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FUNCTIONAL ANALYSIS OF LINEAR PASSIVE NETWORKS

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ABSTRACT

Using distributional kernels the properties of linear passive time-variable networks are investigated, primarily through the scattering matrix. In particular the scattering matrix is shown to be a measure satisfying an energy form constraint. Lossless constraints are developed which lead to properties useful for future synthesis of time-variable networks. Networks consisting of a finite number of elements are considered in some detail, and examples illustrating the theory are presented.

1. INTRODUCTION

There are presently available several theories of linear, passive, time-invariant networks, some in the frequency-domain [1], [2], and some in the time-domain [3]. However, when the restriction of time-invariance is removed very few general results are available. Nevertheless, a treatment of linear, passive but time-variable networks seems important in view of the fact that synthesis methods for such networks are becoming available [4], [5].

In this paper some of the most important properties of linear passive networks are developed in terms of the time-varying scattering matrix $\underset{\sim}{s}(t,\tau)$, $\underset{\sim}{s}$ being the most general useful description available. The paper is structured such that Sections 4 and 5 contain the general results. Section 2 essentially serves as a review section where the underlying concepts of interest are defined; such are a network and its properties and distributional kernels. In Section 3 the scattering matrix is introduced. Because of the generality of the results of Section 4, where a complete characterization of linear, solvable, and passive networks is given in terms of $\underset{\sim}{s}$, one is naturally led to the scattering description of networks introduced in Section 3. An alternate characterization of the passive conditions on $\underset{\sim}{s}$ is given in Section 5 in order to allow the formulation of a complete characterization of lossless networks, as given in Section 6. Through Section 6 attention is focused on the external behavior of networks, while an investigation of the internal structure is begun in Section 7, where a general connection of networks, called cascade loading, is covered. Section 8 investigates some of the properties of arbitrary connections of a finite number of linear passive resistors, inductors, capacitors, transformers and gyrators where time-variable elements are allowed. Some important and useful lossless conditions for these finite networks are obtained, both on the scattering matrix and the impedance matrix. The final technical section gives several examples which illuminate various portions of the theory, the one concerning time-variable delay being of especial physical significance while that of the open-circuit loaded transformer is of considerable theoretical interest.

The theory rests heavily on the theory of distributions and distributional kernels of L. Schwartz [6], [7] and the theory of bounded transformations on Hilbert space [8], with which we assume some familiarity. Nevertheless, the deeper results of distribution theory are mainly used in proofs, and, consequently, we would hope that the results will be clear to those with an intuitive feel for the concepts.

2. PRELIMINARY DEFINITIONS AND NOTATION

We begin by introducing appropriate notation from which we proceed to the definition of a network and the various properties of interest for the present study [4].

Let $\mathcal{D}$, $\mathcal{D}_+$, and $\mathcal{D}'$ denote the spaces of real-valued n-vectors in one real variable whose entries are, respectively, infinitely differentiable functions zero outside a bounded set (compact support), infinitely differentiable functions zero until a finite value of the variable (support bounded on the left), and distributions [6]. Most often the real variable will be taken as time t. Letting $\underset{\sim}{f}$ and $\underset{\sim}{g}$ denote n-vectors and using a superscript tilde, $\tilde{}$, for matrix transposition we write

$$\langle \underset{\sim}{f},\underset{\sim}{g}\rangle_t = \int_{-\infty}^{t} \tilde{\underset{\sim}{f}}(\lambda)\underset{\sim}{g}(\lambda)d\lambda \tag{2.1a}$$

which, for all finite t, is well defined if, for instance, $\underset{\sim}{f},\underset{\sim}{g}\in\mathcal{D}_+$. We further write

$$\|\underset{\sim}{f}\|_t^2 = \langle \underset{\sim}{f},\underset{\sim}{f}\rangle_t \tag{2.1b}$$

$$\|\underset{\sim}{f}\| = \|\underset{\sim}{f}\|_\infty \tag{2.1c}$$

and observe that $\|\ \|$ serves as a norm for the Hilbert space $\mathcal{L}_2$ of Lebesgue measurable $\underset{\sim}{f}$ for which $\langle \underset{\sim}{f},\underset{\sim}{f}\rangle_\infty$ is finite.

The norm $\|T\|$ of a bounded linear transformation $T[\ ]$ of $\underset{\sim}{f}\in\mathcal{L}_2$ into $T[\underset{\sim}{f}]\in\mathcal{L}_2$ is defined in the customary manner as

$$\|T\| = \sup_{\|\underset{\sim}{f}\|=1} \|T[\underset{\sim}{f}]\| \tag{2.2}$$

A physical n-port network [9] places <u>constraints</u> $\underset{\sim}{C}_N$ on the components of the n-vector <u>voltages</u> $\underset{\sim}{v}$ and <u>currents</u> $\underset{\sim}{i}$ at its n ports; mathematically we can consider the constraints as a binary relation [10, p. 10] and represent the relationship by $\underset{\sim}{v}\underset{\sim}{C}_N\underset{\sim}{i}$. Given a binary relation $\underset{\sim}{C}_N$, the mathematical representation of an n-port <u>network</u> $\underset{\sim}{N}$, or for our purposes simply $\underset{\sim}{N}$ itself, is defined as the set of voltage and current couples $[\underset{\sim}{v},\underset{\sim}{i}]$ satisfying $\underset{\sim}{v}C_N\underset{\sim}{i}$, that is, $\underset{\sim}{N}$ is the set of $[\underset{\sim}{v},\underset{\sim}{i}]$ allowed at the ports by the network constraints. Precisely

$$\underset{\sim}{N} = \{[\underset{\sim}{v},\underset{\sim}{i}] \mid \underset{\sim}{v}C_N\underset{\sim}{i}\} \tag{2.3}$$

Among the constraints always assumed will be $\underset{\sim}{v},\underset{\sim}{i}\in\underset{\sim}{\mathcal{D}}_+$ and a choice of variables such that $\tilde{\underset{\sim}{v}}\underset{\sim}{i}$ has the physical interpretation of the total instantaneous power entering $\underset{\sim}{N}$.

Using this definition of a network in terms of allowed, or admissible, pairs of current and voltage the important properties of linearity, solvability, passivity, and losslessness can be defined in their most general context.

(a) $\underset{\sim}{N}$ is <u>linear</u> if for every $[\underset{\sim}{v}_1, \underset{\sim}{i}_1]$, $[\underset{\sim}{v}_2, \underset{\sim}{i}_2]\in\underset{\sim}{N}$ and all real constants α and β

$$[\alpha\underset{\sim}{v}_1 + \beta\underset{\sim}{v}_2, \alpha\underset{\sim}{i}_1 + \beta\underset{\sim}{i}_2]\in\underset{\sim}{N} \tag{2.4}$$

(b) $\underset{\sim}{N}$ is <u>solvable</u> if for every $\underset{\sim}{e}\in\underset{\sim}{\mathcal{D}}_+$ there is a unique $[\underset{\sim}{v},\underset{\sim}{i}]\in\underset{\sim}{N}$ such that

$$\underset{\sim}{e} = \underset{\sim}{v} + \underset{\sim}{i} \tag{2.5}$$

This last equation defines an <u>augmented</u> network $\underset{\sim}{N}_a$ by $[\underset{\sim}{e},\underset{\sim}{i}]\in\underset{\sim}{N}_a$ if $[\underset{\sim}{e} - \underset{\sim}{i}, \underset{\sim}{i}]\in\underset{\sim}{N}$ for each $\underset{\sim}{e}\in\underset{\sim}{\mathcal{D}}_+$. Physically $\underset{\sim}{N}_a$ represents $\underset{\sim}{N}$ with a unit resistor connected in series with each port. The connection is illustrated in Fig. 1 where $\underset{\sim}{1}_n$ is the $n \times n$ identity matrix representing the impedance of the unit resistors. If $\underset{\sim}{N}$ is linear then so is $\underset{\sim}{N}_a$, as simple application of Eqs. (2.4), (2.5) shows.

The physical meaning of

$$\mathcal{E}(t) = \int_{-\infty}^{t} \tilde{\underset{\sim}{v}}(\tau)\underset{\sim}{i}(\tau)d\tau = \langle \underset{\sim}{v},\underset{\sim}{i}\rangle_t \tag{2.6a}$$

is the total energy input into $\underset{\sim}{N}$ at time t. This leads to the definition of passivity.

(c) $\underset{\sim}{N}$ is <u>passive</u> if for every $[\underset{\sim}{v},\underset{\sim}{i}]\in\underset{\sim}{N}$ and every finite t

$$\mathcal{E}(t) \geq 0 \tag{2.6b}$$

It is customary to call $\underset{\sim}{N}$ active if it is not passive. If $\underset{\sim}{N}$ is passive then so is $\underset{\sim}{N}_a$ since

$$\langle \underset{\sim}{e}, \underset{\sim}{i} \rangle_t = \| \underset{\sim}{i} \|_t^2 + \langle \underset{\sim}{v}, \underset{\sim}{i} \rangle_t \tag{2.7a}$$

Of more weight is the relation [1, p. 111]

$$\| \underset{\sim}{e} \|_t^2 = \| \underset{\sim}{v} \|_t^2 + \| \underset{\sim}{i} \|_t^2 + 2\langle \underset{\sim}{v}, \underset{\sim}{i} \rangle_t \tag{2.7b}$$

which shows that for a passive $\underset{\sim}{N}$, $\underset{\sim}{e} \in \underset{\sim}{\mathcal{L}}_2$ implies $\underset{\sim}{v}, \underset{\sim}{i} \in \underset{\sim}{\mathcal{L}}_2$, since the left is finite while each term on the right is positive. In such a case $\mathcal{E}(\infty)$ is well defined for the lossless definition.

(d) $\underset{\sim}{N}$ is lossless if it is passive and solvable and if for every $\underset{\sim}{e} = \underset{\sim}{v} + \underset{\sim}{i} \in \underset{\sim}{\mathcal{D}}_+ \cap \underset{\sim}{\mathcal{L}}_2$

$$\mathcal{E}(\infty) = 0 \tag{2.8}$$

Physically Eq. (2.8) states that the energy put into a lossless $\underset{\sim}{N}$ is returned and dissipated in the resistors used to obtain $\underset{\sim}{N}_a$, if the augmenting voltage sources $\underset{\sim}{e}$ vanish suitably fast.

In the sequel we will have heavy use for distributional kernels. These are n x n matrices $\underset{\sim}{k}(t,\tau)$ of real-valued distributions in two real variables [7]. Any distributional kernel defines a linear continuous mapping

$$\underset{\sim}{y} = \underset{\sim}{k} \bullet \underset{\sim}{x} \tag{2.9a}$$

of $\underset{\sim}{x} \in \underset{\sim}{\mathcal{D}}$ (strong topology) into $\underset{\sim}{y} \in \underset{\sim}{\mathcal{D}}'$ (weak topology). The converse is also true, that is, any such linear continuous mapping of $\underset{\sim}{\mathcal{D}}$ into $\underset{\sim}{\mathcal{D}}'$ can be described by a distributional kernel through Eq. (2.9a) [7, p. 223]. If we let $\langle \underset{\sim}{y}, \underset{\sim}{\varphi} \rangle$ denote the scalar product between $\underset{\sim}{y} \in \underset{\sim}{\mathcal{D}}'$, and $\underset{\sim}{\varphi} \in \underset{\sim}{\mathcal{D}}$, analogous to $\langle \, , \, \rangle_\infty$ of Eq. (2.1a), and $\langle\langle \underset{\sim}{k}(t,\tau), \underset{\sim}{\psi}(t,\tau) \rangle\rangle$ the same for two variables, Eq. (2.9a) is made precise by the definition

$$\langle \underset{\sim}{k}\bullet\underset{\sim}{x}, \underset{\sim}{\varphi}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle\langle k_{ij}(t,\tau), x_j(\tau)\rangle, \varphi_i(t)\rangle \tag{2.9b}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle\langle k_{ij}(t,\tau), \varphi_i(t)x_j(\tau)\rangle\rangle \tag{2.9c}$$

By applying another kernel $\underset{\sim}{h}$ to $\underset{\sim}{y}$ of Eq. (2.9a) we obtain the (Volterra) composition $\underset{\sim}{h}o\underset{\sim}{k}$ [7, p. 229]. In particular

$$\underset{\sim}{z} = \underset{\sim}{h}\bullet\underset{\sim}{y} = \underset{\sim}{h}\bullet[\underset{\sim}{k}\bullet\underset{\sim}{x}] = (\underset{\sim}{h}o\underset{\sim}{k})\bullet\underset{\sim}{x} \tag{2.10}$$

defines $\underset{\sim}{h}o\underset{\sim}{k}$ as the unique kernel mapping $\underset{\sim}{x}$ into $\underset{\sim}{z}$, when such a mapping can be performed. Although $\underset{\sim}{h}o\underset{\sim}{k}$ cannot always be formed, we note that if $\underset{\sim}{k}$ and $\underset{\sim}{h}$ both map $\underset{\sim}{\mathcal{D}}_+$ into $\underset{\sim}{\mathcal{D}}_+$ then $\underset{\sim}{h}o\underset{\sim}{k}$ exists and also maps $\underset{\sim}{\mathcal{D}}_+$ into $\underset{\sim}{\mathcal{D}}_+$. Again, composition need not always be associative, but it will be when all kernels map $\underset{\sim}{\mathcal{D}}_+$ into $\underset{\sim}{\mathcal{D}}_+$ [7, p. 229]. Since $\delta\underset{\sim}{1}_n$, with δ the unit impulse, acts as the identity map, $\delta\underset{\sim}{1}_n$ can be composed with any kernel.

