Stability of passive time-variable circuits

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Synopsis

Lyapunov's second method is used to show that circuits composed of a finite connection of linear, passive, time-variable elements are necessarily stable. Such connections may involve transformers and gyrators as circuit elements, as well as the conventional resistors, inductors and capacitors. An equivalent structure is determined, using only time-variable gyrators to interconnect constant elements consisting of unit capacitors and unit resistors. Using this model, asymptotic stability and uniform asymptotic stability of the original circuit may be investigated. Examples are given to illustrate the methods and concepts.

List of principal symbols

b = total number of branches = $\mathfrak{t} + l$
C = cut-set matrix
C_1, C_2, C_3 = link connections in cut-set matrix
c = number of capacitive tree branches
g = number of gyrator tree branches
$i_b(t)$ = $b$vector of all branch currents
l = number of gyrator link branches
$Q(\boldsymbol{v}_c)$ = quadratic energy function
$Q_s(\boldsymbol{v}_s, t)$ = quadratic energy function expressed in terms of a state vector
r = number of resistive tree branches
$T(t)$ = transformation matrix from state space to 'pseudostate' space
t = time
$\mathfrak{t}$ = number of tree branches = $c + r + g$
$V(x, t)$ = scalar function which may be positive-definite; a Lyapunov function
$\boldsymbol{v}_b(t)$ = $b$vector of all branch voltages
$\boldsymbol{v}_c, \boldsymbol{v}_r, \boldsymbol{v}_g$ = vectors representing designated tree-branch voltages
$\boldsymbol{v}_s(t)$ = $n$vector of state-space variables
$\boldsymbol{v}_\mathfrak{t}$ = $\mathfrak{t}$vector of all tree voltages
$W(x)$ = scalar function which may be positive-definite
$x(t)$ = state vector for an arbitrary system
$y_b(t, \tau)$ = time-variable $b \times b$ admittance matrix
$\mathbf{1}_c, \mathbf{1}_r, \mathbf{1}_g$ = unit matrixes of order designated
$\boldsymbol{\Gamma}$ = square matrix of order $g + l$, giving gyrator branch-by-branch contributions to the admittance matrix
$\boldsymbol{\Gamma}_g, \boldsymbol{\Gamma}_{gl}, \boldsymbol{\Gamma}_l$ = partitioned portions of $\boldsymbol{\Gamma}$
γ = gyration resistance
τ = time

1 Introduction

The primary object of this paper is to show that any finite connection of passive, time-variable (and linear) circuit elements is stable in the sense of Lyapunov. This is accomplished by making circuit replacements to obtain 'pseudostate' variables as voltages across graphically independent capacitors. From such an equivalent circuit, a cut-set analysis shows that the energy in the capacitors serves as a cuitable Lyapunov function, as one would expect from physical considerations.

The pseudostate variables satisfy a first-order system of linear differential equations, and, together with suitable linear equations of constraint, describe the equivalent circuit. By appropriate manipulations, these equations could be reduced to a conventional state–space formulation, employing only a minimum of independent variables. This reduction is not required here to investigate the various concepts of stability. A recent paper by Kuh[1] utilised a state–space formulation of the linear-circuit equations to investigate various properties of stability, by means of Lyapunov's second method. This paper extends the work of Kuh and that of previous workers,[2–4] to include transformers and gyrators in the circuits, and also generalises Lyapunov stability to all finite, time-variable, linear circuits which are constructed from passive elements.

The asymptotically stable behaviour of all voltages across resistors in the equivalent circuit is readily deduced. A simple counterexample shows that uniform asymptotic stability is not implied by asymptotic stability in time-variable circuits, as it is, on the other hand, in time-invariant circuits.

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2 Reduction of an equivalent circuit

Consider a finite connection of linear passive time-variable-circuit elements. In general, these will comprise inductors, capacitors, resistors and also ideal transformers and gyrators. The analytic requirements for passivity of these individual elements have been thoroughly explored elsewhere.[5,6] Briefly, these time-variable elements are called passive if the energy into the element, evaluated at any given time, is never negative. It has been demonstrated that time-variable transformers and gyrators are lossless,[7] and that time-variable inductors, capacitors and resistors can be replaced by constant elements of magnitude ± 1, as viewed through transformers of time-variable turns ratio.[8] It has previously been determined that, when a passive element is replaced in this fashion, no negative constant elements are introduced.[8]

2.1 Gyrator insertions

It is well known that a transformer may be decomposed into a cascade connection of two gyrators[9] (see Fig. 1). The gyrator conventions of this paper are shown in Fig. 2. Utilising this concept, the passive, time-variable inductor may be replaced, as shown in Fig. 3, by gyrators and unit-positive

