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REALIZABILITY CONDITIONS FOR DISTRIBUTED RC NETWORKS

by

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ABSTRACT

General theorems are stated presenting necessary and sufficient conditions for the existence of passive n-ports constructed from transformers and "rationally related" uniform RC transmission lines.

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I. Introduction

Because of their presence in microelectronic structures, passive distributed RC networks have become of importance. Consequently, realizability conditions for driving-point impedances of networks consisting of various types and connections of distributed RC lines have been studied in the literature [1], [2], [3], [4], often under somewhat practical but restricted assumptions. For example, Wyndrum [1, p. 1], by the use of a positive-real transformation, obtains the conditions for driving-point synthesis of uniform lines with identical RC products. Similar conditions are obtained by O'Shea [2], both for driving-point and transfer function synthesis, through the use of a nonpositive-real transformation but again for uniform lines with constant RC products.

By considering arbitrary n-port connections of uniform lines of rationally related $\sqrt{rc}\,\ell$ products, and transformers, some general results on the realizability of such networks are outlined here.

II. Preliminaries

The impedance matrix of a 2-port uniform RC line of length $\ell = \ell_o$, whose capacitance and resistance per unit length are c_o and r_o, is readily calculated as [5, p. 361]

$$\mathfrak{z}(s) = \frac{z_o}{\sqrt{s}} \begin{bmatrix} \coth \sqrt{s}\sqrt{r_o c_o}\,\ell_o & \operatorname{csch} \sqrt{s}\sqrt{r_o c_o}\,\ell_o \\ \operatorname{csch} \sqrt{s}\sqrt{r_o c_o}\,\ell & \coth \sqrt{s}\sqrt{r_o c_o}\,\ell_o \end{bmatrix} \tag{1}$$

where $z_o = \sqrt{r_o/c_o}$, and $s = \sigma + j\omega$ is the true frequency variable. If one considers the two different positive-real frequency transformations, of the type used by Richards [6] to obtain rational functions,

$$p = \tanh\sqrt{s}\,\frac{d_o}{2}\ ;\ d_o = \sqrt{r_o c_o}\,\ell_o \qquad (2a)$$

$$\Omega = \tanh\sqrt{s}\,d_o = \frac{2p}{p^2+1} \qquad (2b)$$

one finds that the resulting impedance matrices

$$\sqrt{s}\,\mathfrak{Z}(s) = \textcircled{Z}(\Omega) = z_o \begin{bmatrix} \frac{1}{\Omega} & \frac{\sqrt{1-\Omega^2}}{\Omega} \\ \frac{\sqrt{1-\Omega^2}}{\Omega} & \frac{1}{\Omega} \end{bmatrix} \qquad (3a)$$

$$= Z(p) = z_o \begin{bmatrix} \frac{p^2+1}{2p} & \frac{p^2-1}{2p} \\ \frac{p^2-1}{2p} & \frac{p^2+1}{2p} \end{bmatrix} \qquad (3b)$$

are positive-real [7], symmetric, and lossless $[Z(p) = -Z(-p)]$ in Ω and p, with $Z(p)$ being rational in p. One also finds that $\sqrt{s}$ times the driving-point impedance of an infinitely long line is the constant z_o (in s, Ω, and p).

The right hand side of Eq. (3b) describes a physically realizable LC 2-port in the p-plane. Any RC line whose $\sqrt{rc}\,\ell$ product is an integral multiple of $\sqrt{r_o c_o}\,\ell_o$ will also be described by a physically realizable LC two-port imepdance matrix, since the function $\tanh nx$, where n is a positive integer, can be expressed as a rational function of $\tanh x$. This enables us to obtain necessary and sufficient conditions for the driving-point impedance of an arbitrary interconnection of transformers and lines which have their products $\sqrt{r_i c_i}\,\ell_i = d_i$, rationally related to each other. Lines which have their $\sqrt{r_i c_i}\,\ell_i$ products rationally related will be called here, "rationally related lines." In such a network we chose $p = \tanh\sqrt{s}\,\frac{d_o}{2}$, where d_o is such that all the lines in the network have d_i equal to an integral multiple of d_o. The quantities d and z_o completely determine the terminal properties of a line even though its r, c, and ℓ are not unique. The 2-port line with $d = d_o$ will be called here, the unit line for d_o. It is of interest to note the behaviour of a few specially terminated lines in the p-plane. These are shown in the table where $p = \tanh \sqrt{s}\,\frac{d_o}{2}$.

III. Theorems

With the preceeding discussion as a preliminary we can state the following theorems.

Theorem 1: A necessary and sufficient condition that a function $z(s)$ be the driving-point impedance of an arbitrary interconnection of uniformly distributed RC lines of finite length whose $\sqrt{rc}\,\ell$ products are rationally related to each other, and infinitely long lines, is that $Z(p) = \sqrt{s}\, z(s)$, where $p = \tanh \sqrt{s}\,\frac{d_o}{2}$ with $d_o = \sqrt{rc}\,\ell$ of the unit line, is a rational, positive-real (driving-point) function of p (with no explicit dependence on s).

