Impedance synthesis via state-space techniques

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Synopsis

The paper deals with the synthesis of passive networks and relies on general systems theory and control concepts. The network-synthesis problem is first interpreted in state-variable terminology, solved as a control problem and the solution is then translated back into network-theory terms. After a review of state–space formulations, an algebraic theory of synthesis is developed, beginning with a minimal state–space realisation, perhaps obtained through control-theory procedures, from which a synthesis of rational positive-real impedance matrices is obtained through a transformation on the state. The method rests upon an appropriate basis change, in the state–space, obtained by factoring the P matrix of the control-theory positive real lemma. The minimum number of resistors and reactive elements is used. The paper also serves as a review of the 'state-of-the-art' for formal n port synthesis; the results lead to new methods of attacking open problems, as well as to methods of analysis and synthesis via digital computers.

List of principal symbols

F, G, H, J = constant state-variable matrices
$I_k = k \times k$ identity matrix
L = constant matrix for p.r. lemma
m = output dimension
M = coupling-network (constant) impedance matrix
n = input dimension
N = network
p = state dimension
P = constant positive definite matrix for p.r. lemma
r = rank of resistivity matrix = number of resistors
$s = \sigma + j\omega$ = complex frequency variable
T, T_R = state-basis change matrices
u = input n vector
$W(s)$ = general transfer-function matrix
W_0 = factor of resistivity matrix at infinity
x = state p vector
y = output m vector
$Z(s), \hat{Z}(s)$ = prescribed impedance matrices
$\delta[q]$ = McMillan degree of $[q]$
Σ = sign matrix
A prime indicates transposition

1 Introduction

The disciplines of network theory, control systems, and general systems have much in common; for example, network functions (or matrices) are generally particular cases of transfer functions (or matrices); again, networks may profitably be examined from the state–space point of view, which is essentially a general-systems concept, primarily introduced to study control systems.

It is surprising, therefore, to find that there are not more links between the disciplines. Nevertheless, it is possible to point to an ever increasing number of isolated examples; e.g. References 1–7, which are concerned with developing a state–space description of a network, or References 8 and 9, which discuss positive real functions and matrices from a control viewpoint. Groundwork for a control viewpoint of the scattering-matrix synthesis problem is discussed in Reference 10, with further results reported in Reference 11.

This paper is an attempt to lay another bridge across the gap. It is concerned with giving a passive-network synthesis via the application of general systems theory and control concepts to the impedance matrix.

The early work of Cauer,[12] Brune,[13] Darlington[14] and others, and later Bott and Duffin,[15] represented some of the first successful attempts to establish synthesis procedures for 1-port networks. Generally, the problem they considered was that of synthetising a network, given a mathematical description of it, usually a positive real function.

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The desire to extend network theory to multiport situations led to the study of positive-real matrices. An $n \times n$ positive real matrix $A(s)$ fulfills the following conditions [References 16 (p. 217) and 17] (the asterisk denotes complex conjugation and the prime denotes matrix transposition):

(a) $A(s)$ is analytic in the strict right-hand halfplane
(b) $A^*(s) = A(s^*)$ in the strict right-hand halfplane
(c) $A(s) + A'(s^*)$ is a nonnegative definite matrix in the strict right-hand halfplane.

This definition is a natural extension of the definition of a positive-real function (Reference 18, p. 67).

It is not difficult to show that, if it exists, the impedance matrix of a multiport network which is linear, finite, time-invariant and passive is a positive real matrix of rational functions (Reference 19, p. 153). It is, however, considerably harder to establish the converse, namely, that to a rational positive real matrix there corresponds a linear, finite, time-invariant, passive network with the given matrix as the impedance matrix of the network.

This impedance-matrix synthesis problem, or, what amounts to a variant of it, the scattering-matrix synthesis problem, has been solved in various ways by a number of workers (Reference 16, Pt. II).[19–23] Both reciprocal syntheses (those using resistors, inductors, capacitors and transformers, but no gyrators) and nonreciprocal syntheses (those using also gyrators) have been considered, and they are summarised in References 24–27. None of these syntheses could be construed as depending on general systems or control-theory techniques for its establishment, though there are available methods of realising lossless networks from their state-variable description.[28]

Our approach in this paper is to express the network-synthesis problem in state-variable terminology as customarily applied to the theory of control, to solve the resulting control problem and then to reinterpret this solution in network-theoretic terms.

In Section 2, we outline briefly, but it is hoped fully, the necessary state-variable preliminaries. The principal idea is that of a realisation of a matrix of rational transfer functions, which is essentially a collection of four constant matrices describing the transfer-function matrix. The theory of minimal realisations (where exactly what is minimal will be explained in Section 2) is also considered. Section 3 poses the impedance-synthesis problem in general systems-theory language, reducing it to a search for a realisation possessing certain properties (corresponding to the passivity of a resistive-coupling network).

Section 4 is concerned with explaining an interesting control-theory lemma, which characterises the concept of positive reality in terms of the matrices of a minimal realisation. Section 5 shows that this characterisation allows ready selection of a realisation possessing the properties mentioned as being sought after in Section 3, so that a passive synthesis can then be given. In this Section, the details of a synthesis procedure are also discussed, and it is shown that the synthesis uses the minimal number of reactive and resistive elements.

Examples of the synthesis procedure are discussed in Section 7, while Section 6 discusses reciprocal synthesis with special emphasis on *RL* networks. Section 8 discusses some of the remaining problems of the control-systems–network-theory interface.

2 State-variable preliminaries

Before turning our attention to the main problem in hand, we digress in this Section to point out some pertinent results of a general (continuous) systems nature. Linear, time-invariant, multivariable, finite-dimensional dynamical systems can be characterised by an $m \times n$ transfer-function matrix $W(s)$ whose elements are rational functions of the variable s.[29] The matrix $W(s)$ relates the Laplace transform of the input n vector $U(s)$ to the Laplace transform of the output m vector $Y(s)$ through

$$Y(s) = W(s)U(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It will be sufficient, for most of the material following, to restrict consideration to the case where $W(s)$ has no pole at infinity, i.e. $W(\infty)$ is finite. If $W(s)$ does have a pole at infinity, its extraction can be made following standard techniques (eqn. 11, Section 3).

Under these conditions, it is possible to describe the system via a time-domain state–space representation. In this representation, the input $\boldsymbol{u}$ and output $\boldsymbol{y}$ are mathematically related via an intermediate variable, the state $\boldsymbol{x}$. The relevant equations are

$$\dot{x} = Fx + Gu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

$$y = H'x + Ju \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

In these equations $\boldsymbol{x}$, $\boldsymbol{u}$, and $\boldsymbol{y}$ are vector functions of time, rather than Laplace transforms, as in eqn. 1; $\dot{x}$ is the time derivative of x. The vector x has a dimension p (which we shall not specify for the moment), while the matrices F, G, H and J are all constant and of appropriate dimensions, respectively, $p \times p$, $p \times n$, $p \times m$ and $m \times n$.

By taking the Laplace transform of eqn. 2 and eliminating $X(s)$, it is straightforward to obtain, here I_p is the $p \times p$ identity matrix,

$$Y(s) = [J + H'(sI_p - F)^{-1}G]U(s) \quad . \quad . \quad . \quad . \quad (3)$$

and it follows, by comparing eqn. 1 and eqn. 3, that the matrix $W(s)$ of rational functions of s is related to the four constant matrices F, G, H and J by

$$W(s) = J + H'(sI_p - F)^{-1}G \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Note that many authors use H where we use H'.

