CHANGE OF LOSSLESS NATURAL FREQUENCIES DUE TO CAPACITOR ADDITION*

It is shown that the addition of passive capacitors to a passive LC structure cannot increase the natural frequencies of the structure.

A recent look[1] at the Fujisawa conditions[2] has suggested that the natural frequencies of a passive LC structure should decrease when passive capacitors are added. Here, using a result of classical mechanics and vector spaces,[3] we show that this is actually the case.

In order to be somewhat precise, let us call a circuit constructed from only passive (linear and time-invariant) inductors, capacitors and transformers an LC structure or, if the inductors are absent, a C structure. For concreteness we will assume the existence of a nodal admittance matrix Y_{nn} (p. 19 of Reference 4), which, for an LC structure, takes the form

$$Y_{nn} = (\Gamma/p) + pC \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where Γ and C are positive semidefinite symmetric matrixes of order m for m node pairs. Paralleling two m node-pair networks is, of course, the operation of connecting corresponding node pairs, and then the two nodal admittance matrixes add.

As will be described in the theorem's proof, the natural frequencies can be determined from Y_{nn} and are all imaginary ($p_i = j