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SYNTHESIS OF LUMPED-DISTRIBUTED RC n-PORTS

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ABSTRACT

Based upon the two-variable reactance matrix synthesis of Koga a method of synthesizing reciprocal networks constructed from uniformly distributed RC lines, lumped capacitors, lumped resistors, and transformers is outlined. The synthesis extends to that of nonreciprocal lumped-distributed lossless n-ports.

SYNTHESIS OF LUMPED-DISTRIBUTED RC n-PORTS*

Realizability conditions for passive uniformly distributed RC lines have been studied [1], [2], [3], [4], [5], [6], [7] mainly by means of transformations of the frequency variable resulting in the analysis and synthesis of lumped networks in the transformed plane. Wyndrum [1], [2], by the use of a positive-real transformation of the type used by Richards [8] obtained the conditions for driving point synthesis of uniform lines with identical RC products. Both driving point and transfer function synthesis have been given by O'Shea [3] for an arbitrary interconnection of uniform lines with constant RC products. Some general results on the realizability of arbitrary n-port connections of uniform lines of rationally related $\sqrt{rc}\,\ell$ products are presented in [7].

The above mentioned analysis and synthesis techniques apply to networks consisting of uniform RC lines only and do not in general allow introduction of lumped elements in the network. Some of the results of the study of two-variable reactance matrices [9] are applied here to the synthesis of networks consisting of an arbitrary interconnection of uniformly distributed RC lines with rationally related $\sqrt{rc}\,\ell$ products, lumped resistors, capacitors and ideal transformers.

The admittance matrix of a 2-port uniformly distributed RC line of length ℓ_o, whose capacitance and resistance per unit length are c and r, is [10]

$$\underset{\sim}{y}_{rc}(s) = \sqrt{s}\, y_o \begin{bmatrix} \operatorname{ctnh}\sqrt{s}\sqrt{rc}\,\ell_o & -\operatorname{csch}\sqrt{s}\sqrt{rc}\,\ell_o \\ -\operatorname{csch}\sqrt{s}\sqrt{rc}\,\ell_o & \operatorname{ctnh}\sqrt{s}\sqrt{rc}\,\ell_o \end{bmatrix} \qquad (1)$$

where $y_o = \sqrt{c/r}$ and $s = \sigma + j\omega$ is the true frequency variable. If we consider the positive-real frequency transformation

$$p = \operatorname{ctnh}\sqrt{s}\,d \quad , \quad 2d = \sqrt{rc}\,\ell_o \qquad (2)$$

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of the type used by Richards [8], we find that the resulting admittance matrix can be written as

$$\frac{1}{\sqrt{s}}\underset{\sim}{y}_{rc}(s) = \underset{\sim}{Y}_{rc}(p) = y_o \begin{bmatrix} \frac{1+p^2}{2p} & \frac{1-p^2}{2p} \\ \frac{1-p^2}{2p} & \frac{1+p^2}{2p} \end{bmatrix} \qquad (3)$$

Consequently $\underset{\sim}{Y}_{RC}(p)$ is lossless, symmetric, positive-real, and rational in p. The admittances of lumped capacitors and resistors can be expressed as

$$\frac{1}{\sqrt{s}}y_c(s) = Y_c(\sqrt{s}) = c\sqrt{s} \ , \ \frac{1}{\sqrt{s}}y_r(s) = Y_r(\sqrt{s}) = \frac{1}{r\sqrt{s}} \qquad (4)$$

In other words $Y_c(\sqrt{s})$ and $Y_r(\sqrt{s})$ are as admittances of capacitors and inductors in the $\sqrt{s}$ plane.