It is clear that any quadruple $\{F, G, H, J\}$ determines a $W(s)$ which is a matrix of rational functions of s, having $W(\infty)$ finite. The converse, however, that $W(s)$ determines a quadruple $\{F, G, H, J\}$, is not obvious immediately. From eqn. 4, it follows that J is determined as $W(\infty)$, but otherwise the existence of F, G and H is not *a priori* guaranteed.

None the less, as is discussed for example in References 29–31, any $W(s)$ does determine an infinity of triples $\{F, G, H\}$, such that eqn. 4 is satisfied with $J = W(\infty)$. These references, with the work of Ho[30] being especially significant, discuss methods of determining the triples, and consider in particular the question of determining all triples when one is known.

Any quadruple $\{F, G, H, J\}$ satisfying eqn. 4 is termed a realisation of $W(s)$, while the triple $\{F, G, H\}$ is termed a *realisation* for $W(s) - W(\infty)$, since J in the quadruple is zero.

The dimensions of the various possible F matrices which can occur in the triples are not the same; but it is true that there is a minimal dimension for the set of all matrices F appearing in the realisations of a prescribed $W(s)$. For example, if $W(s)$ is a constant matrix, it is clear from eqn. 4 that this dimension is zero, or if $W(s)$ is a scalar of the form a/s, it is clear that this dimension is 1.

A realisation $\{F, G, H, J\}$ for which F has a minimal dimension is termed a *minimal realisation*.

A most important feature of minimal realisations is that they are uniquely determined by $W(s)$, except for arbitrary prescription of the basis vectors of the state–space.[29] What concerns us more, however, is the way that this arbitrary prescription affects $\{F, G, H\}$. Reference 29, p. 157, shows that, if $\{F, G, H\}$ is a minimal realisation of $W(s) - W(\infty)$, any other minimal realisation is of the form $\{T^{-1}FT, T^{-1}G, T'H\}$, where T is an arbitrary nonsingular matrix. Thus, if $\{F_1, G_1, H_1\}$ and $\{F_2, G_2, H_2\}$ are both minimal, the existence is guaranteed of a nonsingular T, such that

$$F_2 = T^{-1}F_1T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

$$G_2 = T^{-1}G_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

$$H_2 = T'H_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5c)$$

The dimension of a minimal realisation, i.e. the dimension of the associated state-space or the order of the square matrix F, is termed the *degree* of $W(s)$, written $\delta[W]$.

The history of the concept of degree in network and control theory is an interesting one. Tellegen's definition of the order of a network (Reference 32, p. 322) proceeds on physical grounds by defining the order as the maximum number of natural frequencies obtainable by embedding the given network in an arbitrary passive network. This order definition agrees with the mathematical definition of McMillan (Reference 16, pp. 543 and 592) of the degree of a square matrix $Z(s)$, which is shown to imply that $\delta[Z]$ is the minimal number of reactive elements in any passive synthesis of $Z(s)$ when $Z(s)$ is a positive real impedance matrix (Reference 26, p. 322). Since we can conceive of deriving a state–space representation of $Z(s)$ by associating a state variable with each reactive element in a network synthetising $Z(s)$,[1,2,4] it is not surprising to find that McMillan's definition is essentially the same as the one we have given. Still another mathematical definition of degree, motivated by a different set of physical concepts, is given in Reference 33. Because of the corresponding physical meanings of these, it is therefore fortunate to find (Reference 34, p. 542) that these definitions are mathematically the same thing, provided that poles at infinity are suitably dealt with.

We shall be especially interested in the fact that the minimal number of reactive elements in a synthesis of an impedance matrix $Z(s)$, i.e. McMillan's $\delta[Z(s)]$, is the same thing as the dimension of a minimal (control-systems) realisation, provided that $Z(\infty)$ is finite.

3 Impedance-synthesis problem in systems-theory language

Our solution of the synthesis problem is a control-theoretic one, and, to achieve the solution, it is necessary to express the synthesis problem in systems-theoretic language.

Formally, the synthesis problem is: given a positive real $n \times n$ matrix $Z(s)$ (whole elements are rational functions of s), find a finite circuit connection of passive network elements synthetising $Z(s)$.

To motivate the synthesis procedure presented, it will be necessary to make some apparently restrictive assumptions concerning the final form of the synthesis. These assumptions include more than merely the assumption of the existence of a synthesis; they will, however, be shown to be valid as a result of the synthesis techniques presented.

A synthesis may contain any of the following types of linear, passive, time-invariant network elements: resistors, gyrators (ideal), transformers, inductors and capacitors. The first three classes are nondynamic, or memoryless. The last two classes are dynamic, and thus not memoryless; the behaviour of an individual element can, if desired, be specified with the aid of state variables.

It is possible by a simple replacement to entirely eliminate one of these classes, namely the capacitors. It is now reasonably well known that, if a unit gyrator of impedance matrix

$$Z_g = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is terminated at one port in a unit inductance, the impedance viewed at the other port is that of a unit capacitance (Fig. 1).

Consequently, all capacitors in a circuit may be replaced by gyrators and inductors.

Therefore, in any passive $n$port network N synthetising $Z(s)$, it is possible to always assume that the only dynamic

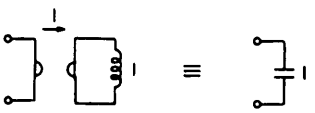


Fig. 1
Capacitor replacement

elements used are inductors, and positive unit inductors at that, since transformers may be used to provide the normalisation.

By dividing the elements of N into two classes, the nondynamic elements and the unit inductors, assumed to be p in number, it is possible to regard N as an interconnection of two networks N_1 and N_2, where N_1 is an $(n+p)$port, consisting of the nondynamic elements of N, and N_2 is simply p unit inductors, uncoupled from one another. One of these inductors loads each of the last p ports of N_1, as shown in Fig. 2.

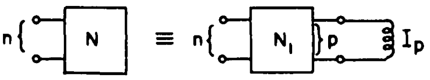


Fig. 2
Inductor extraction

Although N possess an impedance matrix $Z(s)$ by assumption, there is no guarantee that N_1 will possess an impedance matrix. For our purposes it will suffice to simply assume that an impedance matrix does exist for N_1; this will indeed be the case for the synthesis to be considered. Because N_1 consists of purely nondynamic elements, this impedance matrix is constant; it is also positive-real, when N_1 consists of purely passive elements. The port partition of N_1 determines a corresponding partition of its impedance matrix, which we write as

$$M = \left[\begin{array}{c|c} z_{11} & z_{12} \\ \hline z_{21} & z_{22} \end{array}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here the matrices z_{11}, z_{12}, z_{21} and z_{22} have dimensions, respectively, $n \times n$, $n \times p$, $p \times n$ and $p \times p$.

It is now possible to express the input impedance at the first n ports of N_1 (when the latter is terminated in the unit inductors) in terms of the z_{ij} and the impedance matrix N_2, i.e. sI_p. The result, which may be derived by straightforward calculation, is

$$Z(s) = z_{11} - z_{12}(sI_p + z_{22})^{-1}z_{21} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eqn. 8 bears a striking similarity to eqn. 4; in fact, we observe that one possible realisation of $Z(s)$, in the sense of Section 2, is given by

$$\{F, G, H, J\} = \{-z_{22}, z_{21}, -z'_{12}, z_{11}\} \quad . \quad . \quad . \quad . \quad (9)$$

This appears to have been first recognised by Youla (Reference 35, p. 30).