It is appropriate to define a left inverse $\underset{\sim}{k}_\ell^{-1}$ and a right inverse $\underset{\sim}{k}_r^{-1}$ under composition by

$$\underset{\sim}{k}_\ell^{-1}o\underset{\sim}{k} = \delta\underset{\sim}{1}_n \quad (= \delta(t-\tau)\underset{\sim}{1}_n) \tag{2.11a}$$

$$\underset{\sim}{k}o\underset{\sim}{k}_r^{-1} = \delta\underset{\sim}{1}_n \tag{2.11b}$$

If $\underset{\sim}{k}_r^{-1} = \underset{\sim}{k}_\ell^{-1}$, we call these the inverse $\underset{\sim}{k}^{-1}$. Depending upon the domain of definition considered one kernel may have several inverses. Consequently, we will assume, unless otherwise mentioned, that if $\underset{\sim}{k}$ is a mapping of $\underset{\sim}{\mathcal{D}}_+$ into $\underset{\sim}{\mathcal{D}}_+$ then $\underset{\sim}{k}^{-1}$ is also a mapping of $\underset{\sim}{\mathcal{D}}_+$ into $\underset{\sim}{\mathcal{D}}_+$. For such a mapping Eq. (2.11a) has the meaning that for any $\underset{\sim}{x}\in\underset{\sim}{\mathcal{D}}_+$, $\underset{\sim}{k}^{-1}\bullet(\underset{\sim}{k}\bullet\underset{\sim}{x}) = (\underset{\sim}{k}^{-1}o\underset{\sim}{k})\bullet\underset{\sim}{x} = \underset{\sim}{x}$.

It is often useful to be able to fall back on the intuitive meanings for o and $\bullet$. In the case of functions this is seen through the relationships

$$\underset{\sim}{y} = \underset{\sim}{k} \bullet \underset{\sim}{x} = \int_{-\infty}^{\infty} \underset{\sim}{k}(t,\tau)\underset{\sim}{x}(\tau)d\tau \tag{2.12a}$$

$$\underset{\sim}{h} o \underset{\sim}{k} = \int_{-\infty}^{\infty} \underset{\sim}{h}(t,\lambda)\underset{\sim}{k}(\lambda,\tau)d\lambda \tag{2.12b}$$

In much of the following we will use these integral defintions and often will display the variables, as in $\underset{\sim}{y}(t) = \underset{\sim}{k}(t,\tau) \bullet \underset{\sim}{x}(\tau)$. We will also customarily drop the bold-face type when considering the 1-port, $n = 1$, case.

3. THE SCATTERING MATRIX

From this point on, unless otherwise mentioned, we assume that $\underline{N}$ is linear, solvable, and passive. Under these assumptions we show in this section that $\underline{N}$ possesses a scattering matrix $\underline{s}(t,\tau)$.

By the solvability of $\underline{N}$ there exists a transformation $\mathcal{Y}_a[\ ]$ mapping each $\underline{e}\in\underline{\mathcal{D}}_+$ into a unique $\underline{i}\in\underline{\mathcal{D}}_+$

$$\underline{i} = \mathcal{Y}_a[\underline{e}] \tag{3.1a}$$

By the linearity of $\underline{N}$, $\mathcal{Y}_a[\ ]$ is a linear transformation, as is easily checked from Eq. (2.5) and the linear definition. By the passivity of $\underline{N}$ this transformation is continuous from $\underline{e}\in\underline{\mathcal{D}}$ (strong topology) into $\underline{i}\in\underline{\mathcal{D}}'$ (weak topology). This continuity is seen by noting that $\underline{\mathcal{D}}\subset\underline{\mathcal{D}}_+\cap\underline{\mathcal{L}}_2$. Then consider a sequence $\{\underline{e}_j\}$, $\underline{e}_j\in\underline{\mathcal{D}}$ with all $\underline{e}_j$ of fixed support and converging with all derivatives uniformly to $\underline{0}$ (strong convergence in $\underline{\mathcal{D}}$). In conjunction with passivity, Eq. (2.7b) shows that $\underline{i}_j = \mathcal{Y}_a[\underline{e}_j]$ converges to $\underline{0}$, this convergence in fact implying $\lim_{j\to\infty} \langle \underline{i}_j, \underline{\varphi}\rangle_\infty = 0$ for all $\underline{\varphi}\in\underline{\mathcal{D}}$, or weak convergence to $\underline{0}$ in $\underline{\mathcal{D}}'$. The mapping $\mathcal{Y}_a[\ ]$ is then a linear continuous mapping of $\underline{e}\in\underline{\mathcal{D}}\subset\underline{\mathcal{D}}_+$ into $\underline{i}\in\underline{\mathcal{D}}_+\subset\underline{\mathcal{D}}'$ and therefore has a kernel representation [7, p. 223] [11, p. 143]

$$\underline{i} = \underline{y}_a \bullet \underline{e} \tag{3.1b}$$

From $\underline{e} = \underline{v} + \underline{i}$ we then obtain

$$\underline{v} = (\delta \underline{1}_n - \underline{y}_a)\bullet \underline{e} \tag{3.1c}$$

We call $\underline{y}_a(t,\tau)$ the <u>augmented admittance matrix</u> and note, since any kernel can be composed with the unit impulse, that the physical meaning of the (i,j) entry of $\underline{y}_a$ is the current at port i at time t due to a unit impulse of voltage applied to port j of $\underline{N}_a$ at time τ, all other such port voltages being zero. Although $\underline{y}_a$ has been properly defined only as a map of $\underline{\mathcal{D}}$ into $\underline{\mathcal{D}}_+$, the properties of $\mathcal{Y}_a[\ ]$ allow $\underline{y}_a$ to be immediately considered as a map of $\underline{\mathcal{D}}_+$ into $\underline{\mathcal{D}}_+$ [7, p. 224]. This extension of the domain of definition of $\underline{y}_a$ will always be assumed, and allows $\underline{y}_a$ to be applied to $\underline{v}$ of Eq. (3.1c) and $\delta\underline{1}_n - \underline{y}_a$

to $\underset{\sim}{i}$ of Eq. (3.1b) to obtain

$$\underset{\sim}{y}_a \bullet \underset{\sim}{v} = (\delta \underset{\sim}{1}_n - \underset{\sim}{y}_a) \bullet \underset{\sim}{i} \qquad (3.2)$$

If y_a^{-1} exists then an <u>impedance matrix</u> $\underset{\sim}{z}$ can be defined through $\underset{\sim}{v} = \underset{\sim}{z} \bullet \underset{\sim}{i}$, and similarly $\underset{\sim}{i} = \underset{\sim}{y} \bullet \underset{\sim}{v}$ defines an <u>admittance matrix</u> $\underset{\sim}{y}$ if $(\delta \underset{\sim}{1}_n - \underset{\sim}{y}_a)^{-1}$ exists:

$$\underset{\sim}{z} = \underset{\sim}{y}_a^{-1} - \delta \underset{\sim}{1}_n \qquad (3.3a)$$

$$\underset{\sim}{y} = (\delta \underset{\sim}{1}_n - \underset{\sim}{y}_a)^{-1} \circ \underset{\sim}{y}_a \qquad (3.3b)$$

Because neither z nor $\underset{\sim}{y}$ need exist, as shown by the ideal transformer [4, p. 10], and because we can obtain the most general and complete results in simple form, we make a change of variables to <u>incident</u> and <u>reflected</u> voltages $\underset{\sim}{v}^i$ and $\underset{\sim}{v}^r$, respectively,

$$2\underset{\sim}{v}^i = \underset{\sim}{v} + \underset{\sim}{i} = \underset{\sim}{e} \qquad (3.4a)$$

$$2\underset{\sim}{v}^r = \underset{\sim}{v} - \underset{\sim}{i} \qquad (3.4b)$$

or on solving

$$\underset{\sim}{v} = \underset{\sim}{v}^i + \underset{\sim}{v}^r \qquad (3.4c)$$

$$\underset{\sim}{i} = \underset{\sim}{v}^i - \underset{\sim}{v}^r \qquad (3.4d)$$

Inserting these latter into Eq. (3.2) serves to define the <u>scattering matrix</u> $\underset{\sim}{s}(t,\tau)$ in terms of $\underset{\sim}{y}_a$ through

$$\underset{\sim}{v}^r = \underset{\sim}{s} \bullet \underset{\sim}{v}^i \qquad (3.5a)$$

$$\underset{\sim}{s} = \delta \underset{\sim}{1}_n - 2\underset{\sim}{y}_a \qquad (3.5b)$$

As we have seen earlier, every linear, solvable, passive $\underset{\sim}{N}$ possesses an augmented admittance matrix, which by the comment after Eq. (2.7b) maps $\mathcal{D}_+\cap\mathcal{L}_2$ into $\mathcal{D}_+\cap\mathcal{L}_2$. Consequently, we conclude from Eq. (3.5b) that every linear, solvable, passive N possesses a time-variable scattering matrix s which is a linear continuous map of incident voltages $v^i\epsilon\mathcal{D}_+$ into reflected voltages $v^r\epsilon\mathcal{D}_+$. By the properties of y_a, s defines a map of $\mathcal{D}_+\cap\mathcal{L}_2$ into $\mathcal{D}_+\cap\mathcal{L}_2$.

In terms of the new variables the energy of Eq. (2.5a) takes the useful form

$$\mathcal{E}(t) = \|v^i\|_t^2 - \|v^r\|_t^2 \qquad (3.6)$$

If we precompose Eq. (3.2) with any kernel c having an inverse under composition we can obtain a slightly more general description

$$a\bullet v = b\bullet i \qquad (3.7)$$

In Table 3.1 we summarize for easy reference the various interrelations between the descriptions introduced. The various matrices composed internally in the table commute, for instance, $s = (z - \delta 1_n)o(z + \delta 1_n)^{-1}$.

Table 3.1

Summary of Description Interrelations

	$a\bullet v=b\bullet i$			
	$s=(b+a)^{-1}o(b-a)$	$y=b^{-1}oa$	$z=a^{-1}ob$	$y_a=(b+a)^{-1}oa$
$s =$	s	$(\delta 1_n+y)^{-1}o(\delta 1_n-y)$	$(z+\delta 1_n)^{-1}o(z-\delta 1_n)$	$\delta 1_n-2y_a$
$y =$	$(\delta 1_n-s)o(\delta 1_n+s)^{-1}$	y	z^{-1}	$y_a o(\delta 1_n-y_a)^{-1}$
$z =$	$(\delta 1_n+s)o(\delta 1_n-s)^{-1}$	y^{-1}	z	$y_a^{-1}-\delta 1_n$
$y_a =$	$\frac{1}{2}(\delta 1_n-s)$	$yo(\delta 1_n+y)^{-1}$	$(z+\delta 1_n)^{-1}$	y_a

By definition N_d is the dual of N if every $[v_d, i_d]\epsilon N_d$ is of the form $[i_d, v_d]\epsilon N$. Equations (3.4a,b) then show that $v_d^i = v^i$, $v_d^r = -v^r$ and hence

$$\underset{\sim}{s}_d = -\underset{\sim}{s} \tag{3.8a}$$

Similarly

$$\underset{\sim}{z}_d = \underset{\sim}{y}, \tag{3.8b}$$

which, by Table 3.1 checks $\underset{\sim}{s}_d = -\underset{\sim}{s}$.

To conclude this section we point out that the foregoing definitions and formulae all include the well-known time-invariant results when $(t,\tau) = (t-\tau)$.

4. PROPERTIES OF $\underset{\sim}{s}$

In this section are developed necessary and sufficient conditions on the scattering matrix to guarantee $\underset{\sim}{N}$ passive. From these, some useful properties are obtained, such as the fact that $\underset{\sim}{s}(t,\tau)$ is a measure in both variables and $\underset{\sim}{s}_2 \circ \underset{\sim}{s}_1$ is passive with $\underset{\sim}{s}_1$ and $\underset{\sim}{s}_2$ passive.

We begin with some preliminaries, the first of which concerns the support of $\underset{\sim}{s}$. If we let $\underset{\sim}{0}_n$ denote the $n \times n$ zero matrix then

$$\boxed{\underset{\sim}{s}(t,\tau) = \underset{\sim}{0}_n \quad \text{when} \quad t < \tau} \tag{4.1}$$

The same result holds for $\underset{\sim}{y}_a$, by Eq. (3.5b). In essence Eq. (4.1) states that $\underset{\sim}{N}$ is in some sense causal, at least when viewed through the augmenting resistors. More precisely $\underset{\sim}{s}$ is antecedal [12].

