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A CAUER SYNTHESIS OF RL n-PORTS

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ABSTRACT

A continued-fraction first-Cauer synthesis for resistor-inductor-transformer n-ports is described. A feature, important for state-space developments, is that, when a prescribed impedance matrix has no poles at infinity, the resulting structure takes the form of a resistor-transformer network, possessing an impedance matrix, cascade loaded in the minimum number of unit inductors.

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I. Introduction

As a generalization of his reactance function synthesis Cauer [1, p. 549], [2, p. 651] has presented an n-port continued-fraction synthesis of reciprocal lossless networks. Although this can be generalized to the non-reciprocal case [3, Chap. 7], the corresponding n-port RL synthesis is perhaps of more interest because of its use in state-space synthesis [4]. We therefore present here the n-port first-Cauer synthesis of an RL impedance matrix $Z(p)$. From this synthesis we then make the important deduction that when $Z(p)$ has no pole at infinity it can be realized as a cascade load connection [5] of a resistor-transformer coupling network possessing an impedance matrix terminated in unit inductors. The number of inductors required is the minimum possible [6, p. 536], being equal to the degree of $Z(p)$.

II. Preliminaries

We recall that a real $m \times n$ matrix T defines an $(n+m)$-port transformer through [7, p. 233]

$$V_1 = \tilde{T}V_2 , \quad I_2 = -TI_1 \tag{1}$$

where the tilde denotes matrix transposition and V_1, I_1 and V_2, I_2 are voltage and current n- and m-vectors; such a transformer is conveniently denoted by the box labelled T in Fig. 1a). If this transformer is loaded by an impedance Z_ℓ, Fig. 1a), or an admittance Y_ℓ, Fig. 1b), one finds

$$Z_i = \tilde{T}Z_\ell T \tag{2a}$$

$$Y_i = TY_\ell\tilde{T} \tag{2b}$$

for the respective configurations.

By definition a symmetric, rational, positive-real $n \times n$ matrix $Z(p)$ is <u>RL</u> if $\tilde{x}Zx$ is an RL driving-point impedance for all real, constant n-vectors x [8, p. 149]. Thus one can extract the poles at infinity, pL,

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from $Z(p)$, or the constant at infinity, G, from $Y(p)$, to obtain positive-real RL remainders Z_r and Y_r [9, p. 268]:

$$Z(p) = pL + Z_r(p) \quad (3a)$$

$$Y(p) = G + Y_r(p) \quad (3b)$$

Moreover, L and G are positive semidefinite matrices. Of special interest is the fact that if Z, or Y, is singular, then there exists a real, constant, nonsingular matrix T such that [10, p. 155]

$$Z(p) = \tilde{T}[Z_{ns}(p) \dotplus 0_{n-r}]T \quad (4a)$$

$$Y(p) = T[Y_{ns}(p) \dotplus 0_{n-r}]\tilde{T}$$

where $\dotplus$ denotes the direct sum, r is the rank of Z or Y and order of the nonsingular matrices Z_{ns} or Y_{ns}, and 0_{n-r} is the $(n-r)x(n-r)$ zero matrix.

III. Synthesis

Let $Z(p)$ be an $n \times n$ RL impedance matrix; then the first-Cauer synthesis proceeds somewhat as in the 1-port case. By Eqs. (3) and (4) we can extract the pole at infinity from $Z(p)$, make the positive-real remainder nonsingular, invert and subtract the constant at infinity, make the remaining positive-real admittance nonsingular, and repeat the process until it terminates. The equations for the ith inductor extraction are

$$\ldots + \tilde{T}_{i-1}[Z_{i-1}(p) \dotplus 0_{r_{i-2}-r_{i-1}}]T_{i-1} \quad (5a)$$

$$Y_{i-1}(p) = Z_{i-1}(p)^{-1} = G_{i-1} + T_i[Y_i(p) \dotplus 0_{r_{i-1}-r_i}]\tilde{T}_i \quad (5b)$$

$$Z_i(p) = Y_i(p)^{-1} = pL_i + \tilde{T}_{i+1}[Z_{i+1}(p) \dotplus 0_{r_i-r_{i+1}}]T_{i+1} \quad (5c)$$