Let us review the significance of eqn. 9. If $Z(\infty)$ is finite, there are many quadruples $\{F, G, H, J\}$ constituting a realisation in the sense of Section 2. If we have on hand a synthesis of $Z(s)$, and the nondynamic part of this synthesis possesses a constant impedance matrix M, this impedance matrix determines one particular realisation through eqn. 9. Drawing further on the material of Section 2, if the synthesis uses a minimal number of reactive elements, the realisation is a minimal one. Thus each minimal-reactive-element synthesis yields, via M, a minimal realisation. This fact is not especially significant for our purposes here; we know how to construct minimal realisations without the necessity of synthetising a network first.[30]

What is significant, however, is that eqn. 9 implies that each minimal realisation yields a minimal-reactive-element synthesis. Thus, given an impedance matrix $Z(s)$ with $Z(\infty)$ finite, we can determine a minimal realisation by the known methods.[30, 34, 35] This minimal realisation determines the impedance matrix of a network N_1, through eqn. 9, such that, if N_1 is synthetised and its last p ports are terminated in unit inductors, the resulting $n$port has an impedance matrix $Z(s)$. The difficulty arises, however, in that, given an arbitrary minimal realisation, the impedance matrix of N_1

$$M = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \quad . \quad . \quad . \quad . \quad (10)$$

may not be positive real. If it is not, we cannot synthetise the corresponding N_1 using only passive elements, even though the given $Z(s)$ is positive real. If M is positive real, the synthesis problem is easy (Reference 25, pp. 255–261), being that of synthetising a purely resistive network. Furthermore, we achieve thereby a synthesis of $Z(s)$ with the minimum number of reactive elements.

A second apparent difficulty, that of requiring $Z(\infty)$ to be finite, is easily resolved. It is well known (see, for example, Reference 16 for the reciprocal case, and Reference 36, p. 3, for the general case) that a positive-real $\hat{Z}(s)$ can be written as

$$\hat{Z}(s) = s\hat{L} + Z_0(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $\hat{L}$ is a nonnegative definite-constant symmetric matrix and $Z_0(s)$ is positive-real with $Z_0(\infty)$ finite. The matrix $\hat{Z}(s)$ can be synthetised as the series connection of transformer-coupled inductors (of impedance matrix $s\hat{L}$) and a network N_0 [of impedance matrix $Z_0(s)$]. It is, moreover, true that (Reference 36, p. 4)

$$\delta[\hat{Z}(s)] = \delta[s\hat{L}] + \delta[Z_0(s)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where we are using the degree definition of McMillan; in other words, eqn. 12 says that we can achieve a minimal reactive-element synthesis of $\hat{Z}(s)$ by series connecting two minimal-reactive-element syntheses, one of $s\hat{L}$ and one of $Z_0(s)$. We note also that, since $Z_0(s)$ is free of poles at infinity, $\delta[Z_0(s)]$ is also the dimension of a minimal (state-variable) realisation of $Z_0(s)$.

In the case where $Z_0(s)$ has finite poles on the $j\omega$axis, it is possible (but not actually necessary) to further simplify the synthesis problem by writing (Reference 36, p. 3)

$$Z_0(s) = Z_1(s) + Z(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $Z_1(s)$ and $Z(s)$ are both positive real, $Z_1(s)$ has poles only on the $j\omega$axis and $Z(s)$ has poles in the strict left-hand halfplane. The matrix $Z_1(s)$ can be synthetised by known methods (References 19, p, 155, and 37, p. 27), as a series connection of transformer-coupled tuned circuits, possibly in conjunction with gyrators. Alternatively, if $j\omega$axis poles are indeed separately extracted, $Z_1(s)$ can be synthetised by an application of the state-variable technique to it.[38]

It is, moreover, true that

$$\delta[Z_0(s)] = \delta[Z_1(s)] + \delta[Z(s)] \quad . \quad . \quad . \quad . \quad . \quad (14)$$

implying that a minimal reactive-element synthesis of $Z_0(s)$ derives from a minimal reactive-element synthesis of $Z_1(s)$ and $Z(s)$. A minimal reactive-element synthesis of $Z_1(s)$ is a result of the procedures mentioned.

As a consequence, we shall feel free to restrict attention to the problem of synthetising a positive real $Z(s)$ which is finite at $s = \infty$ and has poles only in the strict left-hand half plane. Moreover, the minimal number of reactive elements in a synthesis of $Z(s)$ is the dimension of a minimal realisation of $Z(s)$.

Returning now to the mainstream of the argument, we note that the problem of giving a minimal reactive-element synthesis for a rational positive-real $Z(s)$ reduces to the following problem: Given a minimal realisation $\{F, G, H, J\}$ of $Z(s)$, assumed to be positive-real with $Z(\infty)$ finite, and to have all poles in the strict left-hand halfplane, find a nonsingular T, such that the realisation $\{T^{-1}FT, T^{-1}G, T'H, J\}$ has

$$M = \begin{bmatrix} J & -H'T \\ T^{-1}G & -T^{-1}FT \end{bmatrix} = (I_n \dotplus T^{-1}) \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} (I_n \dotplus T) \quad . \quad . \quad . \quad . \quad (15)$$

positive real (where $\dotplus$ denotes the direct sum), or, alternatively, such that

$$M + M' \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The notation $\geqslant 0$ is shorthand for nonnegative definite.

Note that, from eqn. 5, all minimal realisations will be of the form $\{T^{-1}FT, T^{-1}G, T'H, J\}$ for some T.

We remark that, if $Z(s)$ is not positive real, there is certainly no possibility that a suitable T will exist to obtain M positive real. Even if $Z(s)$ is positive real, the existence of T is not guaranteed *a priori*; this is because the existence of T is equivalent to the existence of an impedance matrix M for N_1. In the next Sections we shall show how to find such a T.

4 Positive-real constraint as a control-theory concept

The existence of T in eqn. 15, such that eqn. 16 is satisfied, is hopefully a consequence of $\{F, G, H, J\}$ satisfying some set of conditions, and hopefully this set of conditions will be satisfied if $Z(s)$ is positive real. Accordingly, we ask: what constraint is placed on the matrices in a minimal realisation $\{F, G, H, J\}$ of a transfer function $Z(s)$ if the transfer function is constrained to being positive real?

The answer to this question is contained in the following control-theory lemma:[9]

Positive real lemma. Let $Z(s)$ be an $n \times n$ matrix of rational transfer functions with $Z(\infty)$ finite. Let $\{F, G, H, J\}$ be a minimal realisation for $Z(s)$. Let all poles of $Z(s)$ either be in the left-hand halfplane, or be simple on the $j\omega$ axis. Then necessary and sufficient conditions for $Z(s)$ to be positive real are: there exist a symmetric positive definite matrix P, and matrices W_0, L, such that

$$PF + F'P = -LL' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17a)$$

$$PG = H - LW_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17b)$$

$$W_0 W_0 = J + J' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17c)$$

While we shall not attempt to prove this result here, we shall make several remarks about it by way of giving a partial outline of the proof. The result was first established for the case $n = 1$ in Reference 39, and for the case of arbitrary n in Reference 9. Reference 8 states, but does not prove, a less general theorem applying for arbitrary n.

The fact that eqns. 17 imply that $Z(s)$ is positive real is not hard to establish; the converse is considerably more difficult, however, and depends for its proof on the following decomposition, valid for positive real $Z(s)$, which is established in Reference 40. For positive real $Z(s)$, there exists a matrix $W(s)$, unique to within multiplication by a constant orthogonal matrix, such that

$$Z(s) + Z'(-s) = W'(-s)W(s) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

with $W(s)$ having several additional properties.

