where $P_N = \sum_{k=1}^{N} p_k$ is a polynomial of degree $\leq 2N$ since each p_k is of degree no higher than $2N$. Equation (12) may then be written as

$$(f_a, f_b) - \sum_{k}^{N} (f_a, \varphi_k)(f_b, \varphi_k) = \frac{1}{a+b} \frac{P_N'(a, b; a_1, \cdots, a_N)}{\prod_{i=1}^{N} (a + a_i) \prod^{N} (b + a_i)}. \qquad (27)$$

But from (10), (18), and (19) it is seen that (27) is equal to zero if a or b equals any of the a_i. This means that the polynomial P_N' is divisible by $\prod_{i=1}^{N} (a - a_i)$ and $\prod_{i=1}^{N} (b - a_i)$, or

$$P_N' = A_N \prod_{i=1}^{N} (a - a_i) \prod_{i=1}^{N} (b - a_i). \qquad (28)$$

We then have

$$(f_a, f_b) - \sum_{k}^{N} (f_a, \varphi_k)(f_b, \varphi_k) = (f_a, f_b) A_N \frac{\prod^{N} (a - a_i) \prod^{N} (b - a_i)}{\prod^{N} (a + a_i) \prod^{N} (b + a_i)}. \qquad (29)$$

If now in (29) each a_i is taken arbitrarily large, we see from (23) that the left-hand side of (29) approaches (f_a, f_b). The right-hand side approaches $(f_a, f_b)A_N(-1)^N(-1)^N$; hence, $A_N \rightarrow +1$, which proves the relation given in (12).

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Synthesis of Lumped-Distributed RC n-Ports

Because the substrate for integrated devices inherently yields distributed RC elements in which lumped resistors or capacitors are diffused or deposited, the theory of RC networks containing both lumped and distributed elements is of some importance. Here we point out how the ideas of Youla [1] and Koga [2] can be used to synthesize lumped-distributed RC networks.

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Realizability conditions for passive uniformly distributed RC lines have been studied [3]–[9] mainly by means of transformations of the frequency variable resulting in the analysis and synthesis of lumped networks in the transformed plane. Wyndrum [3], [4], by the use of a positive-real transformation of the type used by Richards [10], obtained the conditions for driving point synthesis of uniform lines with identical RC products. Both driving point and transfer function syntheses have been given by O'Shea [5] for an arbitrary interconnection of uniform lines with constant RC products. Some general results on the realizability of arbitrary n-port connections of uniform lines of rationally related $\sqrt{rc\ell}$ products are presented in [9].

The above-mentioned analysis and synthesis techniques apply to networks consisting of uniform RC lines only and do not in general allow introduction of lumped elements in the network. However, the recent work of Youla [1] concerned with state-variable type realizations of two-variable rational matrices, and more specifically the two-variable reactance synthesis of Koga [2] are applicable, as we show here, to the synthesis of networks consisting of an arbitrary interconnection of uniformly distributed RC lines with rationally related $\sqrt{rcl}$ products, lumped resistors, capacitors, and ideal transformers.

The admittance matrix of a 2-port uniformly distributed RC line of length l_0, whose capacitance and resistance per unit length are c and r, is [11]

$$\mathbf{y}_{rc}(s) = \sqrt{s}\, y_0 \begin{bmatrix} \operatorname{ctnh} \sqrt{s}\sqrt{rc}\, l_0 & -\operatorname{csch} \sqrt{s}\sqrt{rc}\, l_0 \\ -\operatorname{csch} \sqrt{s}\sqrt{rc}\, l_0 & \operatorname{ctnh} \sqrt{s}\sqrt{rc}\, l_0 \end{bmatrix} \qquad (1)$$

where $y_0 = \sqrt{c/r}$ and $s = \sigma + j\omega$ is the true frequency variable. If we consider the positive real frequency transformation

$$p = \operatorname{ctnh} \sqrt{s}\, d, \qquad 2d = \sqrt{rc}\, l_0 \qquad (2)$$

of the type used by Richards [10], we find that the resulting admittance matrix can be written as

$$\frac{1}{\sqrt{s}} \mathbf{y}_{rc}(s) = \mathbf{Y}_{rc}(p) = y_0 \begin{bmatrix} \dfrac{1+p^2}{2p} & \dfrac{1-p^2}{2p} \\ \dfrac{1-p^2}{2p} & \dfrac{1+p^2}{2p} \end{bmatrix}. \qquad (3)$$