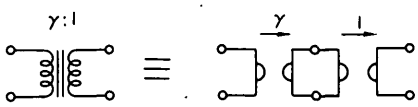

Fig. 1
Gyrator equivalent of the transformer

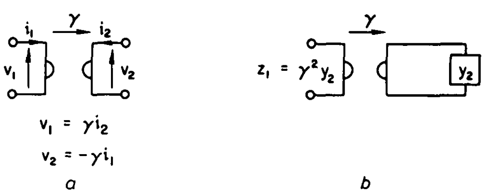

Fig. 2
Gyrator conventions and inverting property
a Gyrator conventions; *b* Inverting property; γ is the gyration resistance

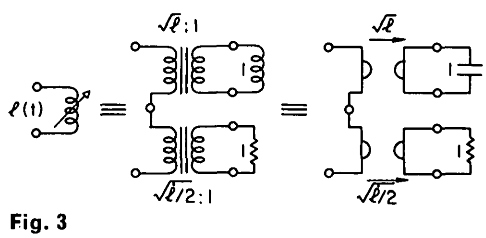

Fig. 3
Gyrator equivalent of a passive time-variable inductance

elements. This substitution may be carried out in three stages. First, the transformer equivalent of Fig. 3 may be substituted. Secondly, each transformer may be replaced by a cascade connection of two gyrators, as shown in Fig. 1. Thirdly, the gyrator structure may be simplified; e.g. a unit gyrator terminated in a unit inductance is a unit capacitor, and a unit gyrator terminated in a unit resistor is just a unit resistor.

In Figs. 4 and 5, the time-variable capacitor and resistor have each been replaced in a manner equivalent to the inductive case.

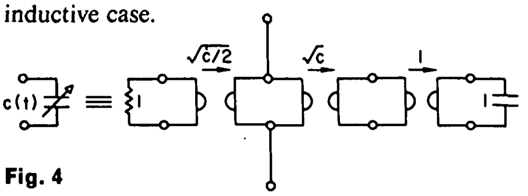

Fig. 4
Gyrator equivalent of a passive time-variable capacitor

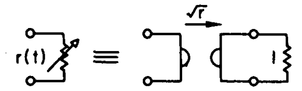

Fig. 5
Gyrator equivalent of a passive time-variable resistor

2.2 Equivalent circuit

Returning to the original passive circuit, each element may be replaced, through the above reasoning, by an appropriate assemblage of unit capacitors, resistors and time-variable gyrators, as required. The new circuit will thus develop the structure shown in Fig. 6.

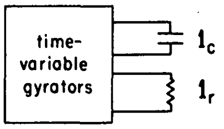

Fig. 6
Final equivalent-circuit structure

3 Formulation of equations

In order to analyse the equivalent circuit, assume that the capacitors and resistors are all in tree branches. This can be done without loss of generality, since any loop connection of resistors and/or capacitors which may remain in the equivalent circuit can be opened into tree branches in the graph by choosing one of the loop elements and inserting an isolating transformer. This procedure, while, of course, not severing the electrical loop, will sever the topological loop. These isolating transformers will then be replaced by gyrators, as previously discussed (see Fig. 1).

The capacitor and resistor tree branches may now be arranged with the capacitors first. Since there may also be some gyrator tree branches, a cut-set matrix[10] can be written, which can be partitioned as follows:

$$C = \begin{bmatrix} \mathbf{1}_c & & & C_1 \\ & \mathbf{1}_r & & C_2 \\ & & \mathbf{1}_g & C_3 \end{bmatrix} \quad \ldots \ldots \ldots \quad (1)$$

The $c \times c$ unit matrix $\mathbf{1}_c$ represents the branch-current connections of the c capacitors in the equivalent circuit, where $c =$ (number of capacitors in the original network) + (number of inductors in the original network). The rank of the unit matrix $\mathbf{1}_r$ is $r =$ (number of resistors in the original circuit) + (number of nonconstant reactive elements in the original circuit). The rank of $\mathbf{1}_g$ is $g =$ number of gyrator tree branches in the equivalent circuit. Now, the total number of branches in the equivalent circuit is $b = t + l$. Letting $\boldsymbol{i}_b$ equal the $b$vector of branch currents, the Kirchhoff current law becomes

$$\mathbf{0} = C\boldsymbol{i}_b(t) \quad \ldots \ldots \ldots \quad (2)$$

Here $\boldsymbol{i}_b$ has been partitioned into tree branches, followed by link branches; so that the last column of the constant $t \times b$ matrix C in eqn. 1 represents the link connections.