The first additional property concerns the size of W, which is $r \times n$, where r is the normal rank of the resistivity matrix $Z(s) + Z'(-s)$. The normal rank of a matrix of rational transfer functions is the rank of that matrix almost everywhere, i.e. throughout the s plane, except perhaps at a finite number of isolated points which result in certain minors of $Z(s) + Z'(-s)$ being zero or infinite at these points only. Note that $r \leqslant n$. We comment that all factorisations of the form of eqn. 18 have the same rank.

The second and third additional properties are that $W(s)$ is analytic in the right-hand halfplane, and that there exists at least one right inverse of W; i.e. a matrix W^{-1}; such that $WW^{-1} = I_r$, with W^{-1} also analytic everywhere in the right-hand halfplane. Equivalently, W has (strict) rank r in the right-hand halfplane. These additional conditions then ensure that W is unique to within multiplication by an arbitrary orthogonal matrix. We comment that it is computationally easy to get one W which satisfies eqn. 18 (Reference 26, p. 168), but to obtain analyticity of W^{-1} in the right-hand halfplane is very difficult.

As pointed out earlier, we can restrict consideration to those $Z(s)$ which have poles with negative real parts. It is then possible to show that this particular $W(s)$ has a minimal realisation $\{F, G, L, W_0\}$, the first two matrices of this realisation being identical to two of the minimal realisation of $Z(s)$. This property will not, in general, be possessed by other $W(s)$ satisfying eqn. 18. The matrix L in this realisation of W is the matrix L of eqn. 17, while, naturally, $W_0 = W(\infty)$.

The proof of the lemma now requires the exhibition of P, and a demonstration that eqns. 17 are satisfied. Eqn. 17c is readily checked, by putting $s = \infty$ in eqn. 18. To define P, we start with any minimal realisation of the $r \times n$ $W(s)$, and transform it so that its system and input matrices F and G are identical with the corresponding matrices of the minimal realisation of Z, thus obtaining L in the quadruple $\{F, G, L, W_0\}$. Eqn. 17a may then be solved for P, since it can be shown to have a unique symmetric positive definite solution. The proof of the lemma concludes by showing that eqn. 17b is automatically satisfied. Details can be found in Reference 9.

If the minimal realisation $\{T_1^{-1}FT_1, T_1^{-1}G, T_1'H\}$ of $Z(s) - Z(\infty)$ is employed instead of $\{F, G, H\}$, a different P and L will be required to satisfy the equations corresponding to eqn. 17. The new P and L in terms of the old P and L may be readily verified to be $T_1'PT_1$ and $T'L_1$. In other words, as a consequence of eqn. 17, there results

$$(T_1'PT_1)(T_1^{-1}FT_1) + (T^{-1}FT_1)'(T_1'PT_1) = (T_1'L)(T_1'L)' \quad (19a)$$

$$(T_1'PT_1)(T_1^{-1}G) = (T_1'H) - (T_1'L)W_0 \quad . \quad . \quad . \quad . \quad (19b)$$

$$W_0'W_0 = J + J' \quad . \quad (19c)$$

With these preliminaries, we can turn to the actual synthesis concepts.

5 Synthesis procedure

We recall (eqns. 15 and 16), that, if $\{F, G, H, J\}$ is a minimal realisation of $Z(s)$, the problem of finding a passive structure synthetising $Z(s)$ reduces to finding a T, such that

$$M = \begin{bmatrix} J & -H'T \\ T^{-1}G & -T^{-1}FT \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

has its symmetric part positive semidefinite,

$$M + M' \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The positive real lemma sets out conditions satisfied by F, G, H and J for $Z(s)$ to be positive real. In particular, the lemma guarantees the existence of a symmetric positive definite matrix P satisfying eqn. 17. For such a matrix, one may define a square root $P^{1/2}$ which is also symmetric and positive definite (Reference 41, p. 76).

Theorem

If $T = P^{-1/2}$, eqn. 15 is satisfied.

Proof

By direct calculation,

$$M + M' = \begin{bmatrix} J + J' & G'P^{1/2} - H'P^{-1/2} \\ P^{1/2}G - P^{-1/2}H & -P^{1/2}FP^{-1/2} - P^{-1/2}F'P^{1/2} \end{bmatrix} \quad . \quad . \quad . \quad . \quad (20)$$

From eqn. 17, one obtains

$$P^{1/2}FP^{-1/2} + P^{-1/2}F'P^{1/2} = -P^{-1/2}LL'P^{-1/2} \quad (21a)$$

and $$P^{1/2}G = P^{-1/2}H - P^{-1/2}LW_0 \quad . \quad . \quad . \quad . \quad . \quad (21b)$$

Using these relationships in eqn. 20, {recall that $r = \text{rank}\,[Z(s) + Z'(-s)]$}

$$M + M' = \begin{bmatrix} W_0'W_0 & (-P^{-1/2}LW_0)' \\ -P^{-1/2}LW_0 & P^{-1/2}LL'P^{-1/2} \end{bmatrix}$$
$$= \begin{bmatrix} W_0' & 0 \\ 0 & -P^{-1/2}L \end{bmatrix} \begin{bmatrix} I_r & I_r \\ I_r & I_r \end{bmatrix} \begin{bmatrix} W_0 & 0 \\ 0 & -L'P^{-1/2} \end{bmatrix} \quad . \quad . \quad . \quad . \quad (22)$$

The latter equality may be verified by direct calculation. From eqn. 22, it is evident that

$$M + M' \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

since the right-hand side of eqn. 22 is of the form $A'BA$, where B is nonnegative definite. This proves the theorem.

Having shown that M is the impedance of a passive network, the question arises as to how to synthetise M. This is discussed, for example, in References 19, p. 156 and 25, p. 261. We use the fact that $2M = M + M' + M - M'$, and the fact that $M + M'$ and $M - M'$ are both positive real impedances (the first because $M + M' \geqslant 0$; the second because it is skew). Then it can be seen that a synthesis of M is obtained by series-connecting transformer-coupled resistors [correspondint to $(M + M')/2$] and transformer-coupled gyrators [corresponding to $(M - M')/2$].

By way of an example, we consider in detail the synthesis of $(M + M')/2$ and show that it uses r resistors. The synthesis of $(M - M')/2$ will use no resistors, and thus we shall be able to conclude that $Z(s)$ can be synthetised with r resistors. Since r is the normal rank of the resistivity matrix $Z'(-s) + Z(s)$, this means we have achieved a synthesis of $Z(s)$ using the minimum number of resistors (References 19, p. 132, and 22, p. 305), as well as a synthesis using the minimum number of reactive elements.

From eqn. 22, it follows, as may be checked by direct multiplication, that

$$\tfrac{1}{2}(M + M') = \tfrac{1}{2}\begin{bmatrix} W_0' \\ -P^{-1/2}L \end{bmatrix} I_r [W_0 \quad -L'P^{-1/2}] \quad . \quad . \quad (23)$$

This equation says that $(M + M')/2$ may be synthetised by terminating a multiport transformer of turns ratio $[W_0 - L'P^{-1/2}]/\sqrt{2}$ in r unit resistors (Reference 25, p. 256).

The procedure for synthetising an arbitrary positive real impedance can now be stated:

(*a*) Separate out the pole at infinity (if any), corresponding to a series extraction of transformer-coupled inductors. The remaining positive-real $Z(s)$ has $Z(\infty)$ finite.

(*b*) (Actually optional.) Separate out poles on the $j\omega$ axis, corresponding to a series extraction of tuned circuits (also transformer-coupled in general). The effect of this is to leave a positive-real $Z(s)$ to be synthetised which is of lower degree than before performing this extraction. Further, this $Z(s)$ has strictly left-hand halfplane poles.