Since $L_i \neq 0_{r_i}$, except possibly at the first step or if $Z_i = 0_{r_i}$, the process does terminate. In actual fact the degree $\delta[\;]$ must then decrease by $\delta[pL_i] = \mathrm{rank}[L_i]$ at the ith extraction since [11, p. 543]

$$\delta[Z_{i-1}] = \delta[Y_{i-1}] = \delta[Y_i] = \delta[Z_i] = \delta[pL_i] + \delta[Z_{i+1}] \quad (6)$$

as seen from $Z_iY_i = 1_{r_i}$, where 1_{r_i} is the $r_i \times r_i$ identity, $\mathrm{rank}[L_i] = r_i$.

Since rank$[L_i]$ inductors are used to realize pL_i, the overall synthesis uses the minimum of $\delta[Z]$ inductors.

To illustrate the structure obtained, let us consider the case of interest to the state-space theory where $Z(p) = Z_{-1}(p)$ has no pole at infinity [4, Sec. 6] The extractions of Eq. (5), for $i = 1$, are shown in Fig. 2 where Eqs. (2) have been used for the terminated transformers. Note that all G_i and L_i are positive semidefinite and hence the resistor and inductor portions can be realized through diagonalization [9, p. 261] by terminating transformers in rank G_i and rank L_i resistors and inductors of unit value.

IV. Existence of a Coupling Impedance

We wish to show that, when $Z(\infty)$ is finite, if all inductors are removed, as shown in Fig. 3, the remaining resistor-transformer coupling network has an impedance matrix Z_c. The iterative nature of the first-Cauer synthesis shows that if there is an impedance matrix for the first iteration, defined through Fig. 2 as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{7a}$$

then an impedance matrix will exist for all such sections in the iteration. Then it is not hard to see that Z_c will exist as an "overlapping direct sum" of the individual section impedances.

To determine the existence of Eq. (7a) first note that $Z_{11} = Z(\infty)$. Second, if Z_{12} exists, then $Z_{21} = \tilde{Z}_{12}$ by reciprocity. But Z_{21} exists since application of I_1 (with I_2 zero) yields a unique $V_1 = Z_{11}I_1$ and hence a unique V_2 since, by Eq. (1),

$$\begin{bmatrix} V_2 \\ \hline \end{bmatrix} = \tilde{T}_1 V_o , \quad \begin{bmatrix} V_o \\ \hline 0 \end{bmatrix} = \tilde{T}_o^{-1} V_1 = \tilde{T}_o^{-1} Z_{11} I_1 \tag{7b}$$

Since $G_o = Y_o(\infty)$ is nonsingular, as seen from $Z_o(\infty)Y_o(\infty) = 1_{r_o}$, we finally see that Z_{22} exists as the upper left $r_1 \times r_1$ corner of $\tilde{T}_1 G_o^{-1} T_1$. Consequently, Z_c exists when $Z(\infty)$ is finite.

V. Conclusion

By removal of terms at infinity a continued-fraction n-port RL synthesis has been given which uses the minimum number, $\delta[Z]$, of inductors. When $Z(\infty)$ is finite, but not otherwise, the resulting structure can be looked upon as a resitor-transformer coupling $(n + \delta[Z])$-port possessing an impedance matrix Z_c cascade loaded in $\delta[Z]$ unit inductors. If $Z(p)$ has a pole at infinity then series inductors are inserted at the input of Fig. 2, and their extraction leads to a new RL impedance matrix $Z_{-1}(p)$ which has $Z_{-1}(\infty)$ finite. Similar results hold for RC networks, and removals at other frequencies than infinity can be made if so desired.

It should be pointed out that similar ideas are mentioned, with scant details, by Hazony, [9, p. 276], and that the method is similar to that for lossless n-ports described by Cauer, but convenient notation, as in Fig. 2, yields considerable insight into the structure.