Let $\tilde{C}$ be the transpose of the matrix C. Then the $b$vector of branch voltages $\boldsymbol{v}_b$ is related to the $t$vector of tree voltages $\boldsymbol{v}_t$ by

$$\boldsymbol{v}_b(t) = \tilde{C}\boldsymbol{v}_t(t) \quad \ldots \ldots \ldots \quad (3)$$

Branch-by-branch nodal equations for the equivalent circuit may now be written in terms of Volterra composition of the branch voltages with the time-domain admittance matrix.[11] Thus,

$$\boldsymbol{i}_b(t) = y_b.\boldsymbol{v}_b = \int_{-\infty}^{\infty} y_b(t, \tau)\boldsymbol{v}_b(\tau)d\tau \quad \ldots \ldots \quad (4)$$

where

$$y_b(t, \tau) = \begin{bmatrix} \delta'(t-\tau)\mathbf{1}_c & & \\ & \delta(t-\tau)\mathbf{1}_r & \\ & & \delta(t-\tau)\boldsymbol{\Gamma}(t) \end{bmatrix} \quad . \quad (5)$$

Here δ and δ' represent the unit impulse and its derivative. The composition given by eqn. 4 is to be interpreted most generally in the distributional sense.[12] The matrix $\boldsymbol{\Gamma}$ in eqn. 5 represents the gyrator branch-by-branch contribution to the admittance matrix. Therefore, $\boldsymbol{\Gamma}$ is skew-symmetric, is equal to $-\tilde{\boldsymbol{\Gamma}}$, and has only one nonzero entry per row. Partitioning this time-variable matrix further into tree (g) and link (l) portions gives

$$\boldsymbol{\Gamma}(t) = \begin{bmatrix} \boldsymbol{\Gamma}_g & \boldsymbol{\Gamma}_{gl} \\ -\tilde{\boldsymbol{\Gamma}}_{gl} & \boldsymbol{\Gamma}_l \end{bmatrix} \quad \ldots \ldots \ldots \quad (6)$$

3.1 Pseudostate equations

A set of first-order linear differential equations may now be generated to describe the behaviour of the equivalent circuit. Using eqns. 2 and 3 in eqn. 4 gives

$$\mathbf{0} = Cy_b\tilde{C}\boldsymbol{v}_t \quad \ldots \ldots \ldots \quad (7)$$

The matrix $Cy_b\tilde{C}$ is calculated as

$$\begin{bmatrix} \mathbf{1}_c & & & C_1 \\ & \mathbf{1}_r & & C_2 \\ & & \mathbf{1}_g & C_3 \\ & & & \end{bmatrix} \begin{bmatrix} \delta'\mathbf{1}_c & & & \\ & \delta\mathbf{1}_r & & \\ & & \delta\boldsymbol{\Gamma}_g & \delta\boldsymbol{\Gamma}_{gl} \\ & & -\delta\tilde{\boldsymbol{\Gamma}}_{gl} & \delta\boldsymbol{\Gamma}_l \end{bmatrix} \begin{bmatrix} \mathbf{1}_c & & \\ & \mathbf{1}_r & \\ & & \mathbf{1}_g \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \delta'\mathbf{1}_c & \mathbf{0} & -\delta C_1\tilde{\boldsymbol{\Gamma}}_{gl} & \delta C_1\boldsymbol{\Gamma}_l \\ \mathbf{0} & \delta\mathbf{1}_r & -\delta C_2\tilde{\boldsymbol{\Gamma}}_{gl} & \delta C_2\boldsymbol{\Gamma}_l \\ \mathbf{0} & \mathbf{0} & \delta\boldsymbol{\Gamma}_g - \delta C_3\tilde{\boldsymbol{\Gamma}}_{gl} & \delta\boldsymbol{\Gamma}_{gl} + \delta C_3\boldsymbol{\Gamma}_l \\ & & & \end{bmatrix} \begin{bmatrix} \mathbf{1}_c & & \\ & \mathbf{1}_r & \\ & & \mathbf{1}_g \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \delta'\mathbf{1}_c + \delta C_1\boldsymbol{\Gamma}_l\tilde{C}_1 & \delta C_1\boldsymbol{\Gamma}_l\tilde{C}_2 & -\delta C_1\tilde{\boldsymbol{\Gamma}}_{gl} + \delta C_1\boldsymbol{\Gamma}_l\tilde{C}_3 \\ \delta C_2\boldsymbol{\Gamma}_l C_1 & \delta\mathbf{1}_r + \delta C_2\boldsymbol{\Gamma}_l\tilde{C}_2 & -\delta C_2\tilde{\boldsymbol{\Gamma}}_{gl} + \delta C_2\boldsymbol{\Gamma}_l\tilde{C}_3 \\ \delta\boldsymbol{\Gamma}_{gl}\tilde{C}_1 + \delta C_3\boldsymbol{\Gamma}_l\tilde{C}_1 & \delta\boldsymbol{\Gamma}_{gl}\tilde{C}_2 + \delta C_3\boldsymbol{\Gamma}_l\tilde{C}_2 & \delta\boldsymbol{\Gamma}_g - \delta C_3\tilde{\boldsymbol{\Gamma}}_{gl} + \delta\boldsymbol{\Gamma}_{gl}\tilde{C}_3 + \delta C_3\boldsymbol{\Gamma}_l\tilde{C}_3 \end{bmatrix} \quad (8)$$