(*c*) Find the four matrices comprising any minimal realisation $\{F, G, H, J\}$ for the impedance $Z(s)$ which remains to be synthetised, using any of the techniques outlined, for instance, in References 29, 30, 34 or 35.

(*d*) Find $W(s)$, using Reference 40, such that $Z(s) + Z'(-s) = W'(-s)W(s)$ with $W(s)$ and $W^{-1}(s)$ analytic in the right-hand halfplane. The rank of $W(s)$ in the right-half plane is equal to the normal rank of $Z(s) + Z'(-s)$.

(*e*) Find a realisation of W of the form $\{F, G, L, W_0\}$ which will be minimal if step (*b*) has been carried out. Thus L is determined.

(*f*) Calculate P as the unique solution of the equation $PF + F'P = -LL'$. This matrix equation can be regarded as $p(p + 1)/2$ linear simultaneous equations for the elements of P, p being the order of F or, what is the same thing, $p = \delta[Z(s)]$. Alternatively, P may be found from

$$P = \int_0^\infty \exp(F't)LL'\exp(Ft)dt$$

(*g*) Using this P, form a new minimal realisation of Z given by $\{P^{1/2}FP^{-1/2}, P^{1/2}G, P^{-1/2}H, J\}$.

(*h*) Synthetise the nonreactive (constant) positive real coupling impedance matrix

$$M = \begin{bmatrix} J & -H'P^{-1/2} \\ P^{1/2}G & -P^{1/2}FP^{-1/2} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

by a series connection of a transformer–resistor network and a transformer–gyrator network, both of $n + \delta[Z]$ ports.

(*i*) Terminate the last $p = \delta[Z(s)]$ ports of this network in unit inductors to obtain a synthesis of $Z(s)$.

Examples of this procedure will be given in Section 7, for which Section 6 is not a prerequisite.

6 Reciprocal *RL* synthesis

In this Section we apply similar techniques to obtain passive reciprocal coupling networks for *RL* (transformer) circuits.

As a preliminary, consider the more general situation where capacitors, but no gyrators, are also present, as illustrated in Fig. 3*a*, where k_1 inductors and k_2 capacitors are assumed.

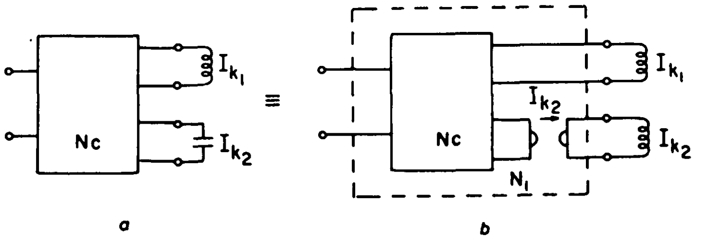


Fig. 3
Extractions for reciprocal networks

The resistive coupling network N_c is described by the symmetric impedance matrix

$$Z_c = Z'_c = \begin{array}{c} \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z'_{12} & z_{22} & z_{23} \\ z'_{13} & z'_{23} & z_{33} \end{bmatrix} \begin{matrix} \}n \\ \}k_1 \\ \}k_2 \end{matrix} \\ \begin{matrix} \;\;n & \;\;k_1 & \;\;k_2 \end{matrix} \end{array} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Here the matrix is partitioned so that the last k_2 rows and columns correspond to the capacitors. By connecting unit gyrators in cascade with each of these final k_2 ports, Fig. 3*a* is seen, from Fig. 1, to be equivalent to Fig. 3*b*; the resulting network N_1 is of the form considered earlier, and has (References 42, pp. 4 and 28)

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m'_{12} & m_{22} & m_{23} \\ -m'_{13} & -m'_{23} & m_{33} \end{bmatrix}, \; m_{ii} = m'_{ii}, \; i = 1, 2, 3 \quad (26a)$$

leading to

$$M = \begin{bmatrix} z_{11} - z_{13}z_{33}^{-1}z'_{13} & z_{12} - z_{13}z_{33}^{-1}z'_{23} & z_{13}z_{33}^{-1} \\ z'_{12} - z_{23}z_{33}^{-1}z'_{13} & z_{22} - z_{23}z_{33}^{-1}z'_{23} & z_{23}z_{33}^{-1} \\ -z_{33}^{-1}z'_{13} & -z_{33}^{-1}z'_{23} & z_{33}^{-1} \end{bmatrix} \quad (26b)$$

It is then important to note that $[I_{n+k_1} \dotplus (-I_{k_2})]M$ is symmetric, and that repeating the gyrator extraction on N_1 yields Z_c from M by equations identical to those (eqn. 26*b*), giving M in terms of Z_c. One also observes, since passivity is unaffected by a gyrator extraction, that Z_c of eqn. 25 will be positive real when (and only when) M is positive real.

One can synthetise N_c, given M of eqn. 26*a*, by synthetising Z_c through a (reciprocal) resistor–transformer network (Reference 25, pp. 256 and 261), at least when m_{33} is nonsingular. If m_{33} is singular, and a scattering matrix S_M exists for M (as when M is positive real), a reciprocal synthesis results through a gyrator extraction from the network N_1 which synthetises S_M. From these arguments, we conclude that a gyratorless minimal synthesis exists (when m_{33} is nonsingular or S_M exists) for a given M (as in eqn. 15) if, and only if, there exists a nonnegative integer k_2, a permutation matrix P_1 (corresponding to a relabelling of inductor–

capacitor ports) and a sign matrix $\Sigma = I_{k_1} \dotplus (-I_{k_2})$; such that $[I_n \dotplus \Sigma][I_n \dotplus P_1]M[I_n \dotplus P'_1]$ is symmetric. It is convenient to call such an M reciprocal, even though M itself is not symmetric.

At this point we apply some of the ideas developed for scattering matrices by Youla and Tissi,[10] referring to their work for omitted proofs. Thus, consider any minimal realisation, $\hat{M}$ of a symmetric $Z(s)$; then there exists a symmetric T, such that (Reference 10, p. 9)

$$\hat{M} = (I_n \dotplus T^{-1})\hat{M}(I_n \dotplus T) \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Since T is symmetric, it can be diagonalised to $+1$s and -1s via a congruency transformation (Reference 41, p. 56):

$$T = T_0 \Sigma T'_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28a)$$

$$\Sigma = I_{k_1} \dotplus (-I_{k_2}),\ p = k_1 + k_2 \quad . \quad . \quad . \quad . \quad (28b)$$

from which we can form (for eqn. 10)

$$M = (I_n \dotplus T_0{}^{-1})\hat{M}(I_n \dotplus T_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

On substituting eqn. 27 into eqn. 29, we find that $[I_n \dotplus \Sigma]M$ is symmetric; furthermore, k_1 and k_2 are unique (Reference 10, p. 7). Thus, when $Z(s)$ is symmetric, there exists a reciprocal M from which a reciprocal synthesis results, at least when m_{33} is nonsingular or S_M exists (certainly when M is positive real). Unfortunately, there seems to be no guarantee that M is positive real. Nevertheless, every other reciprocal M_R results from M of eqn. 29 by

$$M_R = (I_n \dotplus T_R{}^{-1})M(I_n \dotplus T_R) \quad . \quad . \quad . \quad . \quad . \quad (30)$$

with T_R satisfying (Reference 10, p. 7, lemma 6)

$$\Sigma = T_R \Sigma T'_R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

In the RL case, since $k_2 = 0$ and $\Sigma = I_p$, we require T of eqn. 28 to be positive definite and T_R of eqn. 3 to be orthogonal.

