The Padé Table for e^s

Although the coefficients in the general Padé approximant to e^s are well known, the only published proof based on the properties of elementary functions appears to be Padé's proof,[1] which Tuttle[2] quotes as an exercise. This proof involves an expansion of

$$\int_0^1 e^{yx} y^n (1-y)^m \, dy$$

using integration by parts.

The object of this communication is to give an alternative proof, starting from the simultaneous equations for the coefficients in the approximant. The following theorem is required.

Theorem: If $f(x)$ is a polynomial of degree less than m, then

$$S \equiv \sum_{k=0}^{m} \frac{(-1)^k f(k)}{k!\,(m-k)!} = 0. \tag{1}$$

Proof: Let the degree of $f(x)$ be l. By Newton's backward difference formula[3] we have

$$f(x) = \sum_{i=0}^{l} c_i \frac{(m-x)!}{(m-x-i)!}$$

where c_i is the ith backward difference of $f(m)$ divided by $i!$. Hence,

$$S = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \sum_{i=0}^{l} \frac{c_i}{(m-k-i)!}.$$

Changing the order of the summations gives

$$S = \sum_{i=0}^{l} c_i \sum_{k=0}^{m} \frac{(-1)^k}{k!\,(m-k-i)!}.$$

Since $l < m$, and the factorial of a negative integer is infinite, the upper limit of the second sum is effectively $k = m - i$. Hence,

$$S = \sum_{i=0}^{l} c_i \frac{(1-1)^{m-i}}{(m-i)!} = 0.$$

Q.E.D.

The Padé table

The coefficients in the polynomials

$$P_n(s) \equiv a_0 + a_1 s + \cdots + a_n s^n, \qquad a_n \neq 0$$

$$Q_m(s) \equiv b_0 + b_1 s + \cdots + b_m s^m, \qquad b_m \neq 0$$

are to be chosen to make the Maclaurin expansion of P_n/Q_m coincide with that of e^s up to the term in s^N where $N = n + m$. Then,

$$P_n(s) = Q_m(s) e^s \{1 + O(s^{N+1})\}. \tag{2}$$

Equating coefficients of s^r in (2) gives

$$a_r = \sum_{k=0}^{r} \frac{b_k}{(r-k)!}, \qquad r = 0 \text{ to } N, \tag{3}$$

where by definition

$$a_r = 0, \quad r > n; \qquad b_k = 0, \quad k > m.$$

Suppose first that $n \geq m - 1$ (this restriction will be removed later), then the last $m + 1$ equations of (3) are

$$\sum_{k=0}^{m} \frac{b_k}{(r-k)!} = 0, \qquad r = n+1 \text{ to } N. \tag{4}$$

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Comparing (4) with (1), we see that a possible solution for a particular r is

$$b_k = \frac{(-1)^k (r-k)!\, f(k)}{k!\,(m-k)!} \tag{5}$$

where $f(k)$ is a polynomial of degree less than m with coefficients independent of k. To make (5) hold for $r = n + 1$ to N, we must have

$$f(x) = g(x)/(r-x)!$$

for some $g(x)$ independent of both k and r. The only choice yielding a function $f(x)$ which is a polynomial of degree less than m for all these values of r is

$$g(x) = K(N-x)!$$

where K is independent of k and r. Hence, the solution of (4), in terms of an arbitrary b_0, is

$$\frac{b_k}{b_0} = \frac{(-1)^k (N-k)!\, m!}{k!\,(m-k)!\, N!}. \tag{6}$$

If $n < m - 1$ the upper limit of the sum in (4) should be r, as in (3). However, the form (4) may be retained because the factorial of a negative integer is infinite. The analysis then stands, and so (6) is valid for all n, m.

The formula for the a's is obtained by writing (2) as

$$Q_m(-s) = P_n(-s) e^s \{1 + O(s^{N+1})\}. \tag{7}$$

Comparison of (2) and (7) shows that Q_m is P_n with n and m interchanged and s replaced by $-s$. Hence, (6) gives

$$\frac{a_k}{a_0} = \frac{(N-k)!\, n!}{k!\,(n-k)!\, N!}. \tag{8}$$

The complete approximant follows upon taking $a_0 = b_0$.