If we now let

$$\tilde{\boldsymbol{v}}_t = [\tilde{\boldsymbol{v}}_c, \tilde{\boldsymbol{v}}_r, \tilde{\boldsymbol{v}}_g] \quad \ldots \ldots \ldots \quad (9)$$

eqn. 7 becomes

$$\begin{bmatrix} -\dot{\boldsymbol{v}}_c \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} C_1\boldsymbol{\Gamma}_l\tilde{C}_1 & C_1\boldsymbol{\Gamma}_l\tilde{C}_2 & -C_1\tilde{\boldsymbol{\Gamma}}_{gl} + C_1\boldsymbol{\Gamma}_l\tilde{C}_3 \\ C_2\boldsymbol{\Gamma}_l\tilde{C}_1 & \mathbf{1}_r + C_2\boldsymbol{\Gamma}_l\tilde{C}_2 & -C_2\tilde{\boldsymbol{\Gamma}}_{gl} + C_2\boldsymbol{\Gamma}_l\tilde{C}_3 \\ \boldsymbol{\Gamma}_{gl}\tilde{C}_1 + C_3\boldsymbol{\Gamma}_l\tilde{C}_1 & \boldsymbol{\Gamma}_{gl}\tilde{C}_2 + C_3\boldsymbol{\Gamma}_l\tilde{C}_2 & \boldsymbol{\Gamma}_g - C_3\tilde{\boldsymbol{\Gamma}}_{gl} + \boldsymbol{\Gamma}_{gl}\tilde{C}_3 + C_3\boldsymbol{\Gamma}_l\tilde{C}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_c \\ \boldsymbol{v}_r \\ \boldsymbol{v}_g \end{bmatrix} \quad . \quad (10)$$

If the original circuit contains neither capacitive loops nor inductive cut sets, the vector $\boldsymbol{v}_c$ in eqn. 10 is a true state vector for the equivalent circuit.

If the original circuit contains either capacitive loops and/or inductive cut sets, further argument is necessary.

First, because of the possible inclusion of transformers and gyrators, the presence of what are 'effectively' capacitive loops and inductive cut sets in the original circuit may be hard to recognise. However, the presence of such anomalies does not alter either the formation of our equivalent circuit, or the method of analysis. In this latter case, eqn. 10 will exhibit the degeneracy of the components of $\boldsymbol{v}_c$ as a set of linear constraints, of order equal to the number of effective capacitive loops plus the effective number of inductive cut sets. If m represents the number of capacitors in the equivalent circuit, and n is the order of a true state vector, $m > n$. Let $\boldsymbol{v}_s$ be a true state vector, then

$$\boldsymbol{v}_c = \boldsymbol{T}(t)\boldsymbol{v}_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which has terms of dimensions $(m \times 1) = (m \times n)(n \times 1)$, and where the time-variable transformation matrix $\boldsymbol{T}(t)$ is necessarily of rank n.

If $m > n$, let us call $\boldsymbol{v}_c$ a pseudostate vector. If $m = n$, obviously $\boldsymbol{v}_s$ can equal $\boldsymbol{v}_c$, or one can choose $\boldsymbol{T} = \mathbf{1}_n$.