Finally, consider a given symmetric positive real $Z(s)$ with $Z(\infty)$ finite, for which $x'Z(s)x$ satisfies the standard RL 1-port realisability conditions (Reference 18, p. 149) for all real $n$vectors x. By standard $n$port synthesis techniques (Reference 25, p. 270), a structure using transformers and passive resistors and inductors exists, using in fact the minimum number of inductors. By performing this synthesis in continued-fraction form, one can demonstrate the existence of an impedance matrix M,[43] of the positive real type under discussion. From this, or any other reciprocal M, all reciprocal M_R then result from eqn. 30 with T_R orthogonal, or

$$M_R = (I_n \dotplus T'_R)M(I_n \dotplus T_R) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Since such an M_R is positive real, with M being derived through a congruency transformation, we conclude that every minimal reciprocal M_R realising a positive real inductor–resistor $Z(s)$ must itself be positive real. This result is in agreement with a similar one based upon scattering-matrix arguments (Reference 10, p. 14). Of course, by duality, an identical result holds for RCnetworks.

7 Synthesis examples

In this Section, we present two moderately easy examples, different parts of the theory being highlighted by each.

Example 1

Synthesis of the (positive real) impedance

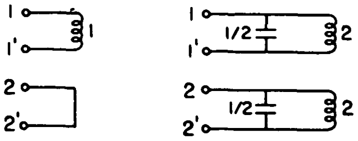
$$\hat{Z}(s) = \begin{bmatrix} s + \dfrac{2s}{s^2+1} + 2 & 4\dfrac{s-1}{s+1} \\ 0 & \dfrac{s^2+1}{2s} + 2 \end{bmatrix} \quad . \quad . \quad (33)$$

The first step is to separate out the term corresponding to the pole at infinity, and then to carry out the (optional) step of removing $j\omega$axis pole terms. Thus

$$\hat{Z}(s) = \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} + \frac{s}{s^2+1}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4\dfrac{s-1}{s+1} \\ & 2 \end{bmatrix} \quad . \quad (34)$$

The first two terms are readily synthetised (see Figs. 4a and b for the separate syntheses). Thus we now consider the positive real

$$Z(s) = \begin{bmatrix} 2 & 4\dfrac{s-1}{s+1} \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{8}{s+1} \\ & 0 \end{bmatrix} \quad . \quad (35)$$



a *b*

Fig. 4
Example 1: jω axis extractions

A minimal realisation for Z is given by

$$F = [-1] \qquad G = [0,\ 1]$$
$$H = [-8,\ 0] \qquad J = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

This may be derived by the techniques described in, for example, Reference 29, or may be found by inspection, since the F matrix is simple. Observe that

$$Z(s) = J + H'(sI - F)^{-1}G \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

is, naturally, satisfied.

We also compute, by inspection or by using Reference 40, that

$$Z(s) + Z'(-s) = 4\begin{bmatrix} 1 & \dfrac{s-1}{s+1} \\ \dfrac{s+1}{s-1} & 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ \dfrac{s+1}{s-1} \end{bmatrix} 2\begin{bmatrix} 1 & \dfrac{s-1}{s+1} \end{bmatrix} \quad . \quad . \quad . \quad . \quad (38)$$

Hence

$$W(s) = 2\begin{bmatrix} 1 & \dfrac{s-1}{s+1} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Furthermore, a realisation for $W(s)$ is given by using F and G as for $Z(s)$, and

$$L = [-4,\ 0] \quad W_0 = [2,\ 2] \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Note that, in the right-hand halfplane, W has strict rank equal to the normal rank of $Z(s) + Z'(-s)$, i.e. unity, and a right inverse is $[1/2,\ 0]'$; W is, moreover, analytic in the right-hand halfplane.

The next step is to form P through

$$PF + F'P = -LL'$$

from which one readily determines that

$$P = [8] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and thus that

$$P^{1/2} = [2\sqrt{2}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Then, although

$$\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

is not positive real, it is true that

$$M = \begin{bmatrix} J & -H'P^{-1/2} \\ P^{1/2}G & -P^{1/2}FP^{-1/2} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2\sqrt{2} \\ 0 & 2 & 0 \\ 0 & 2\sqrt{2} & 1 \end{bmatrix} \quad (44)$$

is positive real. We note that

$$\frac{M + M'}{2} = \begin{bmatrix} 2 & 2 & \sqrt{2} \\ 2 & 2 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 1 \end{bmatrix}[1]\ [\sqrt{2},\ \sqrt{2},\ 1] \quad . \quad . \quad . \quad . \quad (45a)$$

and

$$\frac{M - M'}{2} = \begin{bmatrix} 0 & 2 & -\sqrt{2} \\ -2 & 0 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & \sqrt{2} & -1 \end{bmatrix} \qquad (45b)$$

The network N_1 of impedance matrix M thus has the synthesis of Fig. 5.

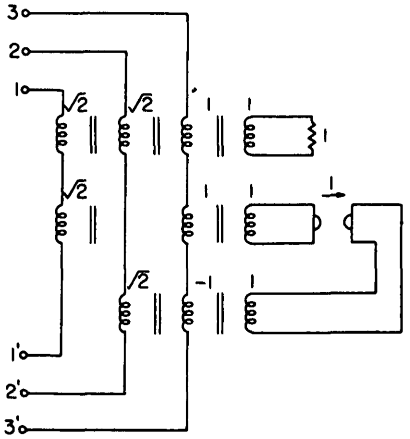


Fig. 5
Synthesis of M for example 1

The network synthetising $Z(s)$ of eqn. 35 is found by terminating port 3 of N_1 in a unit inductance, while the original $\hat{Z}(s)$ has the synthesis of Fig. 6, where the networks

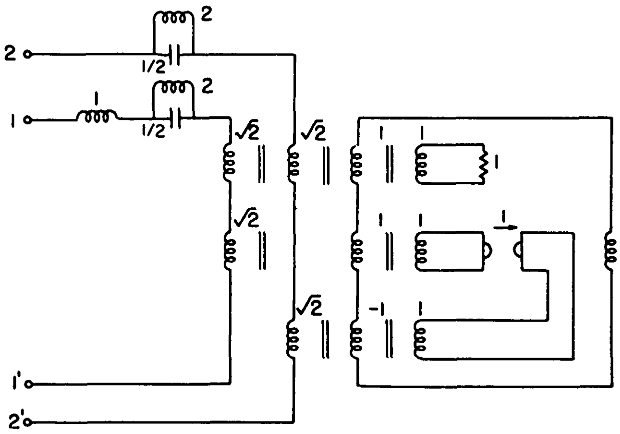


Fig. 6
Circuit synthetising $\hat{Z}$ of eqn. 32

shown in Fig. 4 have been included. It is interesting to compare the terminated N_1 with the similar result using two reactive elements obtained by the Bayard synthesis (Reference 23, p. 88).