At this point we cosider interconnections of the above type elements, and transformers, and, thus, for convenience, define a <u>lumped-distributed RC-network</u> as a network constructed from uniformly distributed RC lines of rationally related $\sqrt{rc}\,\ell_o$ products, lumped capacitors and resistors, as well as transformers. For such a network then, if we let

$$\lambda = \sqrt{s} \qquad (5)$$

there exists a choice of d, for Eq. (2), such that the general branch admittances are of the form $\sqrt{s}\,[Cp+(1/Lp)+c\lambda+(1/\ell\lambda)]$. Consequently [11], any short circuit admittance matrix $\underset{\sim}{y}(s)$ of a lumped-distributed RC-network is such that

$$\frac{1}{\sqrt{s}}\underset{\sim}{y}(s) = \underset{\sim}{y}(p,\lambda) \qquad (6)$$

is a two-variable reactance matrix as defined in [9]. In view of this reactance property of $\underset{\sim}{y}(p,\lambda)$ we can state the following theorem which is based upon Koga's synthesis technique [9] for decomposing a two-variable reactance matrix.

Theorem: The necessary and sufficient conditions for $\underset{\sim}{y}(s)$ to be the short circuit admittance matrix of a lumped-distributed RC-network are that there exist variables p and λ such that $\underset{\sim}{y}(p,\lambda) = \underset{\sim}{y}(s)/\sqrt{s}$ is a rational symmetric, two-variable reactance matrix.

Necessity follows from a proper choice of p and considerations of branch admittances, as outlined above. Sufficiency follows from the fact that an $n \times n$ two-variable reactance matrix $\underset{\sim}{y}(p,\lambda)$ can be decomposed, according to the method of Koga, [9, p. 39], as

$$\underset{\sim}{y}(p,\lambda) = \underset{\sim}{y}_{11}(p) - \underset{\sim}{y}_{12}(p)[\underset{\sim}{y}_{22}(p)+\lambda \underset{\sim}{1}_{\ell}]^{-1}\underset{\sim}{y}_{21}(p) \qquad (7a)$$

and

$$\underset{\sim}{y}(p,\lambda) = \underset{\sim}{ⓨ}_{11}(\lambda) - \underset{\sim}{ⓨ}_{12}(\lambda)[\underset{\sim}{ⓨ}_{22}(\lambda)+p\underset{\sim}{1}_{m}]^{-1}\underset{\sim}{ⓨ}_{21}(\lambda) \qquad (7b)$$

with ℓ and m the degrees of $\varphi(p,\lambda)$, the least common denominator polynomial of the entries in $\underset{\sim}{y}(p,\lambda)$, in λ and p, respectively. Here, for example,

$$\underset{\sim}{Y}_c(p) = \begin{bmatrix} \underset{\sim}{y}_{11}(p) & \underset{\sim}{y}_{12}(p) \\ \underset{\sim}{y}_{21}(p) & \underset{\sim}{y}_{22}(p) \end{bmatrix} \qquad (8)$$

is a lossless, positive-real admittance matrix which when loaded in ℓ λ-plane capacitors yields $\underset{\sim}{y}(p,\lambda)$ through Eq. (7a). Since this decomposition does not in general result in a symmetric $\underset{\sim}{Y}_c(p)$, p-plane gyrators are needed for the realization of $\underset{\sim}{Y}_c$, but p-plane gyrators are active s-plane elements. Consequently, as shown by Koga [9, p. 44], we create from $\underset{\sim}{Y}_c(p)$ a <u>symmetric</u>, lossless, positive-real $(n+2\ell) \times (n+2\ell)$ admittance matrix $\underset{\sim}{Y}_S(p)$ which yields $\underset{\sim}{y}(p,\lambda)$ by loading a reciprocal p-plane realization of $\underset{\sim}{Y}_S(p)$ by unit λ-plane capacitors and inductors at the final 2ℓ ports, as shown in Fig. 1a).

A realization of $\underset{\sim}{Y}_S(p)$ results by applying the standard theory of lossless n-port transmission lines [12], that is by a cascade of $(n+2\ell)$-port unit elements obtained by applying Richard's Theorem for matrices. Finally the s-plane realization of $\underset{\sim}{y}(s)$ results by replacing each of the p-plane $(n+2\ell)$-port unit elements by $n+2\ell$ 2-port RC lines of $\underset{\sim}{y}_{RC}(s)$ as in Eq. (1) but of length $\ell_o/2$ and each λ-plane capacitor and inductor by an s-plane capacitor and resistor, Eq. (4). The final realization is as shown in Fig. 1b). Similar synthesis can be done starting from the decomposition given in Eq. (7b) instead of Eq. (7a). In this case we will have an $(n+2m)$-port network in the λ-plane terminated by RC lines as shown in Fig. 2; here the lines are again of length $\ell_o/2$ terminated in open or short circuits.