3.2 Energy functions

Let us form the quadratic function

$$Q(\boldsymbol{v}_c) = \tfrac{1}{2}\tilde{\boldsymbol{v}}_c\boldsymbol{v}_c > 0, \ \boldsymbol{v}_c \neq \mathbf{0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is a positive-definite function of $\boldsymbol{v}_c$, and represents true instantaneous (reversible) energy storage in the unit capacitors of the equivalent circuit. Substituting from eqn. 11 into eqn. 12, we obtain

$$Q(\boldsymbol{v}_c) = Q_s(\boldsymbol{v}_s, t) = \tfrac{1}{2}\tilde{\boldsymbol{v}}_s\tilde{\boldsymbol{T}}\boldsymbol{T}\boldsymbol{v}_s > 0, \ \boldsymbol{v}_s \neq \mathbf{0} \quad . \quad . \quad (13)$$

The positive-definite character of this last quadratic form in $\boldsymbol{v}_s$ follows from the fact that the $m \times n$ matrix $\boldsymbol{T}$ is of rank n, so that $\tilde{\boldsymbol{T}}\boldsymbol{T}$ is a symmetric matrix of rank n.

Using $Q(\boldsymbol{v}_c) > 0$, we can show, through the Lyapunov theory, that all circuit connections of passive circuit elements are stable. We therefore turn to a brief review of the Lyapunov theory.

4 Lyapunov stability[13–16]

The system of eqns. 10, is generally nonautonomous for time-variable systems. Consider the linear system

$$\dot{x}(t) = A(t)x(t), \text{ for all } t > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $x(t)$ is a state vector of dimension n, and $A(t)$ is an $n \times n$ matrix. Let $||x||$ denote any convenient norm[15] which metrises the space of real $n$vectors, and let $x(t;\, x_0,\, t_0)$ denote that solution (state) vector which has initial value x_0 at time $t = t_0$.

The linear system, eqn. 14, is called zero-state stable if, given any $\epsilon > 0$, there exists a $\delta > 0$ depending on ϵ and t_0, so that, for any $t_0 \geqslant 0$,

$$||x_0|| < \delta \text{ implies } ||x(t;\, x_0,\, t_0)|| < \epsilon, \text{ for all } t \geqslant t_0 \quad . \quad (15)$$

Alternatively, we can say that $x = \mathbf{0}$ is stable in the sense of Lyapunov[15] if eqn. 15 holds.

The scalar function $W(x)$ is called positive-definite if

(*a*) $W(x)$ is continuous, together with its first partial derivatives, in an open region Ω about the origin
(*b*) $W(\mathbf{0}) = 0$
(*c*) $W(x) > 0$, for all x in Ω such that $x \neq \mathbf{0}$.

More generally, the scalar $V(x, t)$ is called positive-definite if

(i) $V(x, t)$ is defined for all x in Ω, $t \geqslant 0$
(ii) $V(\mathbf{0}, t) = 0$, for all $t \geqslant 0$
(iii) there exists a positive-definite function $W(x)$, so that $V(x, t) \geqslant W(x)$ for all x in Ω, $t \geqslant 0$.

If, in addition, $\dot{V}(x, t) \leqslant 0$ for x in Ω, $V(x, t)$ is called a Lyapunov function.[16]

Stability theorem.[14] If there exists a Lyapunov function $V(x, t)$ for the general nonlinear system

$$\dot{x}(t) = f(x, t), \text{ for all } t \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

then, given any $\epsilon > 0$, such that, for any $t_0 \geqslant 0$, we can choose a $\delta > 0$ depending on ϵ, f and t_0, and so that

$$||x_0|| < \delta \text{ implies } ||x(t;\, x_0,\, t_0)|| < \epsilon, \text{ for all } t \geqslant t_0 \quad . \quad (17)$$

We can then say that the general nonlinear system is zero-state stable or stable in the sense of Lyapunov about the origin $x = \mathbf{0}$.

If the system is linear and time-variable, eqn. 17 holds for any $\delta > 0$ if ϵ is chosen greater than δ, and the linear system is Lyapunov-stable for any bounded $||x_0||$; i.e. for linear systems, boundedness and stability of solutions are equivalent.

4.1 The Lyapunov function

Let us now consider the quadratic energy function in eqns. 12 and 13 as a potential Lyapunov function, i.e.

$$\begin{aligned} V(x, t) &= \tfrac{1}{2}\tilde{\boldsymbol{v}}_c\boldsymbol{v}_c > 0, \ \boldsymbol{v}_c \neq \mathbf{0} \\ &= \tfrac{1}{2}\tilde{\boldsymbol{v}}_s\tilde{\boldsymbol{T}}\boldsymbol{T}\boldsymbol{v}_s > 0, \ \boldsymbol{v}_s \neq \mathbf{0} \end{aligned}$$