The validity of Eq. (4.1) follows from Eq. (3.6). Thus, with t_o fixed, if $\underset{\sim}{v}^i(t) = \underset{\sim}{0}$ for $t < t_o$, Eq. (3.6) shows $\underset{\sim}{v}^r(t) = \underset{\sim}{0}$ for $t < t_o$, since $\mathcal{E}(t) \geq 0$ by assumption. Therefore

$$\langle \underset{\sim}{s} \bullet \underset{\sim}{x}, \underset{\sim}{\varphi} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle\langle s_{ij}(t,\tau), \varphi_i(t) x_j(\tau) \rangle\rangle = 0 \tag{4.2}$$

whenever $\underset{\sim}{\varphi}(t) = \underset{\sim}{0}$ for $t > t_o$ and $\underset{\sim}{x}(\tau) = \underset{\sim}{0}$ for $t_o > \tau$, $\underset{\sim}{\varphi}, \underset{\sim}{x} \in \underset{\sim}{\mathcal{D}}$. As this is true irrespective of the values of $\underset{\sim}{\varphi}(t)$ and $\underset{\sim}{x}(\tau)$ for $t < t_o < \tau$, Eq. (4.1) follows [6, vol. I, p. 26]. We remark that it is sufficient to test s_{ij} with separable $\psi_{ij}(t,\tau) = \varphi_i(t)x_j(\tau)$ by the denseness of finite sums of such in nonseparable $\psi_{ij}(t,\tau)$ [6, vol. I, p. 108].

Because $\underset{\sim}{s}$ is a linear continuous transformation on $\underset{\sim}{\mathcal{D}}$ into $\underset{\sim}{\mathcal{D}}'$ it has an adjoint $\underset{\sim}{s}^a$ defined by

$$\langle \underset{\sim}{s} \bullet \underset{\sim}{x}, \underset{\sim}{\varphi} \rangle = \langle \underset{\sim}{x}, \underset{\sim}{s}^a \bullet \underset{\sim}{\varphi} \rangle \tag{4.3a}$$

for all $\underset{\sim}{x}, \underset{\sim}{\varphi} \in \underset{\sim}{\mathcal{D}}$. In fact

$$\boxed{\underset{\sim}{s}^a(t,\tau) = \tilde{\underset{\sim}{s}}(\tau,t)} \tag{4.3b}$$

which follows from Eq. (2.9c) on making an interchange in dummy variables and indices: $\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\langle s_{ij}(t,\tau), \varphi_i(t)x_j(\tau)\rangle\rangle = \langle s\bullet x, \varphi\rangle = \langle x, s^a\bullet\varphi\rangle$

$= \langle s^a\bullet\varphi, x\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle\langle s^a_{ij}(t,\tau), x_i(t)\varphi_j(\tau)\rangle\rangle$

$= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle\langle s^a_{ji}(\tau,t), \varphi_i(t)x_j(\tau)\rangle\rangle.$

Because of Eqs. (4.1) it is clear that $s^a(t,\tau) = 0_n$ for $t > \tau$.

The boundedness of s as a linear continuous transformation on $\mathfrak{L}_2$ can also be obtained. From Eq. (3.6) and the passivity of N

$$\|v^i\|_t^2 \geq \|v^r\|_t^2 = \|s\bullet v^i\|_t^2 \tag{4.4}$$

Choosing $e = 2v^i \epsilon \mathcal{D}_+ \cap \mathfrak{L}_2$ and letting $t = \infty$ shows that s is a bounded linear continuous transformation on a subset of $\mathfrak{L}_2$; we can therefore make another extension [8, p. 298] defining s in a bounded manner for all $v^i \epsilon \mathfrak{L}_2$. Observing Eq. (4.3a) for $x, \varphi \epsilon \mathfrak{L}_2$ shows that s^a is also a bounded linear continuous transformation on $\mathfrak{L}_2$. Then noting that s and s^a have the same norm [8, p. 201] and comparing Eq. (4.4) when $t = \infty$ with the definition of the norm of a transformation shows

$$\|s\| = \|s^a\| \leq 1 \tag{4.5}$$

Equation (4.5) is a necessary condition for the passivity of N; essentially it is also sufficient if N is linear and solvable.

Theorem 1: A linear solvable network N is passive if and only if

(1) A scattering matrix s exists mapping $v^i \epsilon \mathcal{D}_+$ into $v^r \epsilon \mathcal{D}_+$ and

(2) s maps $\mathfrak{L}_2$ into $\mathfrak{L}_2$ and

(3) $s(t,\tau) = 0_n$ for $t < \tau$ and

(4) $\|s\| \leq 1$

Proof: The only if portion has been proven by the reasoning leading to Eq. (4.5). To show the if portion we first observe that if conditions (1), (2) and (3) are not satisfied then N must fail to be either linear or solvable or passive.

As we are assuming $\underset{\sim}{N}$ linear and solvable we are led to consider the existence of an $\underset{\sim}{s}$ satisfying $\|\underset{\sim}{s}\| \leq 1$ but which is not passive. Then there is some pair $[\underset{\sim}{v}_1, \underset{\sim}{i}_1] \epsilon \underset{\sim}{N}$, obtainable from $2\underset{\sim}{v}_1^i = \underset{\sim}{v}_1 + \underset{\sim}{i}_1 \epsilon \underset{\sim}{\mathcal{D}}_+$ and some finite constant T such that $\langle \underset{\sim}{v}_1, \underset{\sim}{i}_1 \rangle_T = \mathcal{E}_1(T) < 0$.

Let a second excitation $\underset{\sim}{v}_2^i \epsilon \underset{\sim}{\mathcal{D}}$ be defined as

$$2\underset{\sim}{v}_2^i = \begin{cases} \underset{\sim}{v}_1 + \underset{\sim}{i}_1, & t \leq T \\ \underset{\sim}{0}, & T + \epsilon \leq t \end{cases} \tag{4.6}$$

for arbitrarily small $\epsilon > 0$ ($\underset{\sim}{v}_2^i$ is defined in an infinitely differentiable manner in $T < t < T + \epsilon$). Then $2\underset{\sim}{v}_2^i = \underset{\sim}{v}_2 + \underset{\sim}{i}_2$; by solvability $\underset{\sim}{v}_2 = \underset{\sim}{v}_1$ and $\underset{\sim}{i}_2 = \underset{\sim}{i}_1$ for $t \leq T$. Then

$$\mathcal{E}_2(T) = \mathcal{E}_1(T) \tag{4.7a}$$

$$\mathcal{E}_2(\infty) \leq \mathcal{E}_2(T) + \gamma(\epsilon) \tag{4.7b}$$

where $\gamma(\epsilon)$ can be made arbitrarily small by properly choosing ϵ since $\mathcal{E}(t)$ is evaluated as an integral. But $\underset{\sim}{v}_2^i \epsilon \underset{\sim}{\mathcal{L}}_2$ and thus also $\underset{\sim}{v}_2^r \epsilon \underset{\sim}{\mathcal{L}}_2$ by $\|\underset{\sim}{v}_2^r\| \leq \|\underset{\sim}{s}\| \|\underset{\sim}{v}_2^i\|$. Equation (3.6) then shows, by $\|\underset{\sim}{s}\| \leq 1$,

$$\mathcal{E}_1(T) + \gamma(\epsilon) \geq \mathcal{E}_2(\infty) = \|\underset{\sim}{v}_2^i\|^2 - \|\underset{\sim}{v}_2^r\|^2 \geq \|\underset{\sim}{v}_2^i\|^2 [1 - \|\underset{\sim}{s}\|^2] \geq 0 \tag{4.7c}$$

Choosing $0 < \gamma(\epsilon) < -\mathcal{E}_1(T)$ shows that the assumption of $\mathcal{E}_1(T) < 0$ is violated, and hence $\underset{\sim}{N}$ must be passive. Q.E.D.

We point out that in this proof the assumption of a solvable $\underset{\sim}{N}$ has allowed the truncated $\underset{\sim}{v}^i$ of Eq. (4.6) in the domain of definition of $\underset{\sim}{N}_a$. Consequently, for the purposes of this paper, $\mathcal{E}(t) \geq 0$ in the definition of passive $\underset{\sim}{N}$, Eq. (2.6b), could be replaced by $\mathcal{E}(\infty) \geq 0$ [1, p. 110]. However, such a modified definition would restrict us to $\underset{\sim}{\mathcal{L}}_2$, in contrast to $\underset{\sim}{\mathcal{D}}_+$, functions, would not generalize to nonsolvable $\underset{\sim}{N}$, and seems physically unappealing. It is worth observing that the -2 ohm resistor has $s(t,\tau) = -3\delta(t-\tau)$ which satisfies conditions (1), (2), (3) of the theorem but has $\|s\| = 3$; this checks the active nature of a negative resistor.

For conciseness it is convenient to call $\underset{\sim}{s}$ passive if it satisfies the conditions of Theorem 1. Note that Eq. (3.8a) shows, with Theorem 1, that $\underset{\sim}{s}_d$ is passive with $\underset{\sim}{s}$, a fact which is also clear from the definition of dual.

A result of some use in synthesis, as well as analysis, is the following.

Theorem 2: If $\underset{\sim}{s}_1$ and $\underset{\sim}{s}_2$ are passive scattering matrices then

$$\underset{\sim}{s} = \underset{\sim}{s}_2 \circ \underset{\sim}{s}_1 \tag{4.8}$$

is passive.

Proof: We must show that $\underset{\sim}{s}$ satisfies Theorem 1. Clearly $\underset{\sim}{s}$ maps $\underset{\sim}{\mathcal{D}}_+$ into $\underset{\sim}{\mathcal{D}}_+$ as the composition of two such scattering matrices; similarly for $\underset{\sim}{\mathcal{E}}_2$ into $\underset{\sim}{\mathcal{E}}_2$. In obvious notation $\underset{\sim}{v}_1^i = \underset{\sim}{v}^i$, $\underset{\sim}{v}_1^r = \underset{\sim}{v}_2^i$, $\underset{\sim}{v}_2^r = \underset{\sim}{v}^r$, or from Eq. (3.6)

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) \geq 0 \tag{4.9a}$$

The argument for Eq. (4.1) then shows that $\underset{\sim}{s}(t,\tau) = \underset{\sim}{0}_n$ for $t < \tau$ and the argument for Eq. (4.5) shows $\|\underset{\sim}{s}\| \leq 1$. This latter fact also follows from

$$\|\underset{\sim}{s}_2 \circ \underset{\sim}{s}_1\| \leq \|\underset{\sim}{s}_2\| \|\underset{\sim}{s}_1\| \leq 1 \tag{4.9b}$$

We observe that $\underset{\sim}{N}$ is defined through Eq. (3.4) and is linear and solvable with $\underset{\sim}{s}$ as its scattering matrix. Q.E.D.

Theorem 2 admits of a nice physical interpretation, which we however postpone for Section 9. The next result is of assistance in exhibiting the form of $\underset{\sim}{s}$ (specifically for finite $\underset{\sim}{N}$ in Section 8), as well as for ruling out nonpassive $\underset{\sim}{s}$.

Theorem 3: A passive $\underset{\sim}{s}(t,\tau)$ is a measure in t for each fixed τ, and a measure in τ for each fixed t.

Proof: The meaning of the theorem is that each component s_{ij} has the stated properties [6, vol. I, p. 15]. By a theorem of L. Schwartz [6, vol. I, p. 25] $\underset{\sim}{s}(t,\tau)$ will be a measure in τ if for each t the sequence of $\mathcal{D}_+$ functions $\{\underline{s}\circ\underline{x}_\alpha\}$ converges to $\underline{0}$ whenever the sequence $\{\underline{x}_\alpha\}$, $\underline{x}_\alpha \in \mathcal{D}$, converges to $\underline{0}$ uniformly irrespective of convergence of the derivatives of $\underline{x}_\alpha$. But for each $\underline{\varphi} \in \mathcal{D}$, we have by Schwarz's inequality [8, p. 198] and Eq. (4.5)

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\langle s_{ij}(t,\tau), x_{\alpha_j}(\tau)\rangle, \varphi_i(t)\rangle\right| = |\langle\underline{s}\circ\underline{x}_\alpha, \underline{\varphi}\rangle| \qquad (4.11a)$$

$$\leq \|\underline{s}\circ\underline{x}_\alpha\|\|\underline{\varphi}\| \leq \|\underline{x}_\alpha\|\|\underline{\varphi}\| \qquad (4.11b)$$

Thus, if $\underline{x}_\alpha$ tends uniformly to $\underline{0}$, by the passivity $\langle\underline{s}\circ\underline{x}_\alpha, \underline{\varphi}\rangle$ tends to 0 for almost all t, as a function. Similarly by the distributional rule for calculating the derivative, denoted here by a prime [6, vol. I, p. 35],

$$|\langle(\underline{s}\circ\underline{x}_\alpha)', \underline{\varphi}\rangle| = |\langle\underline{s}\circ\underline{x}_\alpha, \underline{\varphi}'\rangle| \leq \|\underline{x}_\alpha\|\|\underline{\varphi}'\| \qquad (4.11c)$$

Therefore $(\underline{s}\circ\underline{x}_\alpha)'$ converges to $\underline{0}$ for almost all t; integrating then shows that $\underline{s}\circ\underline{x}_\alpha$ converges to $\underline{0}$ for each t. Considering $\underline{s}^a\circ\underline{x}_\alpha$ we obtain the same result on $\tilde{\underline{s}}^a(\tau,t) = \underline{s}(t,\tau)$, that is, $\tilde{\underline{s}}^a(\tau,t)$ is a measure in t for each fixed τ. Q.E.D.