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The Time-Variable Lattice and Nonreciprocal RLC Networks

Because the lattice network is of fundamental importance in classical network design [1], page 251, [2], page 338, one expects it should also be useful for time-variable synthesis. As this is indeed the case [3], we consider here some useful properties of the lattice which extend to the time-variable case. With a special choice of lattice arms this allows us to extend the examples of [4] and [5] to include a transformerless RLC nonreciprocal network. Likewise this readily allows the synthesis in cascade lattice form of a subclass of passive voltage transfer functions.

Consider the $2n$-port symmetric lattice shown embedded between normalized sources and loads in Fig. 1; here $\mathbf{N}_a$ denotes an n-port with time-variable scattering matrix [6] $\mathbf{s}_a(t, \tau)$ and impedance matrix $\mathbf{z}_a(t, \tau)$, similarly for $\mathbf{N}_b$; $\mathbf{1}_n$ denotes n uncoupled unit resistors, or, in matrix terms, the $n \times n$ identity. By direct calculation one

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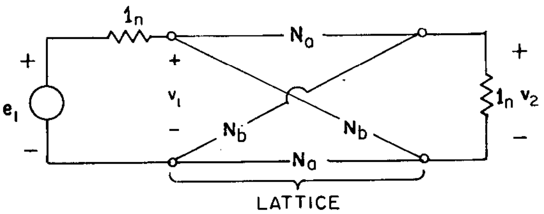

Fig. 1. 2n-port lattice between sources and loads.

finds the lattice described by the $2n \times 2n$ impedance matrix $\mathbf{z} = \mathbf{z}(t, \tau)$

$$\mathbf{z} = \frac{1}{2}\begin{bmatrix} \mathbf{z}_b + \mathbf{z}_a & \mathbf{z}_b - \mathbf{z}_a \\ \mathbf{z}_b - \mathbf{z}_a & \mathbf{z}_b + \mathbf{z}_a \end{bmatrix} \tag{1a}$$

$$= \left\{\frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{1}_n \end{bmatrix}\right\}\begin{bmatrix} \mathbf{z}_a & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{z}_b \end{bmatrix}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{1}_n \end{bmatrix}\right\} \tag{1b}$$

where $\mathbf{0}_n$ is the $n \times n$ zero matrix. Since $\mathbf{z}$ is of the form $\mathbf{z} = \tilde{\mathbf{T}}\mathbf{z}_l\mathbf{T}$, with $\tilde{}$ denoting the transpose and $\mathbf{T}$ orthogonal, $\mathbf{z}$ can be also realized by terminating an orthogonal transformer of turns ratio matrix $\mathbf{T}$ by a load of impedance matrix $\mathbf{z}_l$, where

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{1}_n \end{bmatrix}, \qquad \mathbf{z}_l = \begin{bmatrix} \mathbf{z}_a & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{z}_b \end{bmatrix}. \tag{1c}$$

As a consequence [7], page 11, $\mathbf{s} = \tilde{\mathbf{T}}\mathbf{s}_l\mathbf{T}$ is an expression for the scattering matrix of the lattice, which must then take the same form as $\mathbf{z}$

$$\mathbf{s} = \frac{1}{2}\begin{bmatrix} \mathbf{s}_b + \mathbf{s}_a & \mathbf{s}_b - \mathbf{s}_a \\ \mathbf{s}_b - \mathbf{s}_a & \mathbf{s}_b + \mathbf{s}_a \end{bmatrix}. \tag{2}$$

If the lattice arms are dual, that is, $\mathbf{s}_a = -\mathbf{s}_b$, then

$$\mathbf{s} = \begin{bmatrix} \mathbf{0}_n & \mathbf{s}_b \\ \mathbf{s}_b & \mathbf{0}_n \end{bmatrix}. \tag{3}$$