The positive-definite character of $V(\boldsymbol{v}_c) = V(\boldsymbol{v}_s, t)$ has been established in Section 3.2. The time derivative of V must be investigated. If we premultiply eqn. 10 by $\tilde{\boldsymbol{v}}_t$, this gives

$$-\tilde{\boldsymbol{v}}_c\dot{\boldsymbol{v}}_c = [\tilde{\boldsymbol{v}}_c\tilde{\boldsymbol{v}}_r\tilde{\boldsymbol{v}}_g]\begin{bmatrix} C_1\boldsymbol{\Gamma}_l\tilde{C}_1 & C_1\boldsymbol{\Gamma}_l\tilde{C}_2 & -C_1\tilde{\boldsymbol{\Gamma}}_{gl} + C_1\boldsymbol{\Gamma}_l\tilde{C}_3 \\ C_2\boldsymbol{\Gamma}_l\tilde{C}_1 & \mathbf{1}_r + C_2\boldsymbol{\Gamma}_l\tilde{C}_2 & -C_2\tilde{\boldsymbol{\Gamma}}_{gl} + C_2\boldsymbol{\Gamma}_l\tilde{C}_3 \\ \boldsymbol{\Gamma}_{gl}\tilde{C}_1 + C_3\boldsymbol{\Gamma}_l\tilde{C}_1 & \boldsymbol{\Gamma}_{gl}\tilde{C}_2 + C_3\boldsymbol{\Gamma}_l\tilde{C}_2 & \boldsymbol{\Gamma}_g - C_3\tilde{\boldsymbol{\Gamma}}_{gl} + \boldsymbol{\Gamma}_{gl}\tilde{C}_3 + C_3\boldsymbol{\Gamma}_l\tilde{C}_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_c \\ \boldsymbol{v}_r \\ \boldsymbol{v}_g \end{bmatrix} \quad . \quad . \quad . \quad . \quad (18)$$

Since skew portions do not contribute to quadratic forms, the right-hand side is

$$\tilde{\boldsymbol{v}}_t\begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1}_r & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{v}_t = \tilde{\boldsymbol{v}}_r\boldsymbol{v}_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\text{Thus} \quad \tilde{\boldsymbol{v}}_c\dot{\boldsymbol{v}}_c = -\,\tilde{\boldsymbol{v}}_r\boldsymbol{v}_r \leqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\text{or} \quad \dot{V} = -\,\tilde{\boldsymbol{v}}_r\boldsymbol{v}_r \leqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The derivative in eqn. 21 is thus nonpositive, and represents the (irreversible) rate of energy storage in the loss elements of the equivalent circuit. Furthermore, the quadratic function V satisfies all the conditions for a Lyapunov function as outlined. We have now proved that:

Theorem. All finite, linear circuits which are composed of passive circuit elements are stable in the sense of Lyapunov.

This result is, of course, reasonable and would be expected from physical considerations. Note from eqn. 21 that $\dot{V} = 0$ implies that $\boldsymbol{v}_r = \mathbf{0}$, but this does not imply that either $\boldsymbol{v}_c$ or $\boldsymbol{v}_s$ must equal zero.

4.2 Circuit example

Many of the considerations of this discussion are perhaps best illustrated by a simple example. The passive circuit shown in Fig. 7 includes a capacitive loop and a time-variable resistor. Applying the notation given previously, $\mathbf{0} = \boldsymbol{C}\boldsymbol{i}_b$

$$= \begin{bmatrix} 1 & 0 & & & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & & & 1 & 0 & 0 & 0 & 0 \\ & & 1 & & 0 & 0 & 0 & 0 & 1 \\ & & & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_9 \end{bmatrix} \quad . \quad (22)$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_g & \boldsymbol{\Gamma}_{gl} \\ -\tilde{\boldsymbol{\Gamma}}_{gl} & \boldsymbol{\Gamma}_l \end{bmatrix} = \begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -\alpha & & \\ & & \alpha & 0 & & \\ & & & & 0 & -\beta \\ & & & & \beta & 0 \end{bmatrix} \quad (23)$$

After a little matrix algebra, we obtain, from eqn. 10,

$$\begin{bmatrix} -\dot{v}_{c1} \\ -\dot{v}_{c2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\beta & -\alpha \\ 0 & 0 & 0 & 1 \\ \beta & 0 & 1 & 0 \\ \alpha & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ v_r \\ v_g \end{bmatrix} \quad . \quad . \quad . \quad (24)$$

Eqn. 24 represents the four nodal equations for the equivalent circuit. Further, it depicts the linear relation between the vector components of $\boldsymbol{v}_c$, as introduced by the capacitive loop. This may be checked by inspection of the cascaded-gyrator (transformer) ratio in Fig. 7.