We can further see that $\underline{s}(t,\tau)$ is also a measure in both variables simultaneously over any compact set K of the (t,τ)-plane. Consider

$$\underline{s}_K(t,\tau) = \alpha(t)\underline{s}(t,\tau)\beta(\tau) \qquad (4.12a)$$

where α and β are nonnegative infinitely differentiable functions bounded by unity such that $\alpha\beta = 1$ over K and $\alpha\beta = 0$ outside a square containing K. The scattering matrix $\underline{s}_K$ is passive since Theorem 2 applies to

$$\underline{s}_K = \underline{s}_\alpha \circ \underline{s} \circ \underline{s}_\beta \qquad (4.12b)$$

where

$$\underline{s}_\alpha(t,\tau) = \alpha(t)\delta(t-\tau)\underline{1}_n \qquad (4.12c)$$

$$\underline{s}_\beta(t,\tau) = \beta(t)\delta(t-\tau)\underline{1}_n \qquad (4.12d)$$

Both $\underline{s}_\alpha$ and $\underline{s}_\beta$ come from passive networks (resistors in fact) since for instance

$$\mathcal{E}_\alpha(t) = \|\underline{v}^i\|_t^2 - \|\underline{v}^r\|_t^2 \geq \|\sqrt{1-\alpha^2}\;\underline{v}^i\|_t^2 \geq 0 \qquad (4.13)$$

Now $\underline{s}_K$, having compact support, can be convoluted (denoted by *) with unit step (Heaviside) functions u to give [6, vol. I, p. 114] ($\otimes$ denotes the tensor product)

$$u(t)*\underline{s}_K(t,\tau)*u(\tau) = [u(t)\otimes u(\tau)]*\underline{s}_K(t,\tau) = u(t,\tau)*\underline{s}_K(t,\tau) \qquad (4.14)$$

By Theorem 3, the left side of Eq. (4.14) is a function of bounded variation in both t and τ, consequently so is the right, showing that $\underline{s}_K(t,\tau)$ is a measure jointly in t and τ [6, vol. II, p. 45]. By passivity Eq. (4.1) applies to $\underline{s}_K$ and we can write

$$u*\underline{s}_K*u = \underline{g}_K(t,\tau)u(t-\tau) \qquad (4.15a)$$

where $\underline{g}_K$ consists at least of functions of bounded variation in t and τ. We have immediately

$$\underline{s}_K(t,\tau) = \frac{\partial^2}{\partial t\partial\tau}[\underline{g}_K(t,\tau)u(t-\tau)] \qquad (4.15b)$$

which, on expansion, shows $\underline{g}_K(t,t) = \underline{0}_n$ since $\delta'(t-\tau)$ is not a measure in t for each τ. As shown by the example of $s(t,\tau) = \delta(t-d(t)-\tau)$, to be discussed in Section 9, not much more can be generally concluded concerning $\underline{g}_K$, except that it is infinitely differentiable for finite $\underline{N}$, Section 8.

5. ALTERNATE PASSIVITY CONDITION

Because of the insight it yields, and as a preparation for the lossless constraints, we here express the passivity requirement in terms of a nonnegative form.

By definition a real distributional kernel $\underset{\sim}{k}(\alpha,\beta)$ which is self-adjoint, $\tilde{\underset{\sim}{k}}(\beta,\alpha) = \underset{\sim}{k}(\alpha,\beta)$, is called nonnegative, written $\underset{\sim}{k} \geq 0$, if for every $\underset{\sim}{x}\in\underset{\sim}{\mathcal{D}}$

$$\langle \underset{\sim}{k}\bullet\underset{\sim}{x}, \underset{\sim}{x}\rangle \geq 0 \tag{5.1}$$

The definition differs slightly from that of L. Schwartz because the two variables in $\underset{\sim}{k}$ are not taken as a difference [6, vol. II, p. 131].

Considering Eq. (3.6) we can write the energy as

$$\mathcal{E}(t) = \langle \underset{\sim}{v}^i, \underset{\sim}{v}^i\rangle_t - \langle \underset{\sim}{s}\bullet\underset{\sim}{v}^i, \underset{\sim}{s}\bullet\underset{\sim}{v}^i\rangle_t \tag{5.2a}$$

$$= \langle u(t-\alpha)\delta(\alpha-\beta)\underset{\sim}{1}_n\bullet\underset{\sim}{v}^i(\beta), \underset{\sim}{v}^i(\alpha)\rangle_\infty - \langle u(t-\lambda)\underset{\sim}{s}(\lambda,\beta)\bullet\underset{\sim}{v}^i(\beta), \underset{\sim}{s}(\lambda,\alpha)\bullet\underset{\sim}{v}^i(\alpha)\rangle_\infty \tag{5.2b}$$

where the integral over $(-\infty,t)$ has been replaced by one over $(-\infty,\infty)$ by inserting the unit step function u. We are then able to use the adjoint definition, Eq. (4.3a) with terms reversed, allowing $\underset{\sim}{s}(\lambda,\alpha)$ to be shifted off $\underset{\sim}{v}^i(\alpha)$ in the final portion of Eq. (5.2b); this shows that the composition (in λ) of $\underset{\sim}{s}^a(\alpha,\lambda) = \tilde{\underset{\sim}{s}}(\lambda,\alpha)$ and $u(t-\lambda)\underset{\sim}{s}(\lambda,\beta)$ exists. Collecting all terms composed on $\underset{\sim}{v}^i(\beta)$ we arrive at

$$\mathcal{E}(t) = \langle \underset{\sim}{Q}_t(\alpha,\beta)\bullet\underset{\sim}{v}^i(\beta), \underset{\sim}{v}^i(\alpha)\rangle \tag{5.2c}$$

$$\boxed{\underset{\sim}{Q}_t(\alpha,\beta) = u(t-\alpha)\delta(\alpha-\beta)\underset{\sim}{1}_n - \tilde{\underset{\sim}{s}}(\lambda,\alpha)\circ\{u(t-\lambda)\underset{\sim}{s}(\lambda,\beta)\}} \tag{5.2d}$$

Equation (5.2c) holds for all $\underset{\sim}{v}^i\in\underset{\sim}{\mathcal{D}}_+$, and one easily checks that for each finite t $\underset{\sim}{Q}_t$ is self-adjoint. Consequently, for each finite t $\underset{\sim}{Q}_t$ is nonnegative for a passive $\underset{\sim}{N}$. In such a case $\underset{\sim}{v}^i\in\underset{\sim}{\mathcal{L}}_2$ implies that $\underset{\sim}{Q}_t(\alpha,\beta)\bullet\underset{\sim}{v}^i(\beta)\in\underset{\sim}{\mathcal{L}}_2$, since $\underset{\sim}{s}^a$ and hence each term on the right maps $\underset{\sim}{\mathcal{L}}_2$ into $\underset{\sim}{\mathcal{L}}_2$, and we can again apply the reasoning of Theorem 3 to show that $\underset{\sim}{Q}_t(\alpha,\beta)$ is a measure in α and β because

$$|\langle Q_t \bullet v^i, v^i \rangle| \leq \|v^i\|^2 \qquad (5.3)$$

Since $\mathcal{E}(t)$ is simply related to Q_t and since $v^i \epsilon \mathcal{D}_+$ can be considered as the limit of a sequence of $v^i \epsilon \mathcal{D}$ we can rephrase Theorem 1.

<u>Theorem 4</u>: A linear solvable network N is passive if and only if conditions (1), (2) and (3) of Theorem 1 are satisfied and for all finite t

$$(4') \quad Q_t \geq 0$$

One advantage of the formulation of this section is that it shows the nature of results when z or y, Eqs. (3.3), exist. Thus, using $\mathcal{E}(t) = \langle v, i \rangle_t = \langle i, v \rangle_t$, the manipulations of Eqs. (5.2), and a result similar to Eq. (4.3b), we see that an equivalent statement of condition (4′) is: for each t the form

$$R_t(\alpha,\beta) = u(t-\alpha)z(\alpha,\beta) + u(t-\beta)\tilde{z}(\beta,\alpha) \geq 0 \qquad (5.4)$$

From Eq. (5.4) one sees that the necessary and sufficient condition for a linear solvable N completely described by an impedance matrix z, with $z(t,\tau) = 0_n$ for $t < \tau$, to be passive is the satisfaction of Eq. (5.4). This situation is in contrast to that of Zemanian [3, footnote 17, p. 269] where the time-domain description does not rule out $z(t,\tau) = \delta^{(3)}(t-\tau)$ which is excluded by Eq. (5.4), as proven in Appendix 1.

It is worth noting that

$$Q_\infty = \delta 1_n - s^a o s \qquad (5.5)$$

From this some further insight into the $Q_t \geq 0$ constraint can be gained by observing that when the Laplace transform $S(p) = \mathcal{L}[s(t,0)]$ completely describes N (the time-invariant case) then $S(p)$ must be bounded-real [1, p. 116]; if it is meromorphic then the resistivity matrix $R(p) = 1_n - \tilde{S}(-p)S(p)$ must be positive semidefinite in $\mathrm{Re}\ p > 0$ [13, p. 154]. Excluding consideration of the regions of convergence, $R(p) = \mathcal{L}[Q_\infty(t,0)]$ follows from Eq. (5.5) since $\tilde{S}(-p) = \mathcal{L}[s^a(t,0)]$.

6. LOSSLESS $\underset{\sim}{N}$

Here we show that a necessary and sufficient condition for a passive scattering matrix $\underset{\sim}{s}$ to be lossless is that the adjoint $\underset{\sim}{s}^a$ be a left inverse of $\underset{\sim}{s}$ under composition.

For simplicity we will call $\underset{\sim}{s}$ <u>lossless</u> if it represents a lossless $\underset{\sim}{N}$. Then we first investigate the behavior of $\|\underset{\sim}{s}\|$. By Eqs. (2.8), (3.6) and (4.5) we have for all $v^i \epsilon \underset{\sim}{\mathcal{L}}_2$

$$\mathcal{E}(\infty) = 0 = \|\underset{\sim}{v}^i\|^2 - \|\underset{\sim}{s}\bullet\underset{\sim}{v}^i\|^2 \geq \|\underset{\sim}{v}^i\|^2[1 - \|\underset{\sim}{s}\|^2] \geq 0 \quad (6.1)$$

(we recall that the domain of $\underset{\sim}{s}$ was extended to $\underset{\sim}{\mathcal{L}}_2$ below Eq. (4.4)). Therefore, a passive $\underset{\sim}{s}$ is lossless only if

$$\|\underset{\sim}{s}\| = 1 \quad (6.2)$$

Unfortunately $\|\underset{\sim}{s}\| = 1$ is not a sufficient condition for losslessness, as is seen by any network which over some interval of time behaves in a "nondissipative" manner, irrespective of its behavior over all time. For example the 1-port resistor of resistance $r(t) = u(t)u(1-t)\exp[-1/t^2]\exp[-1/(t-1)^2]$ has $s(t,\tau) = \delta(t-\tau)[r(t)-1]/[r(t)+1]$ with $\|\underset{\sim}{s}\| = 1$.

Consequently we turn to the more useful result obtained from $\underset{\sim}{Q}_t$. For all $\underset{\sim}{v}^i \epsilon \underset{\sim}{\mathcal{D}}_+ \cap \underset{\sim}{\mathcal{L}}_2$ Eqs. (2.8) and (5.2c) yield

$$\mathcal{E}(\infty) = \langle \underset{\sim}{Q}_\infty \bullet \underset{\sim}{v}^i, \underset{\sim}{v}^i \rangle = 0 \quad (6.3)$$

Therefore [6, vol. I, p. 26], $\underset{\sim}{Q}_\infty(\alpha,\beta)\bullet\underset{\sim}{v}^i(\beta) = 0$ for all such $\underset{\sim}{v}^i$ and thus $\underset{\sim}{Q}_\infty(\alpha,\beta)$ is independent of α. Being self-adjoint it is independent of β and we conclude that $\underset{\sim}{Q}_\infty(t,\tau) = \underset{\sim}{0}_n$ is a necessary and sufficient condition for a passive $\underset{\sim}{s}$ to be lossless. The conclusion $\underset{\sim}{Q}_\infty = \underset{\sim}{0}_n$ also follows from a similar result on Hilbert spaces [14, p. 267]. Evaluating $\underset{\sim}{Q}_\infty = \underset{\sim}{0}_n$ from Eq. (5.5) gives the following result which is of considerable importance for synthesis [5].

