Example 2

Let us add the pole $(s+4)$ to the above example

$$p(s) = \frac{2(s+1)}{(s+2)(s+3)(s+4)} = \frac{b_1}{s+2} + \frac{b_2}{s+3} + \frac{b_3}{s+4}.$$

By conventional calculation

$$b_1 = -1, \qquad b_2 = 4, \qquad b_3 = -3.$$

Applying *Theorem 1* ($\alpha = 2$) we get

$$B = b_1 + b_2 + b_3 = -1 + 4 - 3 = 0.$$

Applying *Theorem 2* we get

$$B_0 = \frac{b_1}{2} + \frac{b_2}{3} + \frac{b_3}{4} = \frac{(-1)}{2} + \frac{(4)}{3} + \frac{(-3)}{4} = (2)\left(\frac{1}{24}\right) = kk'.$$

Remarks

1) *Theorem 2* is independent of the value of α, whereas *Theorem 1* depends on α.
2) Using one theorem only will tell if there is an error in the coefficients, however, the theorem cannot guarantee the correctness of the coefficients. But if the two theorems are applied, they will give more satisfactory results.
3) Although there are many ways of checking these coefficients, the above two theorems have proven to be convenient and useful.

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A Canonical Simulation of a Transfer Function Matrix

I. INTRODUCTION AND PRELIMINARIES

The need to simulate on an analog computer a specified transfer function matrix $W(s)$, whose elements are ratios of polynomials in s, has given rise to the theory of *minimal realizations* of a transfer function, where the precise definition of a minimal realization is given below. A minimal realization at once yields a *minimal simulation* on an analog computer, which is a simulation of $W(s)$ using the minimum number of integrators. The desire to achieve such minimal simulations is not merely an economic one; nonminimal simulations incorporate uncontrollable or unobservable states (or both), which may often have a deleterious effect on any application to which the simulation is put.

The techniques of network theory often permit minimal realizations for restricted classes of transfer functions. Here minimality is normally characterized by the number of reactive elements being minimum in some circuit realization.

Many of these network theory results can be fruitfully applied to systems theory problems; our purpose here is to indicate one such application which yields a minimal simulation of $W(s)$.

We shall make the reasonable assumptions that $W(s)$ is real for real s and has no pole at $s = \infty$. One approach to achieving a simulation is to consider $W(s)$ to be the transfer function matrix corresponding to a linear, constant, finite-dimensional, dynamical system [1]. The system is then described by equations of the form

$$\frac{dx}{dt} = Fx + Gu \tag{1a}$$

$$y = Hx + Ju \tag{1b}$$

where x is an n-vector (the state), u is a p-vector (the control or input), y is an m-vector (the output), and F, G, H, and J are constant matrices of appropriate dimension. The *transfer function* matrix $W(s)$, relating the Laplace-transformed control to the Laplace-transformed output, is

$$W(s) = H(sI - F)^{-1}G + J \tag{2}$$

where I is the identity matrix (here of order n). Unless otherwise stated, $W(s)$ will be assumed to be a stable transfer function matrix, as precisely defined in (3) below.

A set of matrices $\{F, G, H, J\}$ yielding $W(s)$ by (2) is termed a *realization of* W; a realization where F has the lowest possible dimension is termed a *minimal* realization [1], [2]. Further, knowledge of $\{F, G, H, J\}$ provides an immediate recipe for setting up $W(s)$ on an analog computer; the number of integrators used in the simulation equals the order of the square matrix F. The minimal number is equal to a number defined by W, called its *degree* $\delta[W]$. A proof is given in Kalman,[1] and a definition and discussion of the concept of degree is given in McMillan.[2]

Given a rational W, realizations always exist, and nonminimal ones are easy to find. The construction of minimal realizations from nonminimal ones can be complex, however, as shown by the two distinct approaches in Kalman.[3]

Here we shall bypass the *theoretical* problem of finding a minimal $\{F, G, H, J\}$ and derive directly the solution to the more *practical* problem of defining a minimal analog computer simulation yielding $W(s)$. From each such simulation F, G, H, and J can be calculated if so desired. As pointed out in Kalman,[4] the practical problem has an immediate solution if $W(s)$ has no multiple poles; our solution however permits the multiple pole case, but differs from Kalman's two approaches to the multiple pole problem.

II. CONSTRUCTION OF ANALOG COMPUTER CONFIGURATION

The material of this section may be summarized as follows. We pass from a given rational $W(s)$, by bordering it with zeros, to a square matrix $\hat{W}(s)$. From $\hat{W}(s)$ we derive a bounded-real matrix $S(s)$, and from $S(s)$ we derive a lossless bounded-real matrix $\Sigma(s)$, corresponding to the scattering matrix of a lossless multiport. Using results established in Belevitch [4] and Youla [5], we can then perform a useful decomposition of $\Sigma(s)$ into a product of factors $\Sigma_i(s)$, all of degree 1 (or 0). Simulations of the individual $\Sigma_i(s)$ are easily possible, and we show how to obtain from these a simulation of $W(s)$. The minimality of the configuration is readily established.

In general, $W(s)$ may not be square. We may make it so, however, by defining $\hat{W}(s)$ to be the matrix $W(s)$ with additional rows or columns of zeros (as the case may be), such that $\hat{W}$ is square.[5]

Since in the practical case we shall not normally be interested in simulating an unstable $W(s)$ [that is, a $W(s)$ with right half-plane poles, as mentioned in Section I], we shall assume

> the poles of $W(s)$ are in the strict left half plane or are simple on the $j\omega$-axis. (3)

The same remark is true of $\hat{W}(s)$. Then we may select a non-negative, real, scalar, constant k large enough so that[6]

$$Z(s) = kI + \hat{W}(s) \tag{4}$$

is a positive-real matrix,[7] that is, it is the impedance of a linear, passive, finite, time-invariant network. The matrix $Z(s)$ then defines a *bounded-real* matrix $S(s)$ through[8]

$$S(s) = [Z(s) + I]^{-1}[Z(s) - I] \tag{5a}$$

or

$$Z(s) = I + 2(I - S)^{-1}S. \tag{5b}$$

The matrix $S(s)$ is the scattering matrix of a linear, passive, time-invariant multiport.[9] Such a matrix allows an extension by embedding to a *lossless* scattering matrix $\Sigma(s)$,[10]

$$\Sigma(s) = \left[\begin{array}{c|c} S(s) & \Sigma_{12}(s) \\ \hline \Sigma_{21}(s) & \Sigma_{22}(s) \end{array}\right]. \tag{6}$$

$\Sigma(s)$ is the scattering matrix of a lossless network, being rational and bounded-real and satisfying $\tilde{\Sigma}(-s)\Sigma(s) = I$, where $\sim$ denotes matrix transposition.

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[1] Kalman [2], p. 536.
[2] McMillan [3], p. 580.
[3] Kalman [1], p. 175, and [2], p. 530.
[4] Kalman [1], p. 180.
[5] Youla [6], p. 29.
[6] Youla [6], p. 29, and Oono [7], p. 483.
[7] Belevitch [4], p. 277.
[8] *Ibid.*, p. 278.
[9] Oono and Yasuura [8], p. 153–165.
[10] *Ibid.*, pp. 164–165.

Simplified computational procedures for determining the matrices Σ_{12}, Σ_{21}, and Σ_{22} are now well documented.[11] It is also possible to determine these matrices such that (with $\delta[\ ]$ the degree)[12]

$$\delta[\Sigma(s)] = \delta[S(s)]. \tag{7}$$

We note also that (using various results)[13]

$$\delta[W(s)] = \delta[\hat{W}(s)] = \delta[Z(s)] = \delta[S(s)] = \delta[\Sigma(s)]. \tag{8}$$

Belevitch [4] and Youla [5] establish (detailing simple factorization procedures) that

$$\Sigma(s) = \Sigma_0\Sigma_1(s)\Sigma_2(s)\cdots\Sigma_\delta(s) \tag{9}$$

where Σ_0 is actually a constant orthogonal matrix, and the $\Sigma_i(s)$ $(i=1, 2, \cdots, \delta)$ are matrices of degree 1 of the form

$$\Sigma_i(s) = I - \frac{2\sigma_0 u\tilde{u}^*}{s+s_0} \tag{10a}$$

where * indicates the complex conjugate, u is a vector, $s_0=\sigma_0+j\omega_0$ is a point in Re $s>0$ and

$$\tilde{u}^*u = 1. \tag{10b}$$

Further, the number of nonconstant terms in the product (9) is defined by

$$\delta = \delta[\Sigma(s)] = \delta[W(s)]. \tag{11}$$

Some of the Σ_i may involve complex quantities, but in this case the factorization can be accomplished such that successive pairs when multiplied together yield real degree 2 transfer functions. For simplicity however, following Kalman,[14] we shall disregard the minor intricacies caused by complex Σ_i.

Minimal simulations of each Σ_i $(i=1, 2, \cdots, \delta)$ are easy to construct. Taking, for example, $F=-s_0$ (a scalar), $G=\tilde{u}^*$, $H=-2\sigma_0 u$, and $J=I$ yields such a construction.[15] A cascade of the $\Sigma_i(s)$ simulations with a straightforward simulation for Σ_0 yields a simulation of $\Sigma(s)$, by (9). Then $S(s)$ is simulated by merely *ignoring*, by (6), all but the first q inputs and outputs of the Σ simulation (where q is the dimension of $\hat{W}$ and, thus, of S). This can be done by inserting blocks with transfer functions

$$E_1 = \begin{bmatrix} I \\ 0_{p\times q} \end{bmatrix} \quad \text{and} \quad E_2 = [I \ \ 0_{q\times p}]. \tag{12}$$

Here I is $q\times q$, p is the order of Σ_{22} in (6), and $0_{p\times q}$ is the $p\times q$ zero matrix. The realization of $S(s)$ is shown in Fig. 1.

Using (5b) we can construct a simulation of Z in terms of S; this equation is the mathematical formulation of the equivalence in Fig. 2.

Finally, from $Z(s)$ we can construct $W(s)$. Equation (4) shows that we must subtract kI from Z to obtain $\hat{W}$, while to obtain W from $\hat{W}$ we must neglect either a number of the inputs, Fig. 3(a), or a number of the outputs, Fig. 3(b). In Fig. 3, E_3 and E_4 are, respectively, similar to E_1 and E_2.

Between them, Figs. 1 to 3 outline the interconnections necessary for the simulation; (4), (5a), (6), and (9) outline the necessary calculations. The simulation will use the minimal number of integrators possible. This is because the minimal number is known to be $\delta[W(s)]$ and by (8) and (9) this is the number used in the simulation.

[11] Newcomb [9], ch. 9.
[12] Oono and Yasuura [8], p. 167, and Newcomb [9], Appendix.
[13] Kalman [2], p. 540 and McMillan [3], p. 543.
[14] Kalman [1], p. 181.
[15] Zadeh [10], p. 409.

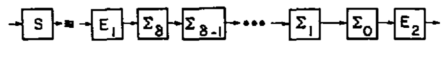

Fig. 1. Simulation of $S(s)$ in terms of degree 1 sections.

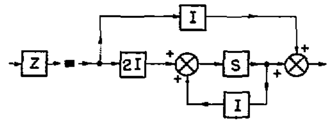

Fig. 2. Simulation of $Z(s)$ derived from simulation of $S(s)$.

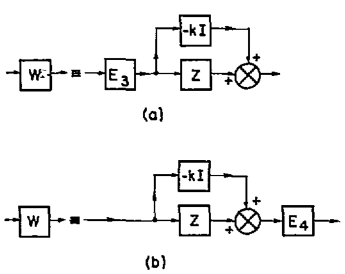

Fig. 3. Simulation of $W(s)$ for (a) added columns of $\hat{W}(s)$ or (b) added rows of $\hat{W}(s)$.

III. Conclusion

We have exhibited another technique for achieving a simulation of a transfer function matrix $W(s)$, using a minimal number of integrators. The calculations required to describe the simulation are somewhat lengthy, but are straightforward, using existing algorithms of the network theory literature.

Although discussion has been restricted to stable transfer functions only, it is possible to tackle the problem of producing a minimal simulation of a rational unstable $W(s)$. In Oono[16] it is pointed out that $W(s+\epsilon)$, for some sufficiently large non-negative real constant ϵ, will be stable. Thus, we may synthesize $W(s+\epsilon)$ using the method outlined above, and then by changing the feedback around integrators derive a simulation of $W(s)$. One can also give a procedure for simulating $W(s)$ with poles at infinity; such details will be discussed elsewhere.

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[16] Oono [7], p. 483.

Investigation on Some Characteristic Features of Third-Order Control Systems

The dynamics of a forced third-order closed-loop system is completely characterized by the instantaneous values of the state variables such as error, first and second derivatives of the output.[1] Depending on suitable pole configurations of the transfer function, any two of these variables of the system following a step signal input can attain zero value simultaneously at a finite time from the instant of application of the input. In the third-order system this can occur in three ways which are obtained from three combinations of the state variables. The pole configurations for occurrence of zero value of each of the combinations has been investigated in the present correspondence.

Consider the closed-loop transfer function of a third-order system as

$$\frac{C(s)}{R(s)} = \frac{a_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{1}$$

where $C(s)$ and $R(s)$ are the transformed output and input of the system. It is assumed that two of the three poles of the transfer function are complex conjugates. The real and imaginary parts of the conjugate poles are expressed in terms of the real pole. Let the three poles be $S_1=-\alpha$, $S_2=-\alpha(K_1+jK_2)$, and $S_3=-\alpha(K_1-jK_2)$, K_1 and K_2 being constants. The three coefficients may be written in terms of the pole configuration as

$$a_1 = \alpha^2(K_1^2+K_2^2+2K_1),$$
$$a_2 = \alpha(2K_1+1) \text{ and } a_0 = \alpha^3(K_1^2+K_2^2). \tag{2}$$

For unit step input of the disturbance with all initial conditions zero, the output $C(s)$ is given by

$$C(s) = \frac{1}{s} + \sum_{i=1}^{3} \frac{A_i}{s+s_i} \tag{3}$$

where

$$A_i = \left.\frac{a_0(s+s_i)}{s(s+s_1)(s+s_2)(s+s_3)}\right|_{s=-s_i}.$$

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[1] G. J. Thaler and M. P. Pastel, *Analysis and Design of Nonlinear Feedback Control Systems*. New York: McGraw-Hill, 1962.