Using $\cdot$ to denote composition [8], section 2, we directly calculate for Fig. 1

$$2\mathbf{v}_2(t) = \mathbf{s}_{21}\cdot\mathbf{e}_1 = \int_{-\infty}^{\infty} \mathbf{s}_{21}(t, \tau)\mathbf{e}_1(\tau)\, d\tau \tag{4a}$$

where $\mathbf{s}_{21}$ is the (2, 1) $n \times n$ submatrix of the scattering matrix for any 2n-port in the lattice position. In particular, for the lattice with dual arms we find, from (3) and (4a),

$$\mathbf{v}_2 = \tfrac{1}{2}\mathbf{s}_b\cdot\mathbf{e}_1 = \mathbf{s}_b\cdot\mathbf{v}_1 \tag{4b}$$

or, the *voltage transfer matrix* $\frac{1}{2}\mathbf{s}_{21}$ is one-half the scattering matrix of the cross arms, $\frac{1}{2}\mathbf{s}_b$ in this case.

If we now consider the cascade connection, shown in Fig. 2, of lattices with dual arms (denoted by a superscript d), then applying source and load at either end and using (3) and (4) (which show a match at all ports) gives for the network of Fig. 2,

$$\mathbf{s} = \begin{bmatrix} \mathbf{0}_n & \mathbf{s}_1 o \mathbf{s}_2 \\ \mathbf{s}_2 o \mathbf{s}_1 & \mathbf{0}_n \end{bmatrix} \tag{5a}$$

where o denotes Volterra composition [8], section 2, and

$$\mathbf{s}_1 o \mathbf{s}_2 = \int_{-\infty}^{\infty} \mathbf{s}_1(t, \lambda)\mathbf{s}_2(\lambda, \tau)\, d\lambda. \tag{5b}$$

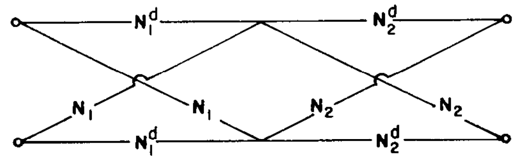

Fig. 2. Cascade of lattices with dual arms.

We conclude that, in contrast to the scalar time-invariant case, the cascade of two or more 2n-port time-variable lattices is generally no longer another lattice. Nevertheless, (5) shows how certain time-variable voltage transfer matrices can be synthesized.

From the above results and the present development of time-variable network synthesis we can give a synthesis of voltage transfer matrices if $\mathbf{s}_{21}$, specified by (4a), is passive [8], section 4, and quasi-lossless, that is, $\mathbf{s}_{21}$ satisfies [3]

$$\mathbf{s}_{21a}^{a} o \mathbf{s}_{21} = \delta\mathbf{1}_n \tag{6a}$$

where $\delta = \delta(t - \tau)$ is the unit impulse and

$$\mathbf{s}_{21}(t, \tau) = \mathbf{A}(t)\delta(t - \tau) + \mathbf{\Phi}(t)\tilde{\mathbf{\Psi}}(\tau)u(t - \tau) \tag{6b}$$

$$\mathbf{s}_{21a}^{a}(t, \tau) = \tilde{\mathbf{A}}(t)\delta(t - \tau) - \mathbf{\Psi}(t)\tilde{\mathbf{\Phi}}(\tau)u(t - \tau) \tag{6c}$$

with u the unit step function and $\mathbf{A}$, $\mathbf{\Phi}$, and $\mathbf{\Psi}$ $n \times n$, $n \times r$, and $n \times r$ matrices, respectively, of infinitely differentiable entries. Equation (6b) states that the network is described by differential equations, while (6a) states that the network, being passive, is nondissipative. When $\mathbf{s}_{21}$ is passive and quasi-lossless, it can be factored into the product of (real) first-degree quasi-lossless passive $\mathbf{s}_i$,

$$\mathbf{s}_{21} = \mathbf{s}_m o \mathbf{s}_{m-1} o \cdots o \mathbf{s}_2 o \mathbf{s}_1 \tag{7}$$

for which each $\mathbf{s}_i$ can be realized using one unit inductor or capacitor and a time-variable transformer bank [3]. Noting the (2, 1) term of (5a) shows that this factorization gives a realization in cascade lattice form of a voltage transfer matrix which is passive and quasi-lossless (after multiplication by 2). Such a synthesis is a complete generalization of time-invariant all-pass lattice synthesis to the time-variable case.