<u>Theorem 5</u>: A passive $\underset{\sim}{s}$ is lossless if and only if

$$\boxed{\underset{\sim}{s}^a \circ \underset{\sim}{s} = \delta \underset{\sim}{1}_n} \quad (6.4)$$

We comment that $s^a o s$ is well defined since, for all $x, \varphi \in \mathcal{D}$, $\langle s \bullet x, s \bullet \varphi \rangle = \langle x, (s^a o s) \bullet \varphi \rangle$ by use of the adjoint definition, Eq. (4.3b), and the definition of composition. Although s^a is a left inverse for s it is clearly not one mapping $\mathcal{D}_+$ into $\mathcal{D}_+$. Consequently associativity does not generally hold in $s o [s^a o s] = s$, and one can not generally conclude that s^a is a right inverse of s under the conditions of Theorem 5. The lossless 1-port delay $s(t,\tau) = \delta(t-d-\tau)$, with constant $d \geq 0$, shows that s^a can map $\mathcal{D}_+$ into $\mathcal{D}_+$ and that s^a can be a right inverse. The unit inductor $s(t,\tau) = \delta(t-\tau)-2(\exp[\tau-t])u(t-\tau)$ has $s^a(t,\tau) = \delta(t-\tau)-2(\exp[t-\tau])u(\tau-t)$ which is a left and a right inverse, but which does not map $\mathcal{D}_+$ into $\mathcal{D}_+$. The example $s(t,\tau) =$

$$\delta(t-\tau)-\varphi(t)\frac{\varphi(\tau)}{\int_\tau^\infty \varphi^2(\lambda)d\lambda}u(t-\tau)$$

with φ any smooth square-integrable function, for instance $\varphi(t) = 1/[1+t^2]$, is lossless, but s^a is not a right inverse.

We observe, from $s^a o s = \delta 1_n$, that a lossless N must have an impulsive term present in s. Further $Q_\infty = 0_n$ can not be used directly to obtain results such as $z + z^a = 0_n$ (which is incorrect) on impedances since z, in contrast to s, does not map $\mathcal{L}_2$ into $\mathcal{L}_2$, as seen by the series tuned circuit described by $z(t,\tau) = \delta'(t-\tau) + u(t-\tau)$. For finite networks a condition for z to be lossless will, however, be obtained in Section 8.

For further insight into the meaning of Eq. (6.4) we note that in the time-invariant case $s^a o s = \delta 1_n$ corresponds to the Laplace transform para-unitary relationship $\tilde{S}(-p)S(p) = 1_n$, when $S(p)$ is meromorphic [1, p. 123].

7. CASCADE LOADING

In order to obtain somewhat more specific results for networks of considerable importance we turn to a useful method of combining networks, that of cascade loading. In particular we calculate the input scattering matrix $\underset{\sim}{s}$ when a certain inverse exists and show that under such a condition $\underset{\sim}{s}$ is passive when the subnetworks are.

Consider an (n+m)-port $\underset{\sim}{N}_{\Sigma}$ whose variables are partitioned as the ports, that is, $\tilde{\underset{\sim}{v}}^i_{\Sigma} = [\tilde{\underset{\sim}{v}}^i_1, \tilde{\underset{\sim}{v}}^i_2]$, $\tilde{\underset{\sim}{v}}^r_{\Sigma} = [\tilde{\underset{\sim}{v}}^r_1, \tilde{\underset{\sim}{v}}^r_2]$ with the subscripts 1 and 2 respectively denoting n- and m-vectors. Then an m-port $\underset{\sim}{N}_{\ell}$ is said to <u>cascade load</u> $\underset{\sim}{N}_{\Sigma}$ if

$$\underset{\sim}{v}^i_{\ell} = \underset{\sim}{v}^r_2 \quad \text{and} \quad \underset{\sim}{v}^r_{\ell} = \underset{\sim}{v}^i_2 \tag{7.1a}$$

where $\underset{\sim}{v}^i_{\ell}$ and $\underset{\sim}{v}^r_{\ell}$ are incident and reflected voltages for $\underset{\sim}{N}_{\ell}$. This connection defines a new n-port $\underset{\sim}{N}$, as illustrated in Fig. 2, whose incident and reflected variables are

$$\underset{\sim}{v}^i = \underset{\sim}{v}^i_1 \quad \text{and} \quad \underset{\sim}{v}^r = \underset{\sim}{v}^r_1 \tag{7.1b}$$

subject to the constraints placed on the coupling network $\underset{\sim}{N}_{\Sigma}$ by loading it with $\underset{\sim}{N}_{\ell}$. Partitioning the scattering matrix $\underset{\sim}{\Sigma}$ of $\underset{\sim}{N}_{\Sigma}$ according to its port partition and defining $\underset{\sim}{s}_{\ell}$ and $\underset{\sim}{s}$ as the scattering matrices of $\underset{\sim}{N}_{\ell}$ and $\underset{\sim}{N}$ lead to

$$\begin{bmatrix} \underset{\sim}{v}^r \\ \underset{\sim}{v}^i_{\ell} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix} \bullet \begin{bmatrix} \underset{\sim}{v}^i \\ \underset{\sim}{v}^r_{\ell} \end{bmatrix} \tag{7.2a}$$

$$\underset{\sim}{v}^r_{\ell} = \underset{\sim}{s}_{\ell} \bullet \underset{\sim}{v}^i_{\ell} \tag{7.2b}$$

$$\underset{\sim}{v}^r = \underset{\sim}{s} \bullet \underset{\sim}{v}^i . \tag{7.2c}$$

where Eqs. (7.1) have been used in $\underset{\sim}{v}^r_{\Sigma} = \underset{\sim}{\Sigma} \bullet \underset{\sim}{v}^i_{\Sigma}$. In order to gain some insight into various of the following manipulations we write these out fully as

$$v^r = \Sigma_{11} \bullet v^i + (\Sigma_{12} \circ s_\ell) \bullet v^i_\ell \qquad (7.3a)$$

$$(\delta 1_m - \Sigma_{22} \circ s_\ell) \bullet v^i_\ell = \Sigma_{21} \bullet v^i \qquad (7.3b)$$

We wish to solve for v^r in terms of v^i, to equate to Eq. (7.2c), by eliminating v^i_ℓ. Doing this when the indicated inverse exists yields

$$s = \Sigma_{11} + \Sigma_{12} \circ s_\ell \circ [\delta 1_m - \Sigma_{22} \circ s_\ell]^{-1} \bullet \Sigma_{21} \qquad (7.4)$$

Unfortunately the inverse needed for Eq. (7.4) may not exist even though N_Σ and N_ℓ are passive. Although an interconnection of the type under consideration need not possess a scattering matrix (see Example 9.3 of Section 9), still the cascade load connection will often be described by s, even though the inverse in Eq. (7.4) may not exist. If we let $\mathcal{R}[\ ]$ denote the range space, then a sufficient condition for the cascade load connection of passive networks to possess a scattering matrix is

$$\mathcal{R}[\delta 1_m - \Sigma_{22} \circ s_\ell] \supset \mathcal{R}[\Sigma_{21}] \qquad (7.5a)$$

To see this we comment that always (see Appendix 2)

$$\mathcal{N}[\delta 1_m - \Sigma_{22} \circ s_\ell] \subset \mathcal{N}[\Sigma_{12} \circ s_\ell] \qquad (7.5b)$$

where $\mathcal{N}[\ ]$ denotes the null space. Thus if v^i is fixed but arbitrary in $\mathcal{D}_+$, then, by Eq. (7.5a) there is at least one $v^i_\ell \in \mathcal{D}_+$ satisfying Eq. (7.3b). If there are two or more such v^i_ℓ their difference is in $\mathcal{N}[\delta 1_m - \Sigma_{22} \circ s_\ell]$ and Eq. (7.5b) then shows that $(\Sigma_{12} \circ s_\ell) \bullet v^i_\ell$ is uniquely determined. Consequently, given any $v^i \in \mathcal{D}_+$ there exists a unique $v^r \in \mathcal{D}_+$ determined by Eqs. (7.5); this shows that s for Eq. (7.2c) is well defined as a map of $\mathcal{D}_+$ into $\mathcal{D}_+$ whenever Eq. (7.5a) holds. Interpreting the -1 of Eq. (7.4) as a type of pseudo-inverse, we will understand Eq. (7.4) to mean the process just described, whenever Eq. (7.5a) holds. Because Eq. (7.5a) always holds in the time-invariant case [15], or in the case where the range spaces are closed, the above considerations are often of importance.

We conclude this section with the physically obvious fact that $\underset{\sim}{N}$ is passive if it is constructed by cascade loading a passive $\underset{\sim}{N}_\Sigma$ by a passive $\underset{\sim}{N}_\ell$.

Theorem 6: If $\underset{\sim}{\Sigma}$ and $\underset{\sim}{s}_\ell$ are passive then the cascaded loaded $\underset{\sim}{N}$ is passive; $\underset{\sim}{s}$ of Eq. (7.4) is passive when it exists.

Proof: Even though $\underset{\sim}{s}$ need not exist, $\underset{\sim}{N}$ is always passive since, in obvious notation we have by Eqs. (7.1) and (3.6)

$$\mathcal{E}(t) = \|\underset{\sim}{v}^i\|_t^2 - \|\underset{\sim}{v}^r\|_t^2 \tag{7.6a}$$

$$= (\|\underset{\sim}{v}_1^i\|_t^2 + \|\underset{\sim}{v}_2^i\|_t^2 - \|\underset{\sim}{v}_1^r\|_t^2 - \|\underset{\sim}{v}_2^r\|_t^2) - (\|\underset{\sim}{v}_\ell^r\|_t^2 - \|\underset{\sim}{v}_\ell^i\|_t^2) \tag{7.6b}$$

$$= \mathcal{E}_\Sigma(t) + \mathcal{E}_\ell(t) \geq 0 \tag{7.6c}$$

By definition $\underset{\sim}{s}$ is then passive when it exists.

Q.E.D.

The generality of the cascade load connection is worth observing. Thus, by suitably manipulating the subnetworks, virtually every interconnection of importance can be represented in cascade form, for example, the series and parallel connections of n-ports [16, p. 127].

8. FINITE NETWORKS

In this section we consider the interconnection of a finite number of common circuit elements, developing a useful set of necessary and sufficient conditions for losslessness of finite networks in terms of the scattering matrix. From such a development a lossless constraint on impedances of a finite network is also obtained. To proceed without undue delay we bypass considerable detail and assume the meaning of various physical concepts to be familiar from classical network theory [16].

We consider the basic elements of interest to be the linear 1-port resistor, inductor and capacitor, the linear 2-port gyrator, and the linear (n+m)-port transformer. We call these <u>circuit elements</u> and define them respectively by

$$v = ri \qquad \text{[resistor]} \tag{8.1a}$$

$$v = d[\ell i]/dt \qquad \text{[inductor]} \tag{8.1b}$$

$$i = d[cv]/dt \qquad \text{[capacitor]} \tag{8.1c}$$

$$\underset{\sim}{v} = \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix} \underset{\sim}{i} \qquad \text{[gyrator]} \tag{8.1d}$$

$$\left.\begin{aligned} \underset{\sim}{v}_1 &= \tilde{\underset{\sim}{T}}\underset{\sim}{v}_2 \\ \underset{\sim}{i}_2 &= -\underset{\sim}{T}\underset{\sim}{i}_1 \end{aligned}\right\} \qquad \text{[transformer]} \tag{8.1e}$$

Here the scalar parameters $r(t)$ [resistance], $\ell(t)$ [inductance], $c(t)$ [capacitance], $\gamma(t)$ [gyration resistance], and the $m \times n$ matrix $\underset{\sim}{T}(t) = [t_{ij}(t)]$ [turns ratios] are infinitely differentiable real-valued functions of time. We assume the transformer ports partitioned as for $\underset{\sim}{N}_\Sigma$ of Section 7 [17, Fig. 1]. The gyrator and transformer are both lossless, while the passivity conditions on the remaining elements are [18] [19, pp. 10-21]

$$r \geq 0 \tag{8.2a}$$

$$\ell \geq 0 \quad \text{and} \quad \ell' \geq 0 \tag{8.2b}$$

$$c \geq 0 \quad \text{and} \quad c' \geq 0 \tag{8.2c}$$

Under these conditions all circuit elements have scattering matrices which are relatively easy to calculate, that for the transformer being [4, p. 11]

$$\underset{\sim}{s}(t,\tau) = \delta(t-\tau)\begin{bmatrix} (\underset{\sim}{1}_n + \tilde{\underset{\sim}{T}}\underset{\sim}{T})^{-1}(\tilde{\underset{\sim}{T}}\underset{\sim}{T} - \underset{\sim}{1}_n) & 2(\underset{\sim}{1}_n + \tilde{\underset{\sim}{T}}\underset{\sim}{T})^{-1}\tilde{\underset{\sim}{T}} \\ 2\underset{\sim}{T}(\underset{\sim}{1}_n + \tilde{\underset{\sim}{T}}\underset{\sim}{T})^{-1} & (\underset{\sim}{1}_m + \underset{\sim}{T}\tilde{\underset{\sim}{T}})^{-1}(\underset{\sim}{1}_m - \underset{\sim}{T}\tilde{\underset{\sim}{T}}) \end{bmatrix} \tag{8.3}$$

where $\underset{\sim}{T} = \underset{\sim}{T}(t)$, and the inverses are those for matrices.