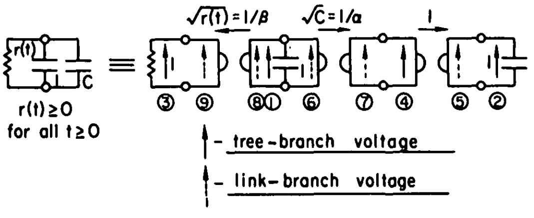

Fig. 7
Passive-time-variable-circuit example

The Lyapunov function becomes

$$V(\boldsymbol{v}_c, t) = \tfrac{1}{2}\tilde{\boldsymbol{v}}_c\boldsymbol{v}_c = \tfrac{1}{2}(v_{c1}^2 + v_{c2}^2)$$
$$= \tfrac{1}{2}(\alpha^2 + 1)v_{c2}^2$$
$$= \tfrac{1}{2}(C^{-1} + 1)v_{c2}^2 \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Furthermore,

$$\dot{V} = \tilde{\boldsymbol{v}}_c\dot{\boldsymbol{v}}_c = -v_r^2 \leqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

5 Asymptotic stability

The linear system, eqn. 14, is called asymptotically stable if (*a*) it is stable in the sense of Lyapunov, and (*b*) there exists a $\delta > 0$ depending on t_0, so that, for any $t_0 \geqslant 0$,

$$||\boldsymbol{x}_0|| < \delta \text{ implies } ||\boldsymbol{x}(t;\, \boldsymbol{x}_0,\, t_0)|| \to 0 \text{ as } t \to \infty \quad (27)$$

Asymptotic-stability theorem.[14] If there exists a Lyapunov function $V(\boldsymbol{x}, t)$ for the general nonlinear system, eqn. 16, so that the derivative

$$\dot{V}(\boldsymbol{x}, t) < 0,\ \boldsymbol{x} \neq 0, \text{ for all } t > t_0 \quad . \quad . \quad . \quad . \quad (28)$$

we can choose a $\delta > 0$ depending on f and t_0 so that

$$||\boldsymbol{x}_0|| < \delta \text{ implies } ||\boldsymbol{x}(t;\, \boldsymbol{x}_0,\, t_0)|| \to 0 \text{ as } t \to \infty. \quad (29)$$

If the system is linear and time-variable, eqn. 29 holds for any $\delta > 0$, and the linear system is asymptotically stable for any bounded $||\boldsymbol{x}_0||$. We say then that we have global asymptotic stability or that we have asymptotic stability in the large.

We saw from eqn. 21 that $\dot{V}$ was not generally a negative-definite function of either $\boldsymbol{v}_c$ or $\boldsymbol{v}_s$, but of $\boldsymbol{v}_r$. Therefore, in each given linear circuit composed of passive elements, the character of $\dot{V}$ must be investigated to determine whether solutions are asymptotic to zero as $t \to \infty$.

The circuit example given in Section 4.2 is asymptotically stable to the zero state. From eqn. 24,

$$v_{c2} = \alpha v_{c1} = -\left(\frac{\alpha}{\beta}\right)v_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Thus, $\dot{V}$, as given in eqn. 26, is a negative-definite function of v_{c1} or v_{c2}. The single-state variable v_s may be identified with any linear combination of v_{c1} and v_{c2}.

5.1 Asymptotic behaviour of $\boldsymbol{v}_r$

While Lyapunov stability does not imply asymptotic stability of $\boldsymbol{v}_c$ or $\boldsymbol{v}_s$, the asymptotic behaviour of $\boldsymbol{v}_r$ is assured. Let

$$V(t_0) = V\{\boldsymbol{v}_c(t_0), t_0\} = \tfrac{1}{2}\tilde{\boldsymbol{v}}_c\boldsymbol{v}_c|_{t_0} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Then, by eqn. 21,

$$V(t_1) - V(t_0) = -\int_{t_0}^{t_1} \tilde{\boldsymbol{v}}_r(t)\boldsymbol{v}_r(t)dt,\ t_0 \leqslant t_1 < \infty \quad (32)$$

and, letting $t_1 \to \infty$,

$$V(\infty) - V(t_0) = -\int_{t_0}^{\infty} \tilde{\boldsymbol{v}}_r(t)\boldsymbol{v}_r(t)dt \quad . \quad . \quad . \quad . \quad (33)$$

If $||\boldsymbol{v}_r||$ does not approach zero as $t \to \infty$,

$$V(\infty) - V(t_0) \to -\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and $\quad V(\infty) \to -\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$

This is, however, impossible, because V is non-negative; $V \geqslant 0$ for all $t \geqslant t_0$. Thus, Lyapunov stability implies that $||\boldsymbol{v}_r|| \to 0$ as $t \to \infty$.