To prove the theorem we note that given an arbitrarily interconnected network of RC lines whose $\sqrt{rc}\,\ell$ products are rationally related we choose a unit line, and the indicated transformation transforms each line of finite length into a passive LC 2-port in the p-plane. Each line of infinite length is transformed into a p-plane resistor. Since the network is an interconnection of these LC 2-ports and resistors in the p-plane any driving-point function is a rational, positive-real function in p. Conversely if $z(s)$ is such that $\sqrt{s}\, z(s) = Z(p)$ is rational and positive-real in p we realize a p-plane RLC network, by standard techniques [8], with $Z(p)$ as the driving-point function, and then the lines in rows 3, 4 and 5 of the table are substituted for the capacitors, inductors and resistors respectively.

Corollary: If only rationally related lines of finite length are allowed in the arbitrary interconnection, a necessary and sufficient condition for $z(s)$ to be a driving-point impedance is that there exists some d_o such that $Z(p)$ is an LC driving-point function in p.

Since each line is transformed into an LC 2-port in the p-plane the resulting network is an LC network. Given $Z(p)$, we realize an LC network with $Z(p)$ as the driving-point impedance and each capacitor and inductor may be replaced by the lines in rows 3 and 4 of the table. It should be noted that neither in the theorem nor in the corollary are transformers needed.

Theorem 2: An $n \times n$ matrix $\mathfrak{Z}(s)$ is realizable as an interconnection of uniformly distributed, rationally related RC lines of finite length and transformers if and only if for some d_o, $Z(p) = \sqrt{s}\,\mathfrak{Z}(s)$ is lossless, positive-real, rational and symmetric.

The content of this theorem is seen to be true from the fact that $Z(p)$ can be realized by standard theories [8, p. 269] as an n-port consisting of inductors, capacitors and transformers in the p-plane.

Theorem 3: An $n \times n$ $\mathfrak{Z}(s)$ satisfying the conditions in theorem 3 can be realized by cascade synthesis using Richards' theorem for matrices [9, p. 142].

The reference cited develops a procedure for such a synthesis using a generalization of Richards' Theorem to Matrices. A special case of this theorem, when $n = 1$, and all d_i = constant, results in the cascade synthesis of Wyndrum [1]. In the matrix case transformers may be needed to consider nondiagonal $Z_o = Z(1)$ or to consider singular matrices met in the iterations needed to obtain cascade form.

IV. Discussion

Since the chosen transformation results in LC or RLC networks in the p-plane the existing wealth of information about these can be utilized in the p-plane. The transformation used by O'Shea is not positive-real and, as a consequence, it is rather difficult to work with and to use in arriving at general conclusions. In the treatment of this note the component lines are not limited to a constant $\sqrt{rc}\,\ell$ product. For p-plane RLC synthesis infinitely long s-plane RC lines are required, but for real frequencies $s = j\omega$, these can be adequately approximated by reasonably long, but finite length, lines [4, p. 75]. Approximation in the p-plane for given s-plane behaviour is of course rather complicated but can be facilitated by curves plotting $p = \tanh\sqrt{s}\,\frac{d_o}{2}$, as given by Scanlan and Rhodes [10].

It is worth commenting that one can form the scattering matrix $S(p) = [Z(p) + 1_n]^{-1}[Z(p) - 1_n]$, where $1_n = n \times n$ identity, and analogous results can be given in terms of $S(p)$. Still the interpretation of $s_{21}(p)$ as a voltage transfer under resistive terminations [11] is not too meaningful since p-plane resistors are s-plane infinite lines. It is also worth commenting that the quantity d_o is somewhat free to be chosen for a given network, but that halving it introduces higher degree (p-plane) functions as seen from rows 1 and 3 of the table.

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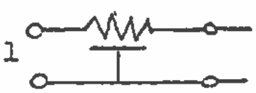
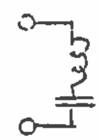
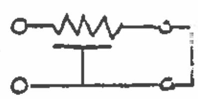
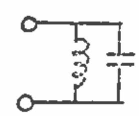
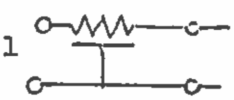
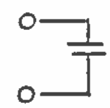
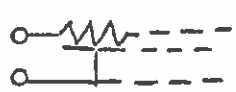
Table

Distributed-Lumped Equivalents

Value of d = $\sqrt{rc}\,\ell$	s-plane Physical Configuration	p-plane equivalent of driving-point impedance at port 1. $z_o = \sqrt{r/c}$
d_o	1 	 $L = \frac{z_o}{2}$ $C = \frac{z_o}{2}$
d_o	1 	 $L = \frac{2}{z_o}$ $C = \frac{z_o}{2}$
$d_o/2$	1 	 $C = \frac{1}{z_o}$
$d_o/2$	1 	 $L = z_o$
$d = \infty$ $0 < rc < \infty$	1 	 $R = z_o$

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