Interconnecting a finite number of circuit elements subject to Kirchhoff's laws yields a finite circuit; attaching ports yields a "network." Therefore, we define a finite network as a network which has a finite circuit representation. Of special interest is the fact that every finite network has a representation in the cascade loaded form of Fig. 2, where $\underset{\sim}{N}_\Sigma$ consists of transformers and $\underset{\sim}{N}_\ell$ consists of constant parameter uncoupled resistors, inductors, capacitors and gyrators. To obtain this representation when all elements are passive, one replaces each element by its equivalent circuit [18] as illustrated in Fig. 3, placing all transformers in the coupling network. Often this leads to a simple method of finding the scattering matrix of a finite network, when it exists, since Eq. (7.4) applies. In particular this shows that $\underset{\sim}{s}$ exists for a finite network constructed from time-invariant circuit elements, that is, with all r, ℓ, c, γ and $\underset{\sim}{T}$ constant [15]. Nevertheless, whether $\underset{\sim}{s}$ exists or not, this shows that a finite network having a circuit representation completely in terms of passive circuit elements is passive, since Theorem 6 applies. This passivity also follows from the additivity of energy in each circuit element.

Equations (8.1) show that a finite passive $\underset{\sim}{N}$ possessing a scattering matrix has $\underset{\sim}{v}^i$ and $\underset{\sim}{v}^r$ related by an ordinary differential equation [19, p. 41]

$$\underline{C}(p,t)\underline{v}^{r}(t) = \underline{D}(p,t)\underline{v}^{i}(t) \qquad (8.4)$$

where $\underline{C}$ and $\underline{D}$ are $n \times n$ matrices, polynomial in the derivative operator $p = d/dt$ with time-varying coefficients. By the measure and causal properties of Eq. (4.15b) and the separability of the impulse responses of differential equations [19, p. 76-86] [20, Chap. 6] we can write

$$\underline{s}(t,\tau) = \underline{A}(t)\delta(t-\tau) + \underline{\Phi}(t)\tilde{\underline{\Psi}}(\tau)u(t-\tau) \qquad (8.5)$$

where in fact $\underline{A}$, $\underline{\Phi}$, and $\underline{\Psi}$ are infinitely differentiable. We comment that the cascade loaded equivalent circuit of a finite network mentioned above virtually guarantees the existence of the general description of Eq. (3.7) which converts to Eq. (8.4) under the change of voltage and current variables to incident and reflected voltages, since distributional kernels have $p^{j} \Longrightarrow \delta^{(j)}$; with this replacement $\underline{s} = \underline{C}^{-1}o\underline{D}$. Further $\underline{s}$ of Eq. (8.5) represents an antecedal impulse response for Eq. (8.4); a nonantecedal response results from changing $u(t-\tau)$ to $-u(\tau-t)$ in Eq. (8.5). Using this idea we note that $\underline{s}^{a}(t,\tau) = \tilde{\underline{s}}(\tau,t)$ can be made <u>antecedal</u> by defining $\underline{s}_{a}^{a}$ through

$$\underline{s}_{a}^{a}(t,\tau) = \tilde{\underline{A}}(t)\delta(t-\tau) - \underline{\Psi}(t)\tilde{\underline{\Phi}}(\tau)u(t-\tau) \qquad (8.6)$$

Turning to the lossless property we note that, by direct integration using Eq. (8.5),

$$\underline{s}^{a}o\underline{s} = \delta 1_{n} = \tilde{\underline{A}}(t)\underline{A}(t)\delta(t-\tau) + \underline{F}(t,\tau)u(t-\tau) + \tilde{\underline{F}}(\tau,t)u(\tau-t) \qquad (8.7a)$$

where

$$\underline{F}(t,\tau) = \tilde{\underline{A}}(t)\underline{\Phi}(t)\tilde{\underline{\Psi}}(\tau) + \underline{\Psi}(t)\left[\int_{t}^{\infty}\tilde{\underline{\Phi}}(\lambda)\underline{\Phi}(\lambda)d\lambda\right]\tilde{\underline{\Psi}}(\tau) \qquad (8.7b)$$

Since a linear, passive $\underline{N}$ is solvable if and only if $\underline{s}$ exists, we conclude the following.

<u>Theorem 7</u>: A finite passive $\underline{N}$ is lossless if and only if

$$\boxed{\begin{array}{c} \tilde{A}(t)A(t) = 1_n \\ \hline F(t,\tau) = 0_n \end{array}} \quad \text{and} \qquad \begin{array}{r} (8.7c) \\ (8.7d) \end{array}$$

Again, by direct calculation

$$s_a^a \circ s = \tilde{A}(t)A(t)\delta(t-\tau) + [F(t,\tau) - \tilde{F}(\tau,t)]u(t-\tau) \qquad (8.7c)$$

$$= \delta 1_n \qquad (8.7f)$$

Consequently, in the case of finite lossless networks s_a^a is an antecedal left inverse of s mapping $\mathcal{D}_+$ into $\mathcal{D}_+$. If an impedance matrix exists we can write, from Table 3.1 and Eq. (8.7f)

$$\delta 1_n = [z_a^a + \delta 1_n]^{-1} \circ [z_a^a - \delta 1_n] \circ [z + \delta 1_n] \circ [z + \delta 1_n]^{-1} \quad (8.8)$$

where we have used $(h \circ k)_a^a = k_a^a \circ h_a^a$ which is valid in the finite case under consideration, as simple calculations show. Since all terms in this last expansion of $\delta 1_n$ are $\mathcal{D}_+$ into $\mathcal{D}_+$ mappings, the product is associative. Precomposing by $z_a^a + \delta 1_n$ and post composing by $z + \delta 1_n$ gives [19, p. 57]

$$\boxed{z_a^a + z = 0_n} \quad \text{[lossless]} \qquad (8.9)$$

as a necessary condition to be satisfied by finite lossless z. Note that we can not use the same argument on $s^a \circ s$ to obtain $z^a + z = 0_n$ (which is false) in view of the equation corresponding to Eq. (8.8) having nonassociative products. It is also to be noted that s_a^a need not exist for nonfinite lossless N, as shown by $s(t,\tau) = \delta(t-d-\tau)$ with constant $d > 0$. Nevertheless, whenever s_a^a and z exist, Eq. (8.9) is valid and shows that even $s \circ s_a^a = \delta 1_n$ for such N. We further remark that Eq. (8.7f) or equivalently Eq. (8.9), is not a sufficient condition for s to be lossless, as nonzero F which are skew under the adjoint operation satisfy Eq. (8.7f). For example $z(t,\tau) = \frac{1}{1+t^2} \cdot \frac{1}{1+\tau^2} \cdot u(t-\tau)$ is not the impedance of a lossless network [as seen by Eq. (9.5b) with $T(t) = 1/(1+t^2)$] but does satisfy Eq. (8.9).

9. EXAMPLES

We begin this section by considering the inductor, determining its descriptions and lossless constraints. Some of the calculations for the inductor carry over to the capacitor loaded transformer of Example 2, which illustrates many points of concern for lossless networks. The next example, that of a transformer of turns ratio falling to zero cascade loaded in an open-circuit, demonstrates that interconnections of networks with scattering matrices need not have scattering matrices. Following this the 3n-port circulator is covered, allowing a physical interpretation of Theorem 2. We close the section with a consideration of the time-variable delay which illustrates many of the differences between finite and nonfinite networks.

Example 1 (Time-Variable Inductor):

The time-variable inductor is described by

$$v = [\ell i]' = \ell i' + \ell' i \tag{9.1a}$$

which gives

$$z(t,\tau) = \ell(t)\delta'(t-\tau) + \ell'(t)\delta(t-\tau) \tag{9.1b}$$

$$y(t,\tau) = \frac{1}{\ell(t)}u(t-\tau) \tag{9.1c}$$

the latter of which follows on integrating the describing equation. Choosing i large but almost constant near a given t and then i small but of large derivative gives, using Eq. (9.1a), the passive constraints $\ell \geq 0$, $\ell' \geq 0$ of Eq. (8.2b). Assuming $\ell > 0$ we can solve $e = \frac{dx}{dt} + \frac{x}{\ell}$, $x = \ell i$, by standard means to get

$x(t) = \int_{-\infty}^{\infty} \{\exp[-\int_{\tau}^{t} \frac{d\lambda}{\ell(\lambda)}]\} u(t-\tau)e(\tau)d\tau$, from which s results by Eq. (3.5b)

$$s(t,\tau) = \delta(t-\tau) - \frac{2}{\ell(t)} e^{-\int_{\alpha}^{t} \frac{d\lambda}{\ell(\lambda)}} \cdot e^{\int_{\alpha}^{\tau} \frac{d\lambda}{\ell(\lambda)}} u(t-\tau) \tag{9.2}$$

Here α is any finite real number. Clearly s exists if $\ell > 0$, as Theorem 1 requires. The lossless constraint follows from Eq. (8.7d) for which

$$F(t,\tau) = -\frac{2}{\ell(t)} e^{-\int_{\alpha}^{t} \frac{dx}{\ell(x)}} e^{\int_{\alpha}^{\tau} \frac{dx}{\ell(x)}} + 4 e^{\int_{\alpha}^{t} \frac{dx}{\ell(x)}} e^{\int_{\alpha}^{\tau} \frac{dx}{\ell(x)}} \int_{t}^{\infty} \frac{1}{\ell^{2}(\lambda)} e^{-2\int_{\alpha}^{\lambda} \frac{dx}{\ell(x)}} d\lambda \tag{9.3}$$

Setting this to zero and differentiating yields $\ell' = 0$ as the lossless constraint; that is, ℓ is a positive constant for a (nonzero) lossless inductor.

Since the capacitor with $c = \ell$ is the dual of the inductor the above results apply to the capacitor with Eq. (3.8a) showing that s of Eq. (9.2) is replaced by its negative for the capacitor.

Example 2 (Unit Capacitor Loaded Transformer):

Consider the 2-port transformer cascade loaded by a capacitor with $c = 1$, as shown in Fig. 4. We calculate, as for Example 1 (see Appendix 3)

$$s(t,\tau) = -\delta(t-\tau) + \left\{ \frac{2T(t)}{\exp[\int_{\alpha}^{t} T^{2}(\lambda)d\lambda]} \cdot T(\tau)\exp[\int_{\alpha}^{\tau} T^{2}(\lambda)d\lambda] \right\} u(t-\tau) \tag{9.4}$$

for any real finite constant α. Note that s exists for any turns ratio $T(t)$, which may even fall to zero (Appendix 3), and, consequently, the combination is linear, solvable, and passive. Note also that Eq. (9.4) agrees with the general passive form of Eq. (8.5) and that the coefficient of δ is orthogonal, Eq. (8.7a). However, the network need not be lossless, as is seen physically if $T(t) = 0$ for $t > t_0 \neq \infty$, in which case charge can be trapped on the capacitor causing $\mathcal{E}(\infty) \neq 0$ with square-integrable incident voltages.

From Eq. (8.7b) one calculates (see Appendix 3)

$$F(t,\tau) = \frac{-2T(t)\exp[\int_{\alpha}^{t} T^2(\lambda)d\lambda]T(\tau)\exp[\int_{\alpha}^{\tau} T^2(\lambda)d\lambda]}{\exp[2\int_{\alpha}^{\infty} T^2(\lambda)d\lambda]} \qquad (9.5a)$$

Since $F(t,\tau) = 0$ is required for losslessness, Eq. (8.7d), one concludes from Eq. (9.5a) that a necessary and sufficient condition for the network of Fig. 4 to be lossless is [21]

$$\int_{\alpha}^{\infty} T^2(\lambda)d\lambda = \infty \qquad (9.5b)$$

for all real finite α.

Example 3 (Open-Circuit Loaded Transformer):

If the turns ratio $T(t)$ falls to zero, as shown in Fig. 5(b), then the 1-port of Fig. 5(a) has rather degenerate behavior. We first observe that the transformer, which is cascade loaded by an open-circuit to form N, is a solvable network described by

$$\underset{\sim}{\Sigma} = \frac{\delta}{T^2+1}\begin{bmatrix} T^2-1 & 2T \\ 2T & 1-T^2 \end{bmatrix} \qquad (9.6a)$$

Since

$$s_{\ell} = \delta \qquad (9.6b)$$

an inverse of $\delta-\Sigma_{22}os_{\ell} = 2T^2\delta/(T^2+1)$ does not exist for $t > t_o$, when $T(t) = 0$. Since any function in the range of $\delta-\Sigma_{22}os_{\ell} = 2T^2\delta/(T^2+1)$ must fall to zero faster than any corresponding function in the range of $\Sigma_{21} = 2T\delta/(T^2+1)$, Eq. (7.5a) does not hold and the given methods of Section 7 fail to yield an s. In actual fact s does not exist because the $\mathcal{D}_+$ constraint requires $e(t_o) = 0$, $e = v+i$; the given network is not solvable in spite of the fact that the transformer and open-circuit are.

Some comments on this behavior are in order. We first observe that one would intuitively say N behaves as a short-circuit $(v \equiv 0)$ for $t > t_o$, which indeed is the case in the formulation being considered. However, one can extend the network to square-integrable v^r and v^i, in which case one essentially has to postulate the behavior of N for $t > t_o$. For instance one can logically postulate that N behaves as an open circuit for $t > t_o$, which in fact is consistent with assuming $T = \epsilon > 0$ for $t > t_o$, using Eq. (7.4) to get

$$s = \frac{\delta}{\epsilon^2+1}\left[\epsilon^2 - 1 + 2\epsilon\left(\frac{1}{2\epsilon^2}\right)2\epsilon\right] = \delta \ , \quad t > t_o \qquad (9.7)$$

and then letting $\epsilon \longrightarrow 0$. Other methods yield other results and we can only conclude that no unique scattering matrix exists for the given network in any reasonable context.