Consequently, $\mathbf{Y}_{rc}(p)$ is lossless (reactive), symmetric, positive real, and rational in p. The admittances of lumped capacitors and resistors can be expressed as

$$\frac{1}{\sqrt{s}} y_c(s) = Y_c(\sqrt{s}) = c\sqrt{s}, \qquad (4a)$$

$$\frac{1}{\sqrt{s}} y_r(s) = Y_r(\sqrt{s}) = \frac{1}{r\sqrt{s}}. \qquad (4b)$$

In other words $Y_c(\sqrt{s})$ and $Y_r(\sqrt{s})$ are as admittances of capacitors and inductors in the $\sqrt{s}$ plane.

At this point we consider interconnections of the above type elements and transformers, and, thus, for convenience, define a lumped-distributed RC network constructed from passive uniformly distributed RC lines of rationally related $\sqrt{rcl_0}$ products, passive lumped capacitors and resistors, as well as transformers. For such a network then, if we let

$$\lambda = \sqrt{s} \qquad (5)$$

there exists a choice of d, for (2), such that the general branch admittances are of the form $\sqrt{s}[Cp + (1/Lp) + c\lambda + (1/l\lambda)]$ with coupling between branches described in terms of $\sqrt{s}\mathbf{Y}_{rc}(p)$ of (3). Consequently, as discussed by Ozaki and Kasami [12], any short-circuit admittance matrix, $\mathbf{y}(s)$, of a lumped-distributed RC network is such that the $n \times n$ matrix

$$\frac{1}{\sqrt{s}}\mathbf{y}(s) = \mathbf{y}(p, \lambda) \tag{6}$$

is a symmetric two-variable reactance matrix as defined by Koga ([2], p. 32), that is, $\mathbf{y}(p, \lambda)$ is a symmetric matrix which is two-variable positive real, and satisfies the reactance condition $\mathbf{y}(p, \lambda) = -\mathbf{y}(-p, -\lambda)$. In view of this reactance property of $\mathbf{y}(p, \lambda)$ we can state the following theorem which is based upon Koga's synthesis technique [2] for decomposing a two-variable reactance matrix.

Theorem

The necessary and sufficient conditions for $\mathbf{y}(s)$ to be the short-circuit admittance matrix of a passive lumped-distributed RC network, are that there exist variables $p = \text{ctnh}\sqrt{s}d$ and $\lambda = \sqrt{s}$ such that $\mathbf{y}(p, \lambda) = \mathbf{y}(s)/\sqrt{s}$ is a rational symmetric, two-variable reactance matrix.

Necessity follows from a proper choice of p and considerations of branch admittances, as outlined above. For sufficiency we first note that $\mathbf{y}(p, \lambda)$ can be decomposed as ([2], p. 34)

$$\mathbf{y}(p, \lambda) = \mathbf{y}_0(p, \lambda) + \mathbf{y}_1(p) + \mathbf{y}_2(\lambda), \tag{7a}$$

where $\mathbf{y}_1(p)$ and $\mathbf{y}_2(\lambda)$ are symmetric reactance matrices in one variable and $\mathbf{y}_0(p, \lambda)$ is a two-variable reactance matrix with poles explicitly depending on both p and λ. Consequently $\mathbf{y}_0(p, \lambda)$ has no poles at infinity ([12], p. 253) and can, therefore, be decomposed according to the methods of Youla ([1], p. 13) or Koga ([2], p. 39) as

$$\mathbf{y}_0(p, \lambda) = \boldsymbol{y}_{11}(p) - \boldsymbol{y}_{12}(p)[\boldsymbol{y}_{22}(p) + \lambda \mathbf{1}_l]^{-1}\boldsymbol{y}_{21}(p) \tag{7b}$$

and

$$\mathbf{y}_0(p, \lambda) = \boldsymbol{y}_{11}(\lambda) - \boldsymbol{y}_{12}(\lambda)[\boldsymbol{y}_{22}(\lambda) + p\mathbf{1}_m]^{-1}\boldsymbol{y}_{21}(\lambda). \tag{7c}$$