With $n = 1$, now consider Fig. 2, redrawn as Fig. 3, with N_1 and N_2 inductors of inductances $l_1(t)$ and $l_2(t)$, respectively. Then, s_1 and s_2 are known as [6]

$$s_i(t, \tau) = \delta(t - \tau) - \frac{2}{l_i(t)}\exp\left[-\int_\tau^t \frac{d\lambda}{l_i(\lambda)}\right]u(t - \tau) \tag{8a}$$

and we calculate from (5b)

$$s_1 o s_2 - s_2 o s_1$$
$$= \frac{4}{l_1(t)}\int_\tau^t \exp\left[-\int_\sigma^t \frac{d\lambda}{l_1(\lambda)}\right]\frac{1}{l_2(\sigma)}\exp\left[-\int_\tau^\sigma \frac{d\lambda}{l_2(\lambda)}\right]d\sigma$$
$$- \frac{4}{l_2(t)}\int_\tau^t \exp\left[-\int_\sigma^t \frac{d\lambda}{l_2(\lambda)}\right]\frac{1}{l_1(\sigma)}\exp\left[-\int_\tau^\sigma \frac{d\lambda}{l_1(\lambda)}\right]d\sigma. \tag{8b}$$

Consequently, for general nonconstant $l_1 \neq l_2$ we have $s_1 o s_2 \neq s_2 o s_1$, and Fig. 2 will have a nonsymmetric scattering matrix (as well as nonsymmetric $\mathbf{z}$). Thus, using a cascade of lattices with inductor and capacitor arms we can obtain a nonreciprocal 2-port (without transformers and gyrators). Intuitively, such a situation can be justified by considering the lattice as approximating delay. A pulse first sent into the right-hand section of Fig. 3 will be delayed differently than one sent into the left-hand section, due to the time variation of the elements. Consequently, transmission in one direction differs from that in the other, showing nonreciprocity. As suggested by Desoer [9], nonreciprocity (without transformers or gyrators) can be also obtained by modulating a resistance value.

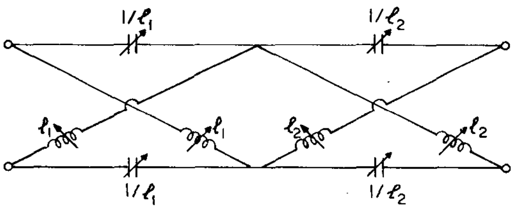

Fig. 3. A generally nonreciprocal 2-port.

We point out that a constant resistance 1-port constructed as a lattice with nonlinear and time-variable arms has been discussed by Desoer and Wong [10], but not with transfer function synthesis in mind. Although the noncommutativity of time-variable systems is known [11], page 395, neither this nor the example of (nonlinear) modulators preceding or following an IF strip is within the framework of RLC interconnections, as discussed here.

In summary, we have shown how some of the useful properties of 2-port lattices extend to the time-variable n-port case. However, not all properties extend and one can obtain nonreciprocity by cascading two time-variable lattices.

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A New Transformation for Microwave Network Synthesis

In 1948, Richards [1] showed the existence of a class of distributed networks which can be treated by methods similar to those used in lumped networks. The driving-point immittance function of such networks was shown to be positive real in the transformed frequency plane p ($p = \tanh(as/2)$) [1]. The purpose of this correspondence is to discuss the synthesis technique of a rational, driving-point impedance in the p domain.

By the Darlington insertion loss method, it can be shown that any driving-point impedance can be realized as a cascade of lossless twoport sections, terminated by a resistance. It has been shown [2]–[5] that the Darlington equivalent sections in the p plane can be realized by simple microwave networks. N. Ikeno was the first to utilize this concept and presented a microwave equivalent of the Brune section. Since then, various papers [3]–[5] have been written on the type C and D sections using different systems for realization. Here, we are primarily concerned with microwave networks which can be realized as an interconnection of strip lines. A Star-Mesh transformation which is very useful for transforming networks into a form suitable for the strip-line realization is presented.

