.

The proof can be split into two parts; we first prove that $z_{11}(s)$ is a reactance function if (2) holds; secondly, we prove that $z_{11}(s)$ can be realized by the nonsymmetric LC lattice section of Fig. 1. In our case the second part of the proof is trivial and therefore will not be considered.

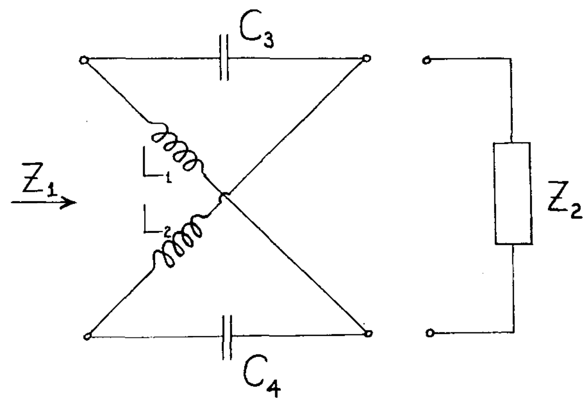

Fig. 1.

In order for $z_{11}(s)$ to be a reactance function, the polynomial $As^4 + s^3 + Bs^2 + Ds + C$ must be Hurwitz. This requires that

1) A, B, C, and D are real and positive, and

$$2)\quad \begin{Vmatrix} 1 & A & 0 \\ D & B & 1 \\ 0 & C & D \end{Vmatrix} = BD - AD^2 - C > 0. \tag{3}$$

The coefficients in (1) were given previously.[1] If we let

$$\begin{aligned} a &= \left.\frac{Z_1(s)}{s}\right|_{s=\omega_0} = \frac{Z_1(\omega_0)}{\omega_0}, \\ b &= Z_1'(s)\,|_{s=\omega_0} = Z_1'(\omega_0), \\ c &= \left.\frac{Z_1(s)}{s}\right|_{s=j\omega_0} = \frac{Z_1(j\omega_0)}{j\omega_0}, \end{aligned} \tag{4}$$

and

$$d = Z_1'(s)\,|_{s=j\omega_0} = Z_1'(j\omega_0),$$

then these coefficients are

$$D = \omega_0^2 \frac{a+b}{a-b}, \tag{5}$$

$$B = \omega_0^2 \frac{a^2+bc}{a-b}, \tag{6}$$

$$A = \frac{a^2 - bc}{2(a-b)}, \tag{7}$$

and

$$C = \omega_0^4 A = \omega_0^4 \frac{a^2 - bc}{2(a-b)} \tag{8}$$

and (2) becomes

$$bd = ac. \tag{2a}$$

All quantities a, b, c, and d are real.

It was shown[1] that $a > |b|$ where a is always positive. Hence, D will always be positive; also d is positive. Hence, it follows from (2a) that b and c must have the same sign. In other words, the product bc is always positive. Consequently, B will be positive too. However, A can become negative if the product bc becomes larger than a^2. Hence, we have to impose the restriction

$$a^2 > bc. \tag{9}$$

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