5.2 Uniform asymptotic stability

The linear system given by eqn. 14 is said to be uniformly asymptotically stable if it is (*a*) stable in the sense of Lyapunov, and (*b*) there exists a $\delta > 0$ independent of t_0 so that, for any $t_0 \geqslant 0$,

$$||\boldsymbol{x}_0|| < \delta \text{ implies } ||\boldsymbol{x}(t;\, \boldsymbol{x}_0,\, t_0)|| \to 0 \text{ uniformly as } t \to \infty \quad . \quad . \quad . \quad . \quad (36)$$

More precisely, there exist both $\delta > 0$ and $\epsilon > 0$ independent of t_0, and $T(\epsilon) > 0$ independent of $\boldsymbol{x}_0$ and t_0; so that

$$||\boldsymbol{x}_0|| < \delta \text{ implies } ||\boldsymbol{x}(t)|| < \epsilon \text{ for } t > t_0 + T, \text{ for all } t_0 \geqslant 0 \quad . \quad . \quad . \quad . \quad (37)$$

It is immediately evident that, for time-invariant circuits (and, in fact, for all autonomous systems), asymptotic stability and uniform asymptotic stability are equivalent. Such is not generally the case for time-variable circuits.

A particular case of the circuit example in Section 4.2, which is not uniformly asymptotically stable, is shown in Fig. 8. Here, $C = 0$ and $r(t) = t$, $t > 0$, for the circuit previously analysed (see Fig. 7). It is obvious from the circuit shown in Fig. 8 that

$$-\dot{v}_c = \frac{1}{t}\,v_c,\ t > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

This is also a degenerate case of eqn. 24. Integrating, and letting $v_c(t_0) = v_0$, gives

$$v_c(t) = \frac{t_0}{t}\,v_0,\ t \geqslant t_0 > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

This solution is obviously asymptotically stable in accordance with the definition in Section 5. However, it is not uniformly asymptotically stable.

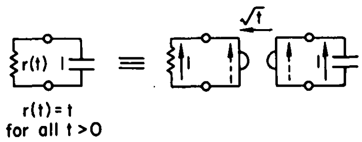

Fig. 8
A particular case of Fig. 7 which does not possess uniform asymptotic stability

Uniform asymptotic stability here implies that, for $|v_0| < \delta$ there exist $\epsilon > 0$ and $T(\epsilon)$ independent of t_0; so that

$$|v_c(t)| < \epsilon, \text{ for all } t \geqslant t_0 + T$$

or, on considering the equality $t = t_0 + T$, this implies that

$$\frac{t_0}{t_0 + T} < \frac{\epsilon}{|v_0|} < \frac{\epsilon}{\delta}, \text{ for all } t_0 > 0 \quad . \quad . \quad . \quad . \quad (40)$$

This is obviously not possible for T independent of t_0, as $t_0 \to \infty$. A system example similar to this has been presented by Zadeh and Desoer,[15] in connection with bounded-input/bounded-output considerations.

6 Conclusions

The stability of general linear circuits has been examined here in considerable detail. It has been shown that those composed of passive elements are always Lyapunov-stable. It is evident that the presence of any negative resistance element in the original circuit would destroy the negative-definite character of the derivative eqn. 21 (as seen from eqn. 19) and hence, also, the necessity for stability. That the same consideration holds for the presence of a negative capacitor (or, equivalently, a negative inductor) can be seen from the well known replacement of a negative capacitor by a positive capacitor, and a tee connection of positive and negative resistors, as shown in Fig. 9.

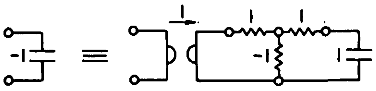

Fig. 9

Negative-capacitor equivalent using passive elements and a negative resistor

Asymptotic stability must generally be investigated in each given circuit example. This can be done by eliminating v_g from eqn. 10, to solve for v_c in terms of v_r. The asymptotic properties of v_c then follow on letting $v_r \rightarrow 0$. The resistor voltages for passive connections will always vanish when the only excitations are initial conditions.

It is evident that the equivalent-network concept of Fig. 6 and the differential eqns. 10 represent a new technique for the formulation of state-space equations for a network.

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