Here $\mathbf{1}_l$ is the identity matrix of order l with l and m the degrees of $\varphi(p, \lambda)$, the least common denominator polynomial of the minors in $\mathbf{y}_0(p, \lambda)$, in λ and p, respectively. Using the method of Koga it has been shown ([2], p. 39) that both

$$\mathbf{Y}_c(p) = \begin{bmatrix} \mathbf{y}_{11}(p) & \mathbf{y}_{12}(p) \\ \mathbf{y}_{21}(p) & \mathbf{y}_{22}(p) \end{bmatrix} \tag{8a}$$

and

$$\boldsymbol{Y}_c(\lambda) = \begin{bmatrix} \boldsymbol{y}_{11}(\lambda) & \boldsymbol{y}_{12}(\lambda) \\ \boldsymbol{y}_{21}(\lambda) & \boldsymbol{y}_{22}(\lambda) \end{bmatrix} \tag{8b}$$

can be guaranteed to be positive real reactance matrices, perhaps after the use of a para-unitary transformation on an original decomposition. Since Youla ([1], p. 23) has shown that any minimal size $\hat{\mathbf{Y}}_c(p)$ results from any other minimal size $\mathbf{Y}_c(p)$ through $\hat{\mathbf{Y}}_c(p) = [\mathbf{1}_n \dotplus \mathbf{T}(p)]\mathbf{Y}_c(p)[\mathbf{1}_n \dotplus \mathbf{T}^{-1}(p)]$, where $\mathbf{T}$ is some nonsingular matrix and $\dotplus$ denotes the direct sum, one can obtain the positive real reactance properties desired by also using Youla's decomposition method with a proper choice of $\mathbf{T}$. In any event, when the network for $\mathbf{Y}_c(p)$ is loaded in l λ-plane capacitors then $\mathbf{y}_0(p, \lambda)$ results from (7b); likewise loading a network for $\boldsymbol{Y}_c(\lambda)$ in m p-plane capacitors also yields $\mathbf{y}_0(p, \lambda)$.

However, since this decomposition does not in general result in a symmetric $\mathbf{Y}_c(p)$, or $\boldsymbol{Y}_c(\lambda)$, some difficulty results when returning to the s-plane because p or λ-plane gyrators are needed to synthesize $\mathbf{Y}_c(p)$ or $\boldsymbol{Y}_c(\lambda)$. That is, when a p or λ-plane gyrator is described in the s-plane it has, by (6), a gyration "resistance", $\sqrt{s}\,\text{ctnh}\sqrt{s}d$ or $\sqrt{s}$, respectively, which require active s-plane circuit elements. Consequently, again following Koga ([2], p. 44) we create a *symmetric* positive real reactance $(n + 2l) \times (n + 2l)$ admittance matrix $\mathbf{Y}_S(p)$ as

$$\mathbf{Y}_S = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12S} & -\mathbf{y}_{12A} \\ \tilde{\mathbf{y}}_{12S} & \mathbf{y}_{22S} & -\mathbf{y}_{22A} \\ \tilde{\mathbf{y}}_{12A} & \tilde{\mathbf{y}}_{22A} & \mathbf{y}_{22S} \end{bmatrix} \tag{9a}$$

where the tilde, $\sim$, denotes matrix transposition and

$$2\mathbf{y}_{12S} = \mathbf{y}_{12} + \tilde{\mathbf{y}}_{21}, \qquad 2\mathbf{y}_{12A} = \mathbf{y}_{12} - \tilde{\mathbf{y}}_{12} \tag{9b}$$

$$2\mathbf{y}_{22S} = \mathbf{y}_{22} + \tilde{\mathbf{y}}_{22}, \qquad 2\mathbf{y}_{22A} = \mathbf{y}_{22} - \tilde{\mathbf{y}}_{22}. \tag{9c}$$

Then $\mathbf{y}_0(p, \lambda)$ results by loading a reciprocal p-plane realization of $\mathbf{Y}_S(p)$ by unit λ-plane capacitors and inductors at the final $2l$ ports, as shown in Fig. 1(a). Identical reasoning applies to obtain $\boldsymbol{Y}_S(\lambda)$ from $\boldsymbol{Y}_c(\lambda)$, and from which $\mathbf{y}_0(p, \lambda)$ also results by loading a reciprocal λ-plane realization of $\boldsymbol{Y}_S(\lambda)$ by unit p-plane capacitors and inductors at the final $2m$ ports, as shown in Fig. 2(a). The reactance matrices $\mathbf{y}_1(p)$ and $\mathbf{y}_2(\lambda)$ are also realized by p or λ-plane reactive elements in the classical manner with these realizations being connected in parallel with that for $\mathbf{y}_0(p, \lambda)$.