Using identical ideas, but with $p = \text{ctnh}\, s\sqrt{LC}\,\ell_o$, $\lambda = s$ and $\underset{\sim}{y}(p,\lambda) = \underset{\sim}{y}(s)$ rational in p and λ, we can give a synthesis of lumped-distributed LC-networks even in the nonreciprocal case [12], where synthesis can proceed from $\underset{\sim}{Y}_C(p)$ if desired to use a minimum number of lumped reactive elements. Further the possibility of a minimal gyrator synthesis exists [13]. This method then gives an alternate, and extension, to that of Youla [14], since synthesis in the lossless case can proceed also when the scattering matrix is prescribed.

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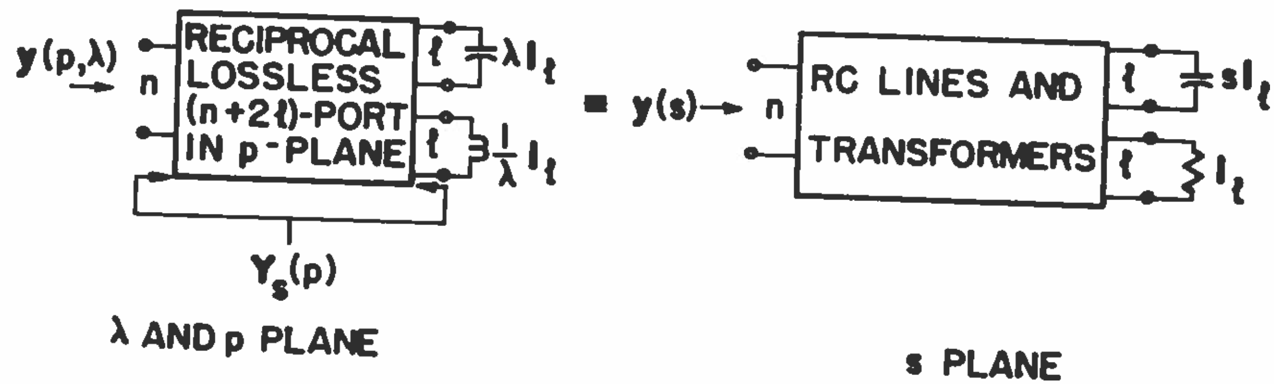
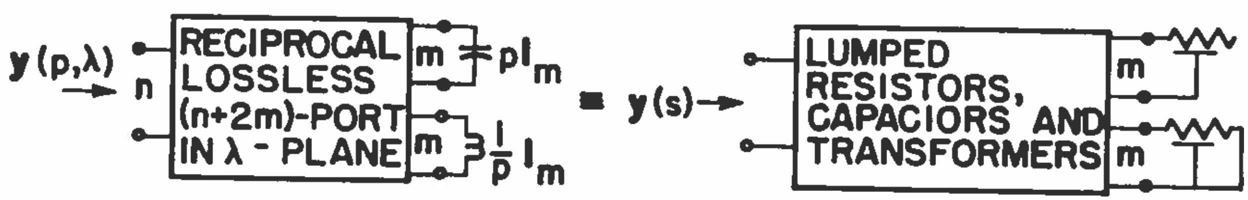
FIGURE CAPTIONS

1. Synthesis from Equation 7a)
2. Synthesis from Equation 7b)



y(p,λ)
n
RECIPROCAL LOSSLESS (n+2ℓ)-PORT IN p-PLANE
ℓ
λ I ℓ
1/λ I ℓ
Y s (p)
λ AND p PLANE
= y(s)
n
RC LINES AND TRANSFORMERS
s I ℓ
I ℓ
s PLANE



y (p,λ)
n
RECIPROCAL
LOSSLESS
(n+2m)-PORT
IN λ - PLANE
m
pI_m
m
1/p I_m
≡ y(s)
LUMPED
RESISTORS,
CAPACIORS AND
TRANSFORMERS
m
m