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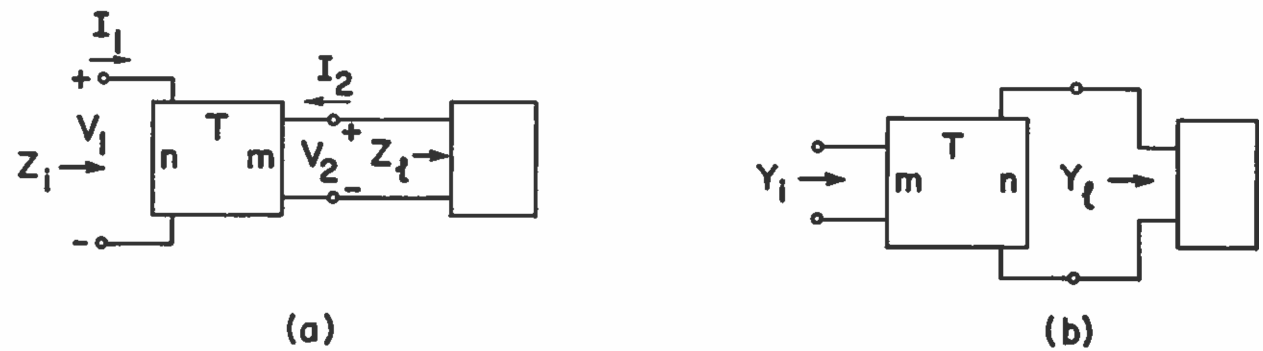
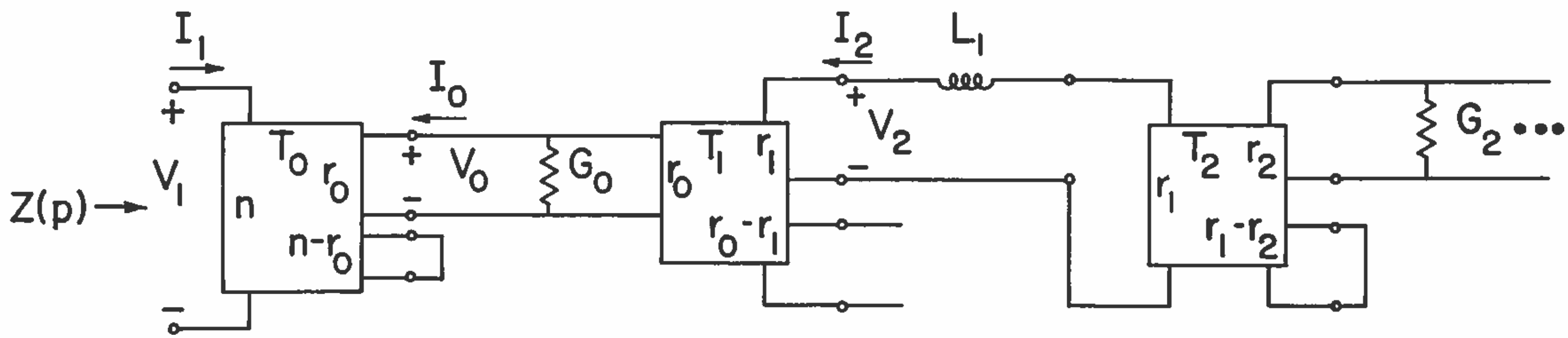
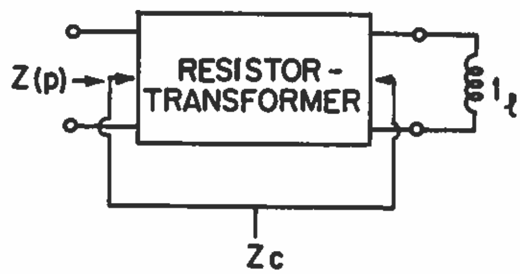
FIGURES TITLES



I1
+
-
V1
Zi
n
T
m
I2
+
V2
-
Zl
(a)
Yi
m
T
n
Yl
(b)



Z(p)
I_1
V_1
+
−
n
T_0
r_0
$n-r_0$
I_0
V_0
G_0
T_1
r_0
r_1
r_0-r_1
I_2
V_2
L_1
T_2
r_1
r_2
r_1-r_2
G_2


RESISTOR -
TRANSFORMER
Z(p)
Zc