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I. A Star-Mesh Transformation

Consider the two threeport networks shown in Fig. 1(a) and Fig. 1(b). One is the star connection of two unit elements and one inductor (p plane), and the other is the mesh connection of four unit elements. The accessible ports (1), (2), and (3) are shown with brackets to indicate that they are ports rather than terminals.

We now introduce a Star-Mesh transformation. Using this transformation, we eliminate the junction of circuit (a) to obtain the mesh structure of circuit (b). This loop network, circuit (b), will subsequently be used as a basic building block. This type of loop structure readily lends itself to strip-line construction techniques.

To prove that the two circuit configurations of Figs. 1(a) and (b) are equivalent, it is sufficient to show that they have the same Y matrix. The circuits are threeport networks, and hence, their Y matrices are 3 by 3 matrices. To find the Y matrix, we short circuit each port in turn and find the Y matrix of the remaining twoport network.

A. Y Matrix of the Circuit, Fig. 1(a)

Since a s/c unit element becomes an inductance in the p plane, the circuits of Fig. 2(a) and Fig. 2(b) are obtained when ports (2) and (3) are short circuited, respectively. The Y matrices of the circuits are

$$Y_{13} = \begin{bmatrix} \dfrac{Y_a Z_b p^2 + (1 + Z_b Y_c)}{(Z_a + Z_b + Z_a Y_c Z_b)p} & \dfrac{-\sqrt{(1-p^2)}\,(Z_b Y_c)}{(Z_a + Z_b + Z_a Y_c Z_b)p} \\ y_{31} & \dfrac{Y_c(Z_a + Z_b)}{(Z_a + Z_b + Z_a Y_c Z_b)p} \end{bmatrix}$$

$$Y_{12} = \begin{bmatrix} \dfrac{Y_a Z_b p^2 + (1 + Z_b Y_c)}{(Z_a + Z_b + Z_a Y_c Z_b)p} & \dfrac{-(1-p^2)}{(Z_a + Z_b + Z_a Y_c Z_b)p} \\ y_{21} & \dfrac{Z_a Y_b p^2 + (1 + Z_a Y_c)}{(Z_a + Z_b + Z_a Y_c Z_b)p} \end{bmatrix}.$$

Since the circuits are reciprocal, $y_{31} = y_{13}$, $y_{21} = y_{12}$. Because of symmetry, Y_{23} can be obtained from Y_{13} by simply interchanging Z_a and Z_b.

B. Y Matrix of the Circuit, Fig. 1(b)

On short circuiting ports (2) and (3), we obtain the circuits of Fig. 3(a) and Fig. 3(b), respectively, the Y matrices of which are given by the following.

Let

$$Y = (1 + Z_6 Y_7 p^2)/(Z_6 + Z_7)p$$
$$= Y_s/p + Y_d p,$$

then

$$Y_{13} = \begin{bmatrix} \dfrac{Y_d Z_4 p^2 + (1 + Z_4 Y_s)}{Z_4 p} & \dfrac{-\sqrt{(1-p^2)}}{Z_4 p} \\ y_{31} & \dfrac{1 + Z_4 Y_5}{Z_4 p} \end{bmatrix}$$

and

$$Y_{12} = \begin{bmatrix} \dfrac{(1 + Y_4 Z_s) + Z_6 Y_7 p^2}{Z_s p} & \dfrac{-(1-p^2)}{Z_s p} \\ y_{21} & \dfrac{(1 + Y_5 Z_s) + Y_6 Z_7 p^2}{Z_s p} \end{bmatrix}.$$

Because of symmetry, Y_{23} can be obtained from Y_{13} by interchanging Z_4 and Z_5, Z_6, and Z_7.

By comparing the two Y matrices of the overall circuits of Figs. 1(a) and 1(b), and equating the coefficients, we obtain twelve