Example 2

Synthesis of the (positive real) impedance

$$Z(s) = \frac{s^2 + 2s + 4}{s^2 + s + 1} \qquad (46)$$

Having no poles on the $j\omega$ axis or at infinity to remove, we write

$$Z(s) = 1 + \frac{s + 3}{s^2 + s + 1}$$

Transfer functions of the form

$$\frac{\sum_{i=0}^{n-1} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

with $a_n = 1$ have a convenient canonical minimal realisation (which does not extend in a straightforward way to the matrix situation). This is given by Reference 29:

$$F = \begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & \cdot & & \\ \cdot & & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \\ -a_0 & \cdot & \cdot & \cdot & \cdot & -a_{n-1} \end{bmatrix},$$

$$G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, H = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \qquad (47)$$

Thus, for the $Z(s)$ under consideration, we have

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} H = \begin{bmatrix} 3 \\ 1 \end{bmatrix} J = [1] \qquad (48)$$

Direct calculation yields

$$Z(s) + Z'(-s) = 2\frac{(s^2 + s + 2)(s^2 - s + 2)}{(s^2 + s + 1)(s^2 - s + 1)} \qquad (49)$$

and then we take

$$W(s) = \sqrt{2}\frac{s^2 + s + 2}{s^2 + s + 1} = \sqrt{2} + \frac{\sqrt{2}}{s^2 + s + 1} \qquad (50)$$

A minimal realisation for W is then

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} L = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} W_0 = [\sqrt{2}]$$

Forming the equation

$$PF + F'P = -LL'$$

we obtain

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad (51)$$

which has

$$P^{1/2} = \begin{bmatrix} \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \qquad (52)$$

The network N_1 has the positive real impedance

$$M = \begin{bmatrix} J & -H'P^{-1/2} \\ P^{1/2}G & -P^{1/2}FP^{-1/2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{7}{5} \\ \frac{2}{\sqrt{5}} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} \qquad (53)$$

We have then

$$\frac{M + M'}{2} = \begin{bmatrix} 1 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} \\ \frac{1}{\sqrt{5}} & -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \qquad (54a)$$

$$= \begin{bmatrix} 1 \\ -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} [1] \begin{bmatrix} 1 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \qquad (54b)$$

while

$$\frac{M - M'}{2} = \begin{bmatrix} 0 & -\frac{3}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ +\frac{3}{\sqrt{5}} & 0 & -1 \\ +\frac{1}{\sqrt{5}} & +1 & 0 \end{bmatrix} \quad . \quad . \quad . \quad . \quad (55a)$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{5}} & -\frac{3}{\sqrt{5}} \\ -1 & 0 \\ 0, & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & -1 & 0 \\ -\frac{3}{\sqrt{5}} & 0 & 1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad (55b)$$

Fig. 7 shows a synthesis for the nonreactive network N_1 derived by series-connecting networks of impedance matrices

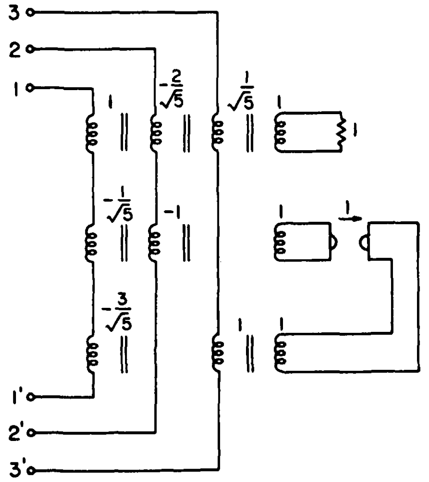


Fig. 7
Synthesis of coupling M for example 2

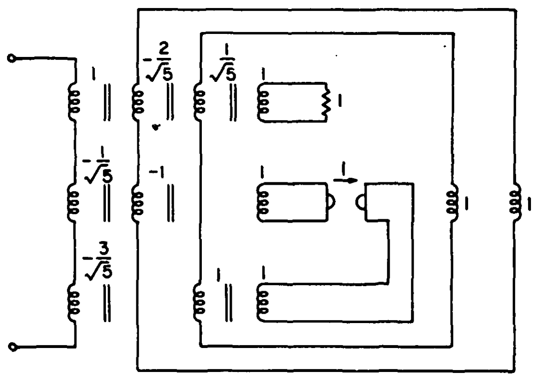


Fig. 8
Final circuit to yield Z of eqn. 45

$(M + M')/2$ and $(M - M')/2$. Terminating the final two ports in unit inductors yields Fig. 8 for the complete synthesis of the original $Z(s)$ of eqn. 46.

Observe that one of the penalties of obtaining a synthesis using simultaneously the minimum number of reactances and resistances is the presence of a gyrator in the realisation of the positive real function $Z(s)$. However, by extracting the resistor and transforming the resulting lossless structure (after adjoining another port for further resistive termination), the somewhat complicated procedures of Oono and Yasuura (Reference 19, pp. 149–153 and 168) yield a gyratorless circuit with the minimum possible number of elements.

8 Conclusions

The material presented highlights the strong interrelation between networks systems and control theory in an elegant manner. One of the classical problems of network theory has been solved by an investigation in terms of the state, using not especially advanced control-theory concepts.

An interesting and important feature of the synthesis is that it is primarily algebraic in character, rather than analytic, as, for example, the Brune synthesis. This is quite proper, for the synthesis problem is evidently in some sense a finite-dimensional one, and thus *a priori* more reasonably attacked by algebra than analysis.

The key point of the synthesis is the translation of the analytical concept of positive reality into algebraic properties of the matrices of a minimal realisation of $Z(s)$. From this point on, the development of the synthesis becomes algebraic.

There are still a number of open problems, however. The present theory must certainly be regarded as incomplete when the synthesis of positive real functions leads to a network containing gyrators. In Section 6, we have attempted to outline some of the difficulties which arise when a reciprocal or, by extension, a minimal-gyrator synthesis is sought. Very possibly, satisfactory results will be achieved by using the algebraic characterisation of reciprocity in Reference 10. Since, however, reciprocal synthesis may often have to use more than the minimum number of resistors (Reference 19, p. 148), further investigations of the effect of positive reality and reciprocity on realisations is in order.

Another pertinent problem is the development of a scattering-matrix synthesis procedure, which uses, in a simple manner, some hitherto unestablished property of minimal realisations of scattering matrices.[11] A very positive step has been made in this direction in Reference 10; Reference 31 discusses the statement of the network problem in control-system terms. Nevertheless, the method given here allows the synthesis of any rational bounded real scattering matrix $S(s)$, since one can form the positive real impedance matrix $Z = 2(I_n - S)^{-1} - I_n$ if $I_n - S$ is nonsingular. If $I_n - S$ is singular of rank ρ, one forms $T_0 S T_0' = S_0 \dotplus I_{n-\rho}$ with T_0 a constant orthogonal matrix (Reference 19, p. 155) (representing transformers) with $I_\rho - S_0$ nonsingular. This yields a realisation through $Z_0 = 2(I_\rho - S_0)^{-1} - I_\rho$, which is a positive real impedance matrix.

The question naturally arises as to how to obtain all passive minimal realisations. From Section 2, we know that every minimal realisation results from applying the transformation of eqn. 5 to a fixed one. In particular, this procedure yields all passive minimal realisations. Nevertheless, except for the *RL* (or *RC*) case treated in Section 6, the restrictions on the transformation T needed to retain passivity cannot as yet be specifically stated.

In a different, but somewhat related, manner, one can obtain all nonminimal realisations by the use of a previous theory.[31]

Some remarks are in order on the computation difficulties of the synthesis described. The major problem is to determine $W(s)$ from $Z(s) + Z'(-s)$. Certainly Reference 40 outlines the procedure, but the actual calculations are long and are considered by Youla to be somewhat inappropriate for programming. The other calculations required in the synthesis are refreshingly easy, and in the 1-port case lead to a fairly simple synthesis through use of the canonical minimal realisation described by eqn. 47. If a simple means of finding $W(s)$ analytic in the right-hand halfplane is found, the method holds excellent promise as a possible means of synthesis via the computer, since the method of finding a minimal realisation[30] could be so programmed. In fact, following ideas very similar to those of this paper, a convenient computer analysis of networks has been developed.[44]

In the field of integrated circuits, the material of this paper has some significant applications, since, on an admittance basis, minimal capacitor[45] and insensitive[46] synthesis techniques result.

