SEL-66-006

AN APPROACH TO THE TIME-VARYING SENSITIVITY PROBLEM

by

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June 1966

TECHNICAL REPORT NO. 6560-1

This work was supported by the Air Force Office of Scientific
Research under Grant AF-AFOSR 337-63 with report
preparation under Nonr 225(83)

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*Talk prepared for the 1966 Fall SIAM meeting on "The Applications of Generalized Functions to System Theory," Stony Brook, September 12-14, 1966.

ABSTRACT

By means of the theory of distributional kernels, a sensitivity matrix $\underset{\sim}{s}(t,\tau)$ is introduced which relates changes in open and closed loop outputs due to changes in a plant parameter. Through equivalence with a passive network scattering matrix, the properties of $\underset{\sim}{s}$ for sensitivity improvement are determined.

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Man, too acute, should perceive
That sensitive hearts have in grown
What's created though varied by time;
Systems are so by construct
But, as with man, little known.
Non sensed, though, in man's purport,
A theory may have some import.

I. INTRODUCTION

The theory of distributions [1] [2] has found wide application in various fields of science, as for example in relativistic quantum mechanics [3], interaction and scattering of elementary particles [4], and network theory [5] [6] [7]. Still, although results are available concerning systems analysis on a distributional basis [8], little use of the rigorous theory of distributions has been made in the area of control system design. Here we investigate one of the fundamental concepts of control systems, that of sensitivity, obtaining results needed for optimal control design [9], in terms of distributions.

One of the classical problems of control theory is to reduce by feedback the sensitivity of a system to variations in the parameters of the plant. As a consequence a rather extensive literature is available concerning pertinent concepts [10], but little which directly discusses time-variable, as opposed to adjustable parameter, systems. Still time-varying, multiple-input, multiple-output systems are appearing in practical environments, by force of circumstances or as a result of implementing an optimal control law. In terms of distributional kernels we here investigate the question of when the sensitivity performance of such time-variable systems is improved by feedback.

The investigation follows the ideas of Cruz and Perkins for the time-invariant case [11] by considering the change in the closed loop response versus a change in the open loop response due to plant parameter changes and with the plant input held fixed. The relation between these open and closed loop response changes is linear and, for physical systems, describable by a distributional kernel, the sensitivity matrix. The main result is that for sensitivity improvement through the application of feedback the sensitivity matrix must be antecedal with a certain form defined by it nonnegative. Such a sensitivity matrix is analogous to the scattering matrix of a passive network, and, consequently, many of the results of passive network theory [12] apply to sensitivity problems.

In Section II we review the necessary distributional background with emphasis placed upon distributional kernels. In Section III we discuss the sensitivity concept introducing the sensitivity matrix as well as the return-difference. In Section IV the required properties of the sensitivity matrix needed for sensitivity improvement with the application of feedback are discussed; these being obtained by the above mentioned network analogy. For convenience we adhere as closely as possible to the notation of Cruz and Perkins [11].

II. PRELIMINARIES

Here we review and introduce those concepts associated with distributional kernels which are necessary to the sequel. Along with this we discuss the physical constraints placed on kernels used in control theory. We assume as known the basic rudiments of distribution theory [1] [2].

Let $\mathcal{D}$, $\mathcal{D}_+$, $\mathcal{L}_2$, and $\mathcal{D}'$ denote the spaces of real-valued n-vectors in one real variable with entries which are, respectively, infinitely differentiable functions zero outside a bounded set (i.e., with compact support), infinitely differentiable functions zero until a finite value of the variable (i.e., with support bounded on the left), square integrable functions on $(-\infty, \infty)$, and distributions. The scalar product between any $y \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ is denoted by $\langle y, \varphi \rangle$ which, on letting $t = \infty$, is the analogue of

$$\langle y, \varphi \rangle_t = \int_{-\infty}^{t} \tilde{y}(\lambda) \varphi(\lambda) \, d\lambda \qquad (2.1a)$$

defined, for instance, when $y, \varphi \in \mathcal{D}_+$; here the superscript tilde denotes matrix transposition. When defined we also write