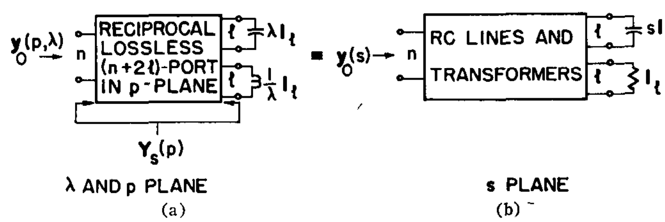

Fig. 1. Synthesis from (7b).

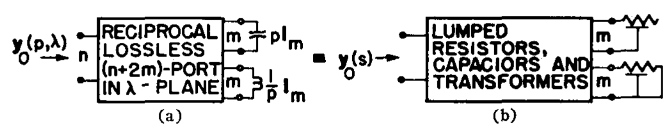

Fig. 2. Synthesis from (7c).

A realization of $\mathbf{Y}_S(p)$ results by applying the standard theory of lossless n-port transmission lines [13], that is by a cascade of $(n + 2l)$-port unit elements obtained by applying Richards' theorem for matrices. Finally the s-plane realization of $\mathbf{y}_0(s) = \sqrt{s}\mathbf{y}_0(p, \lambda)$ results by replacing each of the p-plane $(n + 2l)$-port unit elements by $n + 2l$ 2-port RC lines of $\mathbf{y}_{rc}(s)$ as in (1) but of length $l_0/2$, and each λ-plane capacitor and inductor by an s-plane capacitor and resistor, (4). The final $\mathbf{Y}_S(p)$ realization is as shown in Fig. 1(b). Alternately $\boldsymbol{Y}_S(\lambda)$ can be realized as a lossless, reciprocal $(n + 2m)$ port in the λ-plane, and the final s-plane realization of $\mathbf{y}_0(s)$ would then consist of an $(n + 2m)$-port lumped RC network terminated by RC lines as shown in Fig. 2(b), where the lines are of length $l_0/2$ terminated in open or short circuits. In this case, the RC lines, which are inherently 2-ports, are used as 1-ports and, hence, as the reviewer points out, they may not be used in the most efficient manner.

Using identical ideas, but with $p = \text{ctnh}\ s\sqrt{LC}l_0$, $\lambda = s$, and $\mathbf{y}(p, \lambda) = \mathbf{y}(s)$ rational in p and λ, we can give a synthesis of lumped-distributed LC networks even in the nonreciprocal case [13], where synthesis can proceed from $\mathbf{Y}_C(p)$, if desired, to use a minimum number of lumped reactive elements. This method then gives an alternate to that of Youla [1], since synthesis in the lossless case can proceed also when the scattering matrix is prescribed.

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A Note on the Synthesis of Multivariable Positive Real Functions

The concept of multivariable positive functions has found application[1,2,3] in dealing with variable networks, and in the case of networks composed of transmission lines and lumped elements. Realization methods are available[4] for such functions. The object of the present note is to discuss a few details in regard to these methods.

Let us consider a realizable driving point impedance function of the form

$$Z(s, \lambda, \mu, \cdots \gamma) = \frac{P(s, \gamma, \mu, \cdots \gamma)}{Q(s, \gamma, \mu, \cdots \gamma)} \tag{1}$$

where P and Q are polynomials in each of the finite number of variables $s, \lambda, \mu, \cdots, \gamma$. With reference to (1),