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10 References

1 BASHKOW, T. R.: 'The A matrix, a new network description', *IRE Trans.*, 1957, **CT-4**, pp. 117–120

2 BYRANT, P. R.: 'The explicit form of Bashkow's A matrix', *ibid.*, 1962, **CT-9**, pp. 303–306

3 KUH, E. S.: 'Stability of linear time-varying networks—the state space approach', *ibid.*, 1965, **CT-12**, pp. 150–157

4 KUH, E. S., and ROHRER, R. A.: 'The state-variable approach to network analysis', *Proc. Inst. Elect. Electronics Engrs.*, 1965, **53**, pp. 672–686

5 HILLER, J.: 'The solution of two synthesis problems', University of New South Wales, School of Electrical Engineering Research Report, 1964

6 SILVERMAN, L. M.: 'Representation and realization of time-variable linear systems', Columbia University, Department of Electrical Engineering Technical Report 94, 1966

7 LAYTON, D. M.: 'Equivalent representations for linear systems, with application to N-port network synthesis', Ph.D. dissertation, University of California, 1966

8 KALMAN, R. E.: 'On a new characterization of linear passive systems', Proceedings of the first Allerton conference on circuit and system theory, University of Illinois, 1963, pp. 456–470

9 ANDERSON, B. D. O.: 'System theory criterion for positive real matrices', *SIAM J. Control*, to be published

10 YOULA, D. C., and TISSI, P.: 'N-port synthesis via reactance extraction—Part 1', Electrophysics Memorandum PIBMRI-1309-66, Polytechnic Institute of Brooklyn, 1966

11 ANDERSON, B. D. O.: 'Algebraic description of bounded real matrixes', *Electronics Letters*, 1966, **2**, pp. 464–465

12 CAUER, W.: 'Die Verwirklichung von Wechselstromwiderstanden vorgeschriebener Frequenzabhaungigkeit', *Arch. Elektrotech.*, 1926, **17**, pp. 355–388

13 BRUNE, O.: 'Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency', *J. Math. Phys.*, 1931, **10**, pp. 191–236

14 DARLINGTON, S.: 'Synthesis of reactance four-poles which produce prescribed insertion loss characteristics', *ibid.*, 1939, **18**, pp. 257–353

15 BOTT, R., and DUFFIN, R. J.: 'Impedance synthesis without use of transformers', *J. Appl. Phys.*, 1949, **20**, p. 816

16 MCMILLAN, B.: 'Introduction to formal realizability theory', *Bell Syst. Tech. J.*, 1952, **31**, pp. 217–279 and 1952, **31**, pp. 541–600

17 NEWCOMB, R. W.: 'On network realizability conditions', *Proc. Inst. Radio Engrs.*, 1962, **50**, p. 1995

18 VAN VALKENBURG, M. E.: 'Introduction to modern network synthesis' (Wiley, 1960)

19 OONO, Y., and YASUURA, K.: 'Synthesis of finite passive $2n$-terminal networks with prescribed scattering matrices', *Mem. Fac. Engng. Kyushu Univ.*, 1954, **14**, pp. 125–177

20 BAYARD, M.: 'Résolution du problème de la synthèse des réseaux de Kirchhoff par la détermination de réseaux purement reactifs', *Câbles et Transm.*, 1950, **4**, pp. 281–296

21 BELEVITCH, V.: 'Factorization of scattering matrices with applications to passive networks synthesis', *Philips Res. Rep.*, 1963, **18**, pp. 275–317

22 BELEVITCH, V.: 'Synthèse des reseaux électrique passifs à n paires de bornes de matrice de répartition prédeterminée', *Ann. Télécomm.*, 1951, **6**, pp. 302–312

23 NEWCOMB, R. W.: 'A Bayard-type nonreciprocal n-port synthesis', *IEEE Trans.*, 1963, **CT-10**, pp. 85–90

24 BAYARD, M.: 'Théorié des réseaux de Kirchhoff' (Editions de la Revue d'Optique, 1954)

25 HAZONY, D.: 'Elements of network synthesis' (Reinhold, 1963)

26 NEWCOMB, R. W.: 'Linear multiport synthesis' (McGraw-Hill, 1966)

27 BELEVITCH, V.: 'Classical network theory' (Holden-Day, 1968)

28 YARLAGADDA, R., and TOKAD, Y.: 'Synthesis of LC networks—a state-model approach', *Proc. IEE*, 1966, **113**, (6), pp. 975–981

29 KALMAN, R. E.: 'Mathematical description of linear dynamical systems', *J. SIAM Control*, [A], 1963, **1**, pp. 152–192

30 HO, B. L.: 'On effective construction of realizations from input-output descriptions', Ph.D. dissertation, Stanford University, 1966

31 ANDERSON, B. D. O., NEWCOMB, R. W., KALMAN, R. E., and YOULA, D. C.: 'On the equivalence of linear time-invariant dynamical systems', *J. Franklin Inst.*, 1966, **281**, pp. 371–378

32 TELLEGEN, B. D. H.: 'Synthesis of passive resistanceless four-poles that may violate the reciprocity condition', *Philips Res. Rep.*, 1948, **3**, pp. 321–337

33 DUFFIN, R. J., and HAZONY, D.: 'The degree of a rational matrix function', *SIAM J. Appl. Math.*, 1963, **11**, pp. 645–658

34 KALMAN, R. E.: 'Irreducible realizations and the degree of a matrix of rational-functions', *ibid.*, 1965, **13**, pp. 520–544

35 YOULA, D. C.: 'The synthesis of linear dynamical systems from prescribed weighting functions', *ibid.*, 1966, **14**, pp. 527–549

36 NEWCOMB, R. W.: 'A nonreciprocal n-port Brune synthesis', Report SEL-62-125 (TR 2254–5), Stanford Electronics Laboratories, 1962

37 CARLIN, H. J.: 'Synthesis of nonreciprocal networks', Proceedings of the Polytechnic Institute of Brooklyn symposium on modern network synthesis', 1955, **5**, pp. 11–44

38 ANDERSON, B. D. O., and NEWCOMB, R. W.: 'Lossless N-port synthesis via state-space techniques', Stanford Electronics Laboratories technical report 6558–8, 1967

39 KALMAN, R. E.: 'Lyapunov functions for the problems of lur'e in automatic control', *Proc. Nat. Acad. Sci. USA*, 1963, **49**, pp. 201–205

40 YOULA, D. C.: 'On the factorization of rational matrices', *IRE Trans.*, 1961, **IT-7**, pp. 172–189

41 MACDUFFEE, C. C.: 'The theory of matrices' (Springer, 1933)

42 DUFFIN, R. J., HAZONY, D., and MORRISON, N.: 'Network synthesis through hybrid matrices', *SIAM J. Appl. Math.*, 1966, **14**, pp. 390–413

43 RAO, T. N., NEWCOMB, R. W., and ANDERSON, B. D. O.: 'A cauer synthesis of RL n-ports', Stanford Electronics Laboratories technical report 6558–6, 1966

44 NEWCOMB, R. W., and MILLER, J. A.: 'Formulation of network state–space equations suitable for computer investigations', *Electronics Letters*, 1967, **3**, pp. 307–308

45 NEWCOMB, R. W., and ANDERSON, B. D. O.: 'State variable results for minimal capacitor integrated circuits', Stanford Electronics Laboratories technical report 6558–14, 1966

46 KERWIN, W. J., HUELSMAN, L. P., and NEWCOMB, R. W.: 'State-variable synthesis for insensitive integrated circuit transfer functions', Stanford Electronics Laboratories technical report 6560–10, 1966