$$\|y\|_t^2 = \langle y, y \rangle_t \qquad (2.1b)$$

$$\|y\| = \|y\|_\infty \qquad (2.1c)$$

and observe that $\| \ \|$ serves as a norm for the Hilbert space $\mathcal{L}_2$. The norm of a bounded linear transformation $T[\ ]$ of $u \in \mathcal{L}_2$ into $T[u] \in \mathcal{L}_2$ is defined in the customary manner as

$$\|T\| = \sup_{\|u\|=1} \|T[u]\| \qquad (2.2)$$

By a <u>distributional kernel</u> $k(t,\tau)$ is meant an $n \times m$ matrix of real-valued distributions in two real variables [13, p. 221]. Any linear continuous map of (m-vectors) $u \in \mathcal{D}$ (strong topology) into (m-vectors) $y \in \mathcal{D}'$ (weak topology) defines a distributional kernel k [14, p. 143, the Kernel Theorem]

$$y = k \bullet u \qquad (2.3a)$$

and conversely any distributional kernel defines such a map. If we denote the scalar product in two variables by $\langle\langle\, ,\, \rangle\rangle$, Eq. (2.3a) is made precise by the definition, for all $u, \varphi \in \mathcal{D}$,

$$\langle k \bullet u, \varphi \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \langle k_{ij}(t,\tau), u_j(\tau) \rangle, \varphi_i(t) \rangle \tag{2.3b}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \langle k_{ij}(t,\tau), \varphi_i(t) u_j(\tau) \rangle \rangle \tag{2.3c}$$

Applying another kernel h to y of Eq. (2.3a) we obtain

$$z = h \bullet y = h \bullet (k \bullet u) = (h \circ k) \bullet u \tag{2.4}$$

which serves to define the Volterra composition $h \circ k$ of h and k as the unique kernel mapping u into z, whenever such a mapping exists. Although $h \circ k$ cannot always be formed we note that it does exist and maps $\mathcal{D}_+$ into $\mathcal{D}_+$ whenever h and k both map $\mathcal{D}_+$ into $\mathcal{D}_+$. The composition of a number of kernels is not necessarily associative, but a sufficient condition guaranteeing associativity is that all kernels map $\mathcal{D}_+$ into $\mathcal{D}_+$ [15, p. 120]. With δ the unit impulse and 1_n the $n \times n$ identity matrix, $\delta 1_n = \delta(t-\tau) 1_n$ acts as the identity map under composition and hence can be composed with any kernel.

In the standard manner one defines the inverse k^{-1} under composition by

$$k^{-1} \circ k = k \circ k^{-1} = \delta 1_n \tag{2.5}$$

Depending upon the domain of definition considered one kernel may have several inverses. Consequently, we will assume, unless otherwise stated, that if k is a mapping of $\mathcal{D}_+$ into $\mathcal{D}_+$ then k^{-1} is also a mapping of $\mathcal{D}_+$ into $\mathcal{D}_+$. For such a mapping Eq. (2.5) means that for any $u \in \mathcal{D}_+$, $k^{-1} \bullet (k \bullet u) = (k^{-1} \circ k) \bullet u = u$.

For intuitive reasoning it is convenient to recall the functional meaning of $\bullet$ and $\circ$

$$y = k \cdot u = \int_{-\infty}^{\infty} k(t,\lambda)u(\lambda)\, d\lambda \tag{2.6a}$$

$$h \circ k = \int_{-\infty}^{\infty} h(t,\lambda)k(\lambda,\tau)\, d\lambda \tag{2.6b}$$

Also in the standard manner one defines the adjoint k^a through

$$\langle u, k^a \cdot \varphi \rangle = \langle k \cdot u, \varphi \rangle \tag{2.7a}$$

for all $u, \varphi \epsilon \mathcal{D}$. Through Eqs. (2.3) [12, Sect. 4] one readily finds

$$k^a(t,\tau) = \tilde{k}(\tau,t) \tag{2.7b}$$

and, thus, k^a generally will not map $\mathcal{D}_+$ into $\mathcal{D}_+$ when k does.

Of special interest are the nonnegative kernels [3, p. 45]. By definition, a real self-adjoint distributional kernel is nonnegative, written $k \geq 0$, if for all $\varphi \epsilon \mathcal{D}$

$$\langle k \cdot \varphi, \varphi \rangle \geq 0 \tag{2.8}$$

Turning to more physical notions, a system can be conceived as a transformation, here assumed linear, mapping inputs u into outputs y. Because we wish to treat physical systems we can assume that $u, y \epsilon \mathcal{D}_+$ [16]. Further, discontinuous transformations seem physically out of the question. Consequently, since $u \epsilon \mathcal{D} \subset \mathcal{D}_+$ and $y \epsilon \mathcal{D}_+ \subset \mathcal{D}'$, we find by the Kernel Theorem that a linear physical system is described by a distributional kernel k through $y = k \cdot u$. In actual fact $y = k \cdot u$ is defined for all $u \epsilon \mathcal{D}_+$, with $y \epsilon \mathcal{D}_+$, as the above physical arguments show. For some systems $y = k \cdot u$ can be defined for other distributional inputs than $u \epsilon \mathcal{D}_+$, but such extensions are of minor concern for this work, except for showing that k has the physical interpretation of an impulse response matrix.

III. THE SENSITIVITY MATRIX

In this section we define and interpret the sensitivity matrix.

Consider a fixed linear plant P which takes (m-vector) inputs $\underset{\sim}{u}_o$ into (n-vector) outputs $\underset{\sim}{y}_o$ and which is subject to variations in a parameter x. Then P is described by its (n x m) impulse response matrix $\underset{\sim}{p}_x$, a distributional kernel dependent on x. To obtain desirable transfer characteristics a controller G_1 is customarily inserted before the plant, as shown in Fig. 1, such that actual (p-vector) inputs $\underset{\sim}{r}$ are modified by the (m x p) controller impulse response matrix $\underset{\sim}{g}_1$ to obtain the plant inputs:

$$\underset{\sim}{y}_o = \underset{\sim}{p}_x \bullet \underset{\sim}{u}_o \,,\; \underset{\sim}{u}_o = \underset{\sim}{g}_1 \bullet \underset{\sim}{r} \qquad (3.1a)$$

or

$$\underset{\sim}{y}_o = (\underset{\sim}{p}_x \circ \underset{\sim}{g}_1) \bullet \underset{\sim}{r} \qquad (3.1b)$$

The (n x p) impulse response matrix of the open loop system, Fig. 1, is then $\underset{\sim}{p}_x \circ \underset{\sim}{g}_1$ and one notes that although $\underset{\sim}{y}_o$ depends upon x, $\underset{\sim}{u}_o$ does not since $\underset{\sim}{r}$ and $\underset{\sim}{g}_1$ are assumed free of such variations. However, classical control theory [17, p. 211] recognizes that a redesign of the controller to incorporate feedback, and hence cause the plant input to vary properly with x, can lead to smaller variations in the plant output with x. A general closed loop configuration of this type is shown in Fig. 2 where the controller components G and H are described by their (m x p and p x n) impulse response matrices $\underset{\sim}{g}$ and $\underset{\sim}{h}$, also assumed independent of x. We note that

$$\underset{\sim}{y}_c = \underset{\sim}{p}_x \bullet \underset{\sim}{u}_c \,,\; \underset{\sim}{u}_c = \underset{\sim}{g} \bullet \underset{\sim}{r} - (\underset{\sim}{g} \circ \underset{\sim}{h}) \bullet \underset{\sim}{y}_c \qquad (3.2a)$$

and hence, for the closed loop system

$$\underset{\sim}{y}_c = [(\delta \underset{\sim}{1}_n + \underset{\sim}{p}_x \circ \underset{\sim}{g} \circ \underset{\sim}{h})^{-1} \circ \underset{\sim}{p}_x \circ \underset{\sim}{g}] \bullet \underset{\sim}{r} \qquad (3.2b)$$

For a meaningful design the open and closed loop controllers are of course constructed such that the respective plant outputs are equal,

$y_c = y_o$, for a given input r when the parameter x assumes its design value $x = x_d$. This entails, for $x = x_d$, that $u_c = u_o$ or, from Eqs. (3.1a) and (3.2a)

$$[g-g_1-g \circ h \circ p_x \circ g_1] \cdot r = 0 \tag{3.3}$$

which can be used to design g and h. However, the problem of interest here is the determination of the constraints on g and h such that variations in the closed loop output y_c, due to changes in x, are smaller than the corresponding variations in the open loop output y_o, for a given g_1 and p_x.

For such an investigation let, in contradistinction to Cruz and Perkins [11, p. 217], primed quantities denote the designed situation $x = x_d$, and unprimed quantities the situation for general x; thus $p_x' = p_{x_d}$. We then introduce the open and closed loop output errors, e_o and e_c, through

$$e_o = y_o'-y_o \tag{3.4a}$$

$$e_c = y_c'-y_c \tag{3.4b}$$

Then $e_o = e_c + (y_c-y_o)$ and, from Eqs. (3.1a) and (3.2b) $y_c - y_o = p_x \cdot [g-g_1] \cdot r - p_x \cdot g \cdot h \cdot y_c$, which on subtraction and addition of $p_x \cdot g \cdot h \cdot y_c' = (p_x \circ g \circ h) \cdot p_x' \cdot u_c' = (p_x \circ g \circ h \circ p_x') \cdot u_o = p_x \cdot [g \circ h \circ p_x' \circ g_1] \cdot r$, and the use of Eq. (3.3) (primed), yields

$$e_o = [\delta 1_n + p_x \circ g \circ h] \cdot e_c \tag{3.5a}$$

We note that the feedback factor

$$f = \delta 1_n + p_x \circ g \circ h \tag{3.5b}$$

is the <u>return-difference</u> [18, p. 48], that is the difference between "unit" signal applied to the controller at the input to H and the signal returned to the controller via the feedback path of Fig. 2, when $r = 0$.

Since it is of most interest to evaluate the closed loop changes in terms of the open loop ones we define the <u>sensitivity matrix</u> s as

$$\underset{\sim}{s} = [\delta \underset{\sim}{1}_n + \underset{\sim}{p}_x \circ \underset{\sim}{g} \circ \underset{\sim}{h}]^{-1} \tag{3.6a}$$

for which

$$\underset{\sim}{e}_c = \underset{\sim}{s} \bullet \underset{\sim}{e}_o \tag{3.6b}$$

In summary a linear transformation exists relating the changes in the open loop output to changes in the closed loop output, due to variations in a plant parameter x, the relationship being represented by an $n \times n$ distributional kernel $\underset{\sim}{s}$, the sensitivity matrix. Being the inverse of the return-difference matrix $\underset{\sim}{f}$, $\underset{\sim}{s}$ agrees with the more classical concepts for time-invariant single input-output systems [19, p. 121].

IV. SENSITIVITY IMPROVEMENT CRITERIA

Here we show that the closed loop system yields improved sensitivity performance if and only if the sensitivity matrix is a bounded antecedal map of $\mathcal{L}_2$ into $\mathcal{L}_2$ of norm bounded by unity.

We first limit the inputs to $r \in \mathcal{D}_+$ in which case we know on physical grounds that $e_o, e_c \in \mathcal{D}_+$. Consequently, through Eqs. (2.1), the quadratic performance indices $\langle e_o, e_o \rangle_t$ and $\langle e_c, e_c \rangle_t$ are well defined. A reasonable criteria for improvement of sensitivity performance is then, that, for any given $r \in \mathcal{D}_+$,

$$\mathcal{E}(t) = \|e_o\|_t^2 - \|e_c\|_t^2 \tag{4.1}$$

satisfies, for all finite t,

$$\mathcal{E}(t) \geq 0 \tag{4.2}$$

That is, we will say that sensitivity is improved by feedback if at each instant of time the integral of the sum of squared error components is not increased by the application of feedback.

At this point we note that the situation is analogous to that for passive (linear and solvable) n-port networks. Thus, if we consider e_o as incident voltages, v^i, and e_c as reflected voltages v^r, then s is completely analogous to the scattering matrix of the network with $\mathcal{E}(t)$ the total input energy [12]. Consequently, by choosing $e_o(\lambda) = 0$ for $\lambda < t$, we see that $e_c(\lambda) = 0$ for $\lambda < t$, from Eq. (4.1), which implies that s is antecedal, that is satisfies $s(t,\tau) = 0_n$ for $t < \tau$, where 0_n is the n x n zero matrix. Further, s can be extended to a map of $\mathcal{L}_2$ into $\mathcal{L}_2$ simply by noting that $\|e_o\|$ is defined for $e_o \in \mathcal{L}_2$ thus implying that $e_c \in \mathcal{L}_2$ by Eq. (4.2), in which case $\mathcal{E}(\infty) \geq 0$ implies that $\|e_o\| \geq \|s \cdot e_o\|$ or that $\|s\|$ exists and is bounded by unity. Omitting the particulars which are detailed elsewhere [12, Sect. 4], we then have the main result.

Theorem: Sensitivity is improved by feedback if and only if the sensitivity matrix s satisfies the following conditions:

(1) $\underset{\sim}{s}$ maps $\mathcal{L}_2$ into $\mathcal{L}_2$

(2) $\underset{\sim}{s}(t,\tau) = \underset{\sim}{0}_n$ for $t < \tau$

(3) $\|\underset{\sim}{s}\| \leq 1$

One of the most useful properties that can be determined from the theorem is that $\underset{\sim}{s}(t,\tau)$ is a measure (i.e., at most impulsive) in both variables simultaneously over any compact set of the (t,τ)-plane [12]. Another property is seen by writing Eq. (4.2) in more detail

$$\mathcal{E}(t) = \langle \underset{\sim}{e}_o, \underset{\sim}{e}_o \rangle_t - \langle \underset{\sim}{s} \bullet \underset{\sim}{e}_o, \underset{\sim}{s} \bullet \underset{\sim}{e}_o \rangle_t \tag{4.3a}$$

$$= \langle (\delta \underset{\sim}{1}_n - \underset{\sim}{s}^a \circ \underset{\sim}{s}) \bullet \underset{\sim}{e}_o, \underset{\sim}{e}_o \rangle_t \tag{4.3b}$$

Thus, letting $t \longrightarrow \infty$ with $\underset{\sim}{e}_o \in \underset{\sim}{\mathcal{D}}$ we see that

$$\underset{\sim}{R} = \delta \underset{\sim}{1}_n - \underset{\sim}{s}^a \circ \underset{\sim}{s} \geq 0 \tag{4.3c}$$

or $\underset{\sim}{R}$ is a nonnegative kernel. Note that in some sense the "smaller" $\underset{\sim}{R}$ the less the sensitivity improvement, the limit being for $\underset{\sim}{s}^a = \underset{\sim}{s}^{-1}$. In terms of the return-difference we also have, from Eqs. (3.5) and (3.6),

$$(\underset{\sim}{s}^a)^{-1} \circ \underset{\sim}{R} \circ \underset{\sim}{s}^{-1} = \underset{\sim}{f}^a \circ \underset{\sim}{f} - \delta \underset{\sim}{1}_n \geq 0 \tag{4.3d}$$

If the system is time-invariant [20] then $\underset{\sim}{s}(t,\tau) = \underset{\sim}{s}(t-\tau,0)$, in which case one can take the Laplace transform $\mathcal{L}[\ ]$, [21], to obtain

$$\underset{\sim}{S}(p) = \mathcal{L}[\underset{\sim}{s}(t,0)] \tag{4.4}$$

Again by analogy with the network situation [12], [22, p. 116], $\underset{\sim}{S}(p)$ must be bounded-real, that is satisfy the following corollary, where a superscript asterisk denotes complex conjugation.

Corollary: If $\underset{\sim}{s}(t,\tau) = \underset{\sim}{s}(t-\tau,0)$, then sensitivity is improved by feedback if and only if

(1) $\underset{\sim}{S}(p)$ is holomorphic in $\operatorname{Re} p > 0$

(2) $\underset{\sim}{S}^*(p) = \underset{\sim}{S}(p^*)$ in $\operatorname{Re} p > 0$

(3) $\underset{\sim}{1}_n - \tilde{\underset{\sim}{S}}(p^*)\underset{\sim}{S}(p)$ is positive semidefinite in $\operatorname{Re} p > 0$.

When $\underset{\sim}{S}(p)$ is rational this precisely states the results of Cruz and Perkins [11, p. 219].

V. DISCUSSION

By observing the strict equivalence between the scattering matrix of a passive n-port and the sensitivity matrix of an n-output system for which sensitivity is improved by feedback application the conditions of the theorem have been obtained. The theory rests heavily upon the theory of distributions for its formulation with the theorem showing, however, that no "worse" than impulses appear in $\underset{\sim}{s}$. For example, in the case of a system described by differential equations (a differential system) $\underset{\sim}{s}$ takes the form

$$\underset{\sim}{s}(t,\tau) = \underset{\sim}{A}(t)\delta(t-\tau) + \underset{\sim}{\Phi}(t)\underset{\sim}{\tilde{\Psi}}(\tau)1(t-\tau) \tag{5.1}$$

where $1(\cdot)$ is the unit step function, $\underset{\sim}{A}$ has eigenvalues no greater than one, and $\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}$ are infinitely differentiable matrices subject to Eq. (4.3c).

If one has a finite dynamical (differential) system with H following the plant in the forward loop and unity feedback (i.e., Fig. 2 with $\underset{\sim}{y}_c$ the output of H in place of P) then, under broad conditions it can be shown that an "optimally" designed linear feedback law leads to sensitivity improvement [9]. Conversely, strict sensitivity improvement means that, for a time-invariant finite dynamical system, there is some quadratic loss function for which the feedback system is optimal [23]. Consequently, the results should be of some practical importance. It should however be pointed out that the theory of this paper is based upon starting, at $t = -\infty$, in the zero state; nevertheless, a finite dimensional state space is not assumed in the general arguments.

It is clear that the theory is valid for the most general linear systems of interest, but does not cover general nonlinear systems, even though many of the concepts should carry over to the latter case. It is not so clear, however, that the variation of the disturbing parameter x should be "nonexistent." That is x is essentially fixed for all time in the analysis and two "different" systems compared, one with x arbitrary and one with x at its design value x_d. This implied assumption is inherent in all such work and is physically reasonable for slow variations in x.

The study does point out that for more insight into sensitivity matrices a more detailed study of nonnegative distributional kernels is in order, there being very little presently available [3], [12].

ACKNOWLEDGMENT

The authors are grateful to Barbara Serrano for the excellent preparation of the manuscript.

REFERENCES

1. L. Schwartz, "Théorie des distributions," vol. I, Hermann, Paris, 1957.
2. L. Schwartz, "Théorie des distributions," vol. II, Hermann, Paris, 1959.
3. L. Schwartz, "Application of Distributions to the Study of Elementary Particles in Relativistic Quantum Mechanics," Department of Mathematics, University of California, Berkeley, Technical Report No. 7, NR041-221, March 1961.
4. W. Güttinger, R. Penzl, and E. Pfaffelhuber, "Peratization of Unrenormalizable Field Theories," Annals of Physics, vol. 33, no. 2, June 1965, pp. 246-271.
5. R. W. Newcomb, "Hilbert Transforms and Positive-Real Functions," Proceedings of the IRE, vol. 50, no. 12, December 1962, pp. 2516-2517.
6. R. W. Newcomb, "The Foundations of Network Theory," The Institute of Engineers Australia, Electrical and Mechanical Engineering Transactions, vol. EM6, May 1964, pp. 7-12.
7. M. R. Wohlers and E. J. Beltrami, "Distribution Theory as the Basis of Generalized Passive-Network Analysis," IEEE Transactions on Circuit Theory, vol. CT-12, no. 2, June 1965, pp. 164-170.
8. V. Doležal, Dynamics of Linear Systems, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964.
9. B. D. O. Anderson, "Sensitivity Improvement Using Optimal Design," Proceedings of the IEE, 1966, to appear.
10. P. V. Kokotovic and R. S. Rutman, "Sensitivity of Automatic Control Systems (Survey)," Automatika i Telemekhanika, vol. 26, no. 4, April 1965, pp. 730-750.
11. J. B. Cruz, Jr. and W. R. Perkins, " A New Approach to the Sensitivity Problem in Multivariable Feedback System Design," IEEE Transactions on Automatic Control, vol. AC-9, no. 3, July 1964, pp. 216-223.
12. B. D. O. Anderson and R. W. Newcomb, "Functional Analysis of Linear Passive Networks," International Journal of Engineering Science, to be published.
13. L. Schwartz, "Théorie des noyaux," Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, pp. 220-230.

14. L. Schwartz, "Espaces de fonctions différentiables à valeurs vectorielles," Journal d'Analyse Mathématique, Jérusalem, vol. IV, 1954-1955, pp. 88-148.

15. L. Schwartz, "Théorie des distributions à valeurs vectorielles," Annales de l'Institut Fourier, vol. 7, 1957, Chapter 1, pp. 1-141.

16. R. W. Newcomb, "On the Definition of a Network," Proceedings of the IEEE, vol. 53, no. 5, May 1965, pp. 547-548.

17. H. Chestnut, Systems Engineering Tools, John Wiley & Sons, N.Y., 1965.

18. H. W. Bode, Network Analysis and Feedback Amplifier Design, D. Van Nostrand, N.Y., 1945.

19. J. G. Truxal, Automatic Feedback Control Systems Synthesis," McGraw-Hill, N.Y., 1955.

20. R. W. Newcomb, "Distributional Impulse Response Theorems," Proceedings of the IEEE, vol. 51, no. 8, August 1963, pp. 1157-1158.

21. L. Schwartz, "Transformations de Laplace des distributions," Communications du Séminaire Mathématique de L'Université de Lund, Tome Supplémentaire dédié à Marcel Riesz, 1952, pp. 196-206.

22. D. C. Youla, L. J. Castriota, and H. J. Carlin, "Bounded Real Scattering Matrices and the Foundations of Linear Passive Network Theory," IRE Transactions on Circuit Theory, vol. CT-4, no. 1, March 1959, pp. 102-124.

23. B. D. O. Anderson, "The Inverse Problem of Optimal Control," Stanford Electronics Laboratories, Stanford, California, Technical Report No. TR6560-3, SEL-66-038, May 1966.

FIGURE CAPTIONS

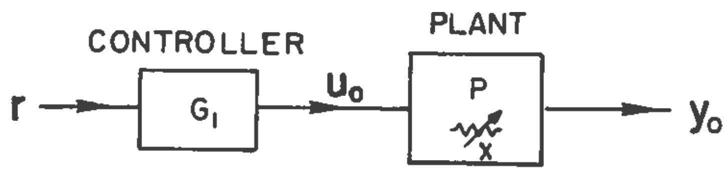

FIGURE 1

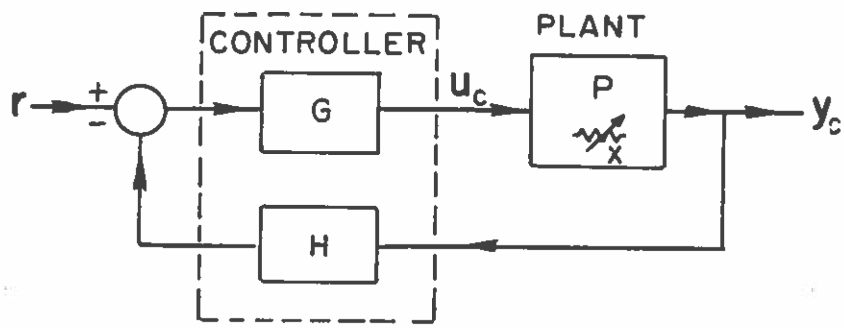

FIGURE 2