Physically this example points up the ideal nature of the transformer. In actual fact such a device is the limit of mutually coupled coils with the mutual inductance infinite [16, p. 174]. To get a turns ratio behavior as shown in Fig. 5(b) the mutual inductance must flip from being infinite before $t = t_o$ to being zero after $t = t_o$. This type of behavior is of course unknown in the physical world, where every network must be solvable. Nevertheless, this example shows the type of behavior which must necessarily be considered when using a workable mathematical theory to model reality.

Example 4 (Realization of Scattering Matrix Product):

We define the 3n-port circulator by

$$\underline{s}(t,\tau) = \delta(t-\tau)\begin{bmatrix} \underline{0}_n & \underline{0}_n & \underline{1}_n \\ \underline{1}_n & \underline{0}_n & \underline{0}_n \\ \underline{0}_n & \underline{1}_n & \underline{0}_n \end{bmatrix} \qquad (9.8)$$

Since $\underset{\sim}{s}^a o \underset{\sim}{s} = \delta \underset{\sim}{1}_{3n}$ this is clearly lossless, as it is passive by Theorem 1. The 3n-port circulator, represented in Fig. 6, can be considered as the juxtaposition of n 3-port circulators, which can be physically realized [22, p. 520].

Now consider Fig. 7 where we first consider $\underset{\sim}{N}_2$ as a cascade load for the circulator. Equation (7.4) gives

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{0}_n & \underset{\sim}{s}_2 \\ \underset{\sim}{1}_n & \underset{\sim}{0} \end{bmatrix} \tag{9.8}$$

which is actually more easily obtained by physically reasoning on the structure. Again applying Eq. (7.4) to $\underset{\sim}{N}_1$ loading $\underset{\sim}{N}_\Sigma$ gives

$$\underset{\sim}{s} = \underset{\sim}{s}_2 o \underset{\sim}{s}_1 \tag{9.10}$$

Consequently, Fig. 7 realizes the product considered in Theorem 2. Note that the passivity of each subnetwork in Fig. 7 makes the passivity of $\underset{\sim}{s}$ physically obvious. This connection has been considered in the time-invariant case by Belevitch [23, p. 280] whose corresponding results indicate that a factorization theory of passive $\underset{\sim}{s}$ may be fruitful for time-variable synthesis.

Using the connection of Fig. 7 one can obtain a physical proof of the associativity of the composition of passive scattering matrices. This is shown in Fig. 8 where a 4n-port circulator has been introduced in (b).

Example 5 (Time-Variable Delay):

Consider a 1-port described by

$$s(t,\tau) = \delta(t-d(t)-\tau) \tag{9.11}$$

where $d(t)$ is a real-valued infinitely differentiable function of time with (for convenience) $d(\pm\infty) \neq \pm\infty$. Since

$$v^r(t) = v^i(t-d(t)) \tag{9.12}$$

the name <u>time-variable delay</u> is descriptive when $d \geq 0$, which must be the case if s is passive by Eq. (4.1). By a change of variable $x = \lambda - d(\lambda)$ we have

$$s^{a}_{os} = \int_{-\infty-d(-\infty)}^{\infty-d(+\infty)} \frac{\delta(x-t)\delta(x-\tau)}{1-d'(a(x))}dx = \frac{\delta(t-\tau)}{1-d'(a(t))} \tag{9.13}$$

where $a(\lambda - d(\lambda)) = \lambda$ is the inverse of the change of variable. We observe from Eq. (9.13) that this is a lossless network if and only if the delay d is constant, since it is passive by the following results.

To find the passivity constraint on s it is easiest to directly calculate the energy in terms of Eq. (3.6).

$$\mathcal{E}(t) = \int_{-\infty}^{t} [v^i(\tau)]^2 d\tau - \int_{-\infty}^{t} [v^i(\tau - d(\tau))]^2 d\tau \tag{9.14a}$$

$$= \int_{-\infty}^{t-d(t)} \frac{-d'(a(t))}{1-d'(a(t))}[v^i(\tau)]^2 d\tau + \int_{t-d(t)}^{t} [v^i(\tau)]^2 d\tau \tag{9.14b}$$

Since $d \geq 0$, the final term in Eq. (9.14b) is nonnegative. Consequently we choose the support of v^i in $[-\infty, t-d(t)]$ and let v^i approach an impulse to see that $-d'/[1-d'] \geq 0$ (for all t) is necessary and sufficient for $\mathcal{E}(t) \geq 0$. Then either $d' \geq 1$ or $d' \leq 0$. In the former case $d(\alpha) - d(t) = \int_t^{\alpha} d'(t)dt \geq (\alpha - t)$ and letting $t \longrightarrow -\infty$ with α fixed contradicts $d \geq 0$. We conclude that s is passive if and only if

$$d(t) \geq 0 \tag{9.15a}$$

$$d'(t) \leq 0 \tag{9.15b}$$

for all t. Physically, relaxing the $d' \leq 0$ constraint would mean that input (incident) signals become stretched in time when being changed into output (reflected) signals, allowing more energy to come out of the network than would have gone in.

Because s is not of the form of Eq. (8.5) we know that the time variable delay, $d > 0$, is not a finite network. Nevertheless it is of value in modeling the reflection characteristics of time-varying media, such as the ionosphere, or of moving targets in radar studies [24].

Along the same lines one can replace $\delta(t-\tau)$ in Eq. (9.8) by $\delta(t-d(t)-\tau)$ to obtain a time-variable circulator, the passive conditions again being as in Eqs. (9.15) with $d(\pm\infty) \neq \pm\infty$. Choosing $n = 1$ and terminating port 3 in a unit resistor yields Fig. 9 which is described by

$$\underline{s}(t,\tau) = \begin{bmatrix} 0 & 0 \\ \delta(t-d(t)-\tau) & 0 \end{bmatrix} \tag{9.16}$$

Such a device is matched at both ports and has one-way transmission which is a variable delay. The presence of the resistor as well as the time-variable circulator explains why s of Eq. (9.16) is not lossless, although it is passive under Eqs. (9.15).

10. Discussion and Conclusions

Because of the generality of the scattering matrix $\underline{s}$, as shown by the conditions for its existence of Section 3, we have concentrated on the properties of passive networks in terms of $\underline{s}$. The most fundamental results are those of Theorem 1, which gives a complete characterization of (linear, solvable, and) passive networks in terms of $\underline{s}$, and Theorem 5, which completely characterizes lossless networks through $\underline{s}$. Since Theorem 1 relies on the difficult calculation of $\|\underline{s}\|$, an alternate characterization in terms of an energy from $\underline{Q}_t$ was given in Theorem 4. Although the properties of $\underline{Q}_t$ may often be equally hard to find, it yields conditions on the impedance $\underline{z}$ and shows how the time-domain approach of this paper reduces to the classical frequency-domain approach used for time-invariant networks.

Since finite networks are of considerable practical importance, general results for this class of time-variable networks were considered, and, in particular, lossless conditions were obtained both on $\underline{s}$ and $\underline{z}$ in Section 8. Of most interest is Theorem 7 which gives a complete characterization of finite lossless networks. From the resulting form, synthesis methods can be developed which should allow the synthesis of all finite passive networks described by a scattering matrix. Because the cascade load is of importance for such synthesis methods and because almost all interconnections of networks can be considered as of the cascade loaded form, the scattering matrix for such a connection was investigated in Section 7, where a method of finding the over-all scattering matrix was given. Although this latter will usually exist for physically meaningful connections, Example 3 of Section 9 shows that care must be used, as somewhat degenerate networks can result from the interconnection of reasonable networks. In particular, the connection of two solvable networks need not be solvable, this in spite of the fact that all physically constructable networks are solvable.

The remaining examples of Section 9 show the ease with which the developed lossless criteria can be used, and various tricks which are useful in avoiding some of the tedious calculations needed in applying passivity conditions. The essential difference between finite and nonfinite networks is illustrated by the time-variable delay which also points out the generality of the theory. Example 2, besides illustrating various calculations, points out that "obvious" results of time-invariant theory can not be carried over to time-variable theory, since an interconnection of lossless networks need not be lossless.

In final summary, the material of this paper prepares a firm foundation for the synthesis of passive time-variable networks and rigorously establishes properties already used in time-varying lossless synthesis [19]. As such it generalizes known results of time-invariant networks [1] [2] [3] while also opening up for investigation other known time-invariant results [25] for extension to time-variable situations. The results, although developed in a network context are of course applicable to any passive scattering system.

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APPENDIX 1 (Activity of $z = \delta^{(3)}$):

We show here that the impedance $z(t,\tau) = \delta^{(3)}(t-\tau)$ fails to satisfy the passivity criterion on R_t of Eq. (5.4). The method used is distinct from an essentially ad hoc procedure of Spaulding [19, p. 48]. We shall evaluate $\langle R_t(\alpha,\beta)\bullet\varphi(\beta), \varphi(\alpha)\rangle$, with $R_t(\alpha,\beta)=u(t-\alpha)\delta^{(3)}(\alpha-\beta)+u(t-\beta)\delta^{(3)}(\beta-\alpha)$, explicitly for arbitrary φ, and observe that the resulting expression is not necessarily positive.

We have

$$
\begin{aligned}
\langle R_t(\alpha,\beta)\bullet\varphi(\beta), \varphi(\alpha)\rangle &= 2\langle u(t-\alpha)\delta^{(3)}(\alpha-\beta)\bullet\varphi(\beta), \varphi(\alpha)\rangle \\
&= 2\langle u(t-\alpha)\varphi^{(3)}(\alpha), \varphi(\alpha)\rangle \\
&= 2\int_{-\infty}^{t}\varphi^{(3)}(\alpha)\varphi(\alpha)d\alpha \\
&= \int_{-\infty}^{t}\frac{d}{d\alpha}[2\varphi''(\alpha)\varphi(\alpha) - \varphi'(\alpha)\varphi'(\alpha)]d\alpha \\
&= 2\varphi''(t)\varphi(t) - [\varphi'(t)]^2
\end{aligned}
$$

since φ and all its derivatives are zero at the lower limit. Clearly this expression has arbitrary sign depending on the selection of φ.

APPENDIX 2 (Null Spaces for Cascade Loading):

Without preamble, we carry over the notation of Section 7. First observe that since $\underset{\sim}{\Sigma}$ and $\underset{\sim}{s}_\ell$ are passive scattering matrices, the same is true of

$$\hat{\underset{\sim}{s}}_\ell = \begin{bmatrix} \delta \underset{\sim}{1}_n & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{s}_\ell \end{bmatrix} \tag{A2.1}$$

and thus also (by Theorem 2)

$$\hat{\underset{\sim}{\Sigma}} = \underset{\sim}{\Sigma} \circ \hat{\underset{\sim}{s}}_\ell = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \circ \underset{\sim}{s}_\ell \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \circ \underset{\sim}{s}_\ell \end{bmatrix} \tag{A2.2}$$

Consequently to demonstrate

$$\mathcal{N}[\delta \underset{\sim}{1}_m - \underset{\sim}{\Sigma}_{22} \circ \underset{\sim}{s}_\ell] \subset \mathcal{N}[\underset{\sim}{\Sigma}_{12} \circ \underset{\sim}{s}_\ell] \tag{7.5b}$$

it is sufficient to show that

$$\mathcal{N}[\delta \underset{\sim}{1}_m - \underset{\sim}{\Sigma}_{22}] \subset \mathcal{N}[\underset{\sim}{\Sigma}_{12}] \tag{A2.3}$$

for any passive $\underset{\sim}{\Sigma}$. To prove this we first establish a preliminary result.

Lemma A_2: If $\underset{\sim}{\Sigma}$ is passive, then in the sense of Eq. (5.1)

$$\delta \underset{\sim}{1}_m - \underset{\sim}{\Sigma}_{22}^{a} \circ \underset{\sim}{\Sigma}_{22} - \underset{\sim}{\Sigma}_{12}^{a} \circ \underset{\sim}{\Sigma}_{12} \geq 0 \tag{A2.4}$$

Proof: From Theorem 4 and Eq. (5.5) applied to vectors of the form $\tilde{\underset{\sim}{y}} = [\tilde{\underset{\sim}{0}}_n, \tilde{\underset{\sim}{x}}]$, $\underset{\sim}{x} \in \underset{\sim}{\mathcal{D}}$, the result follows immediately. Q.E.D.