$$Z_1(s) \triangleq Z(s, k, k, \cdots k), \tag{2}$$

k being an unspecified positive real constant, is realizable since any realization of (1), in which all elements, excepting resistors and those relevant to s, are replaced with appropriately[5] valued resistors, is also a realization of $Z_1(s)$ of (2). This also means that a realization, N_1 of $Z_1(s)$, is possible such that in N_1 the values of all the s elements are independent of k, in the sense that the values are not in terms of k. At this point suppose we have this realization, N_1, starting from $Z_1(s)$ of (2) and we realize $Z(s, \lambda, \mu, \cdots, \gamma)$ on the pattern of N_1. Let this realization of $Z(s, \lambda, \mu, \cdots, \gamma)$ be called N_1'. (The realization N_1' is on the same pattern as that of N_1 in the sense that the procedure used to get N_1' is the same as that used to get N_1. For instance, if a certain N_1 is found to be $(Sk/S + k) + k/2$, the starting point to get N_1' would be to remove from the given impedance a part which is not a function of s; the remainder will have s and some function not having s in parallel.) This N_1' will consist of completely determined s elements plus impedance functions, similar in form to (1), of $\lambda, \mu, \cdots, \gamma$. Let $Z_2(\lambda, \mu, \cdots, \gamma)$ be one such impedance function. Then we synthesize $Z_2(\lambda, k, \cdots, k)$ to get a realization, say, N_2, such that in N_2 the values of λ elements are independent of k. Next $Z_2(\lambda, \mu, \cdots, \gamma)$ is realized on the pattern of N_2 so that in this realization the impedance functions to be decomposed further will be only of $\mu \cdots \gamma$. By successive application of the technique on each $Z_2(\lambda, \mu, \cdots, \gamma)$ it is clearly possible to complete the realization of the given impedance function (1).

Theoretically it is possible to go through the whole procedure without using k. However, the intermediate step, using k, will help in deriving the pattern of realization at every stage because in such a case the function will consist of only one variable and k instead of several variables. If at any stage the function to be realized is of only two variables, introduction of k becomes, of course, immaterial. Another point to be noted is that k should be carried throughout as a symbol and not assigned any particular value so that we are sure that in the realization at every stage the values of elements determined are independent of k.

It is clear that while going from (1) to (2) one might as well get $Z_1(\lambda)$, $Z_1(\mu)$, $\cdots$ or $Z_1(\gamma)$ instead of $Z_1(s)$. Which of these is simpler from the point of view of getting N_1 can be found easily by inspection.

The question of how to get at such a realization of $Z(s, k, k, \cdots k)$, such that in this realization the values of all s elements are independent of k, will not be discussed here since this question has already been dealt with elsewhere.[6,7] The following example will illustrate the discussion of the present note.

Illustrative Example

Let us consider the synthesis of the realizable impedance

$$Z(s, \lambda, \mu, \gamma) = \frac{\left\{\begin{aligned} &s^2[(\lambda\mu + 1)(\gamma\lambda + \mu\lambda + 1 + \lambda) + \mu(\lambda\gamma\mu + \gamma + \lambda\gamma)] + s[(\lambda\mu + 1)\lambda + (\lambda\mu + 1)(\gamma\lambda + \mu\lambda + 1 + \lambda) \\ &\quad + \mu(\mu\lambda + 1 + \lambda) + (\lambda\mu + 1 + \mu)(\lambda\gamma\mu + \gamma + \lambda\gamma)] + [(\lambda\mu + 1 + \mu)(\mu\lambda + 1 + \lambda) + \lambda(\lambda\mu + 1)]\end{aligned}\right\}}{s^2[\mu(\gamma\lambda + \mu\lambda + 1 + \lambda] + s[\mu\lambda + (\lambda\mu + 1 + \mu)(\gamma\lambda + \mu\lambda + 1 + \lambda)] + [\lambda(\lambda\mu + \mu + 1)]}. \tag{3}$$

From (3)

$$Z_1(s) \triangleq Z(s, k, k, k) = \frac{\left\{\begin{aligned} &s^2[(k^2 + 1)(2k^2 + k + 1) + k(k^3 + k^2 + k)] + s[k(k^2 + 1) + (k^2 + 1)(2k^2 + k + 1) \\ &\quad + k(k^2 + k + 1) + (k^2 + k + 1)(k^3 + k^2 + k)] + [(k^2 + k + 1)^2 + k(k^2 + 1)]\end{aligned}\right\}}{s^2[k(2k^2 + k + 1)] + s[k^2 + (k^2 + k + 1)(2k^2 + k + 1)] + k(k^2 + k + 1)}. \tag{4}$$

Examination of the denominator in (4) shows that (4) can be rewritten as

$$Z_1(s) = Z_{12}(s) + Z_{13}(s) \tag{5}$$

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[5] For instance, say, $2/\lambda$ would be replaced by $2/k$.

[6] H. Ozaki and T. Kasami, *op. cit.*, p. 254, Theorem 2, and pp. 256–257.

[7] In finding the factors of a polynomial $P(s, k)$, the pattern is found more easily by considering first $P(k, k)$.