A simple extension using condition (4′) of Theorem 4 is, for any $\underset{\sim}{x} \in \underset{\sim}{\mathcal{D}}_+$,

$$\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle_t - \langle \underset{\sim}{\Sigma}_{22} \bullet \underset{\sim}{x}, \underset{\sim}{\Sigma}_{22} \bullet \underset{\sim}{x} \rangle_t - \langle \underset{\sim}{\Sigma}_{12} \bullet \underset{\sim}{x}, \underset{\sim}{\Sigma}_{12} \bullet \underset{\sim}{x} \rangle_t \geq 0 \tag{A2.5}$$

Now to establish Eq. (A2.3), assume there is a $\underset{\sim}{D}_+$ function $\underset{\sim}{x} \epsilon \mathcal{N}[\delta \underset{\sim}{1}_m - \underset{\sim}{\Sigma}_{22}]$. Then $\underset{\sim}{x} = \underset{\sim}{\Sigma}_{22} \bullet \underset{\sim}{x}$ and thus $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle_t - \langle \underset{\sim}{\Sigma}_{22} \bullet \underset{\sim}{x}, \underset{\sim}{\Sigma}_{22} \bullet \underset{\sim}{x} \rangle_t = 0$. From Eq. (A2.5) it follows that $\langle \underset{\sim}{\Sigma}_{12} \bullet \underset{\sim}{x}, \underset{\sim}{\Sigma}_{12} \bullet \underset{\sim}{x} \rangle_t = 0$, and thus $\underset{\sim}{\Sigma}_{12} \bullet \underset{\sim}{x} = 0$, that is, $\underset{\sim}{x} \epsilon \mathcal{N}[\underset{\sim}{\Sigma}_{12}]$.

APPENDIX 3 (Unit Capacitor Loaded Transformer):

We consider the network shown in Fig. 4, under the assumption that the turns ratio $T(t)$ is never zero. Then the voltage and current at the input port are related by

$$\frac{d}{dt}\left[\frac{v(t)}{T(t)}\right] = T(t)i(t) \qquad (A3.1)$$

Replacing the variables v and i by v^i and v^r with the aid of Eqs. (3.4) we obtain the following relation between v^i and v^r:

$$\dot{v}^r + [T^2(t) - \frac{\dot{T}(t)}{T(t)}]v^r = -\dot{v}^i + [T^2(t) + \frac{\dot{T}(t)}{T(t)}]v^i \qquad (A3.2)$$

Since s maps v^i into v^r, $s(t,\tau)$ is the antecedal impulse response of the above equation. Methods of finding this impulse response are by now well-known, being detailed in, for example, [20, p. 355], and one may easily determine

$$s(t,\tau) = -\delta(t-\tau) + \left\{\frac{2T(t)}{\exp[\int_\alpha^t T^2(\lambda)d\lambda]} \cdot T(\tau)\exp[\int_\alpha^\tau T^2(\lambda)d\lambda]\right\} u(t-\tau) \qquad (9.4)$$

Here the real finite constant α is arbitrary. One can apply the cascade load results to also obtain this, which, although the calculations are somewhat more tedious, shows that Eq. (9.4) still holds when $T(t)$ is permitted to equal zero, since
$\delta-\Sigma_{22}\circ s_\ell = 2[\delta(t-\tau)-(1-T^2(t))e^{-(t-\tau)}u(t-\tau)]/(1+T^2(t))$ is never singular.

The adjoint is found in the usual fashion:

$$s^a(t,\tau) = -\delta(t-\tau) + \left\{2T(t)\exp[\int_\alpha^t T^2(\lambda)d\lambda]\cdot\frac{T(\tau)}{\exp[\int_\alpha^\tau T^2(\lambda)d\lambda]}\right\} u(\tau-t) \qquad (A3.3)$$

The composition product $\underset{\sim}{s}^a \circ \underset{\sim}{s}$ can now be formed; it consists of the sum of a $\delta(t-\tau)$ term, a term multiplied by $u(\tau-t)$, and a term multiplied by $u(t-\tau)$. The last term is given, by inspection of Eqs. (A3.3) and (9.4), as

$$F(t,\tau) = \frac{-2T(t)}{\exp\left[\int_\alpha^t T^2(\lambda)d\lambda\right]} T(\tau)\exp\left[\int_\alpha^\tau T^2(\lambda)d\lambda\right] +$$

$$4T(t)\exp\left[\int_\alpha^t T^2(\lambda)d\lambda\right] \cdot \int_t^\infty \frac{T^2(\sigma)d\sigma}{\exp\left[2\int_\alpha^\sigma T^2(\lambda)d\lambda\right]} \cdot T(\tau)\exp\left[\int_\alpha^\tau T^2(\lambda)d\lambda\right] \quad \text{(A3.4)}$$

Now observe that

$$\int_t^\infty \frac{T^2(\sigma)d\sigma}{\exp\left[2\int_\alpha^\sigma T^2(\lambda)d\lambda\right]} = \int_t^\infty - \frac{1}{2}\frac{d}{d\sigma}\left\{\exp\left[-2\int_\alpha^\sigma T^2(\lambda)d\lambda\right]\right\}d\sigma$$

$$= \frac{1}{2\exp\left[2\int_\alpha^t T^2(\lambda)d\lambda\right]} - \frac{1}{2\exp\left[2\int_\alpha^\infty T^2(\lambda)d\lambda\right]} \quad \text{(A3.5)}$$

and thus

$$F(t,\tau) = \frac{-2T(t)\exp\left[\int_\alpha^t T^2(\lambda)d\lambda\right]T(\tau)\exp\left[\int_\alpha^\tau T^2(\lambda)d\lambda\right]}{\exp\left[2\int_\alpha^\infty T^2(\lambda)d\lambda\right]} \quad \text{(9.5a)}$$

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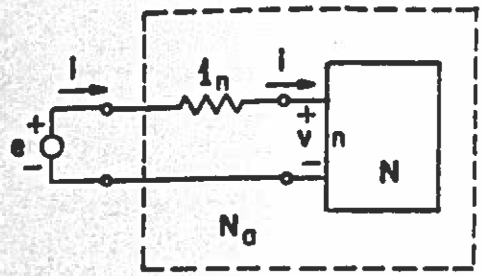

Figure 1
Augmented Network

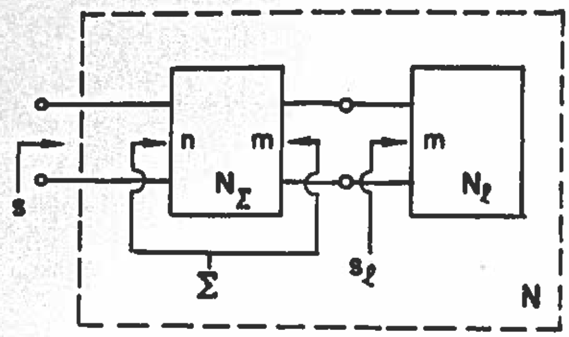

Figure 2
Cascade Loading

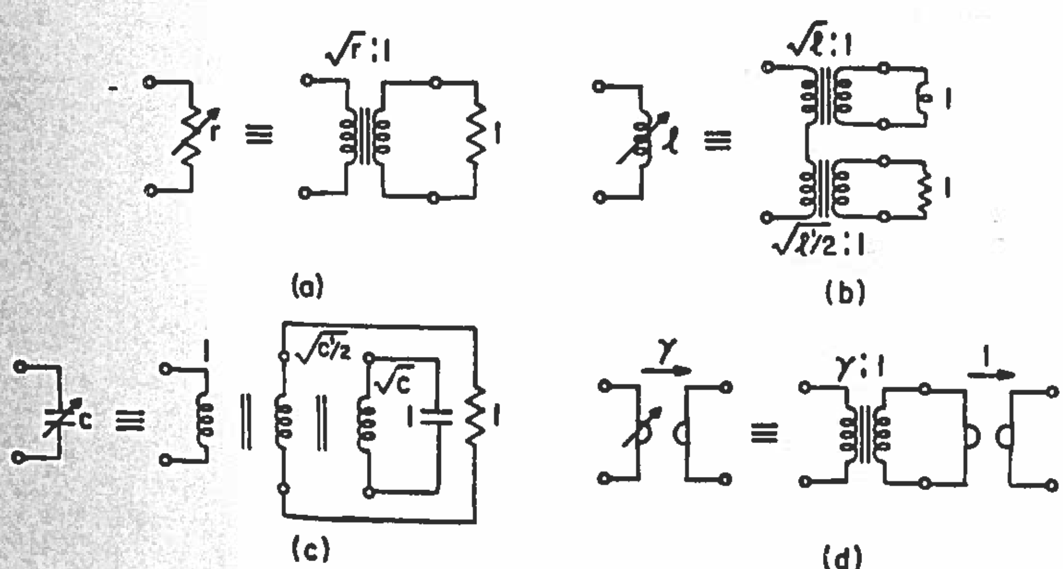

Figure 3
Transformer Replacements

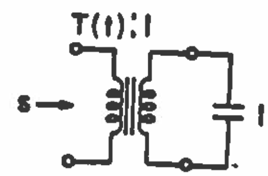

Figure 4
Capacitor Loaded Transformer

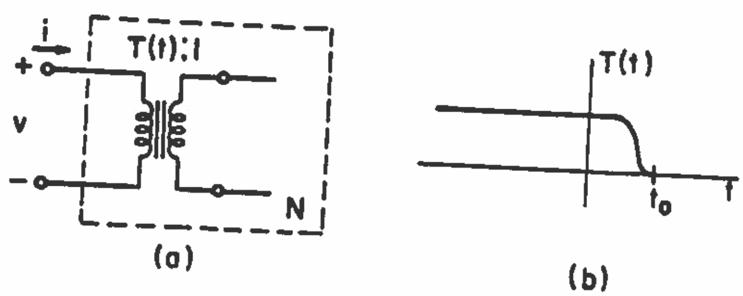

Figure 5
Open Circuit Loaded Transformer

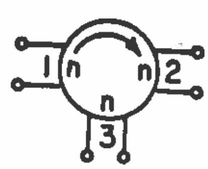

Figure 6
3n-Port Circulator Symbol

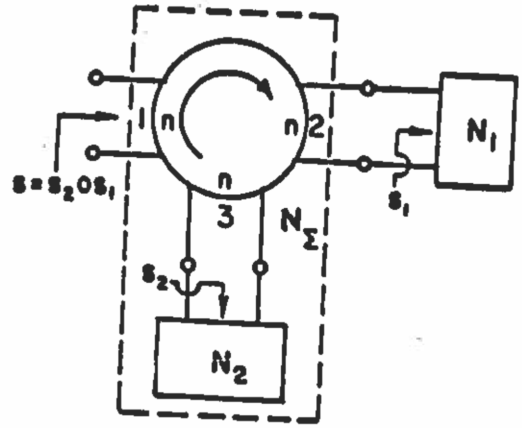

Figure 7
Interpretation of Theorem 2-

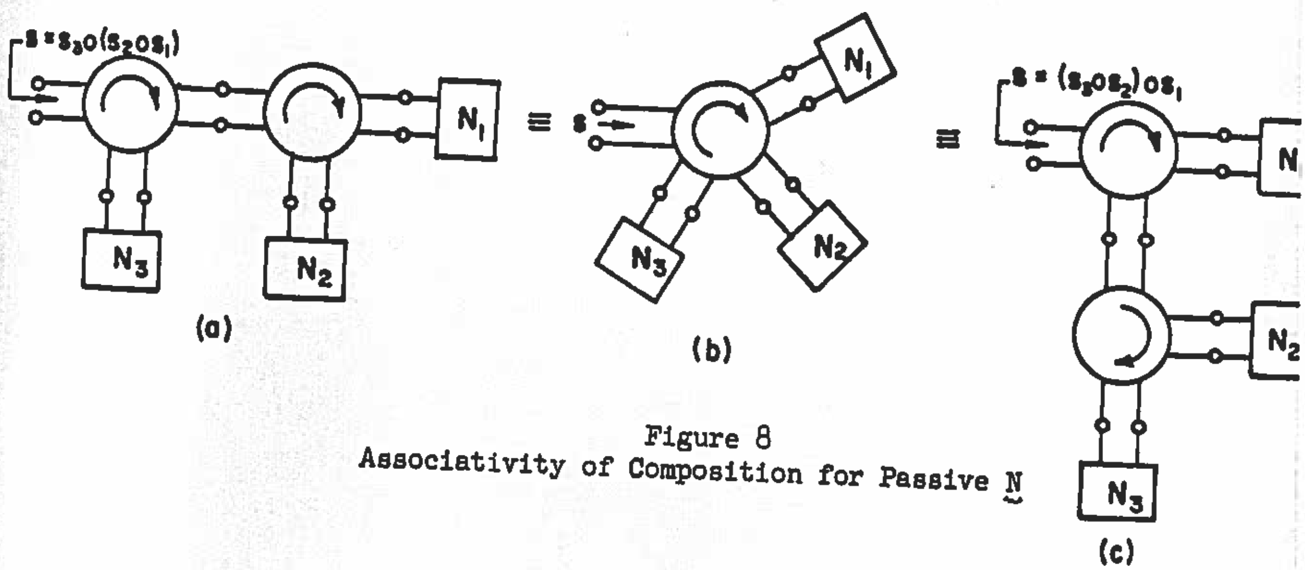

Figure 8
Associativity of Composition for Passive N

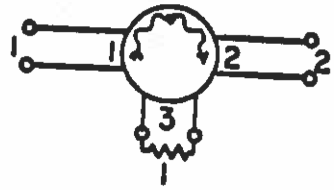

Figure 9
Loaded Time-Variable Circulator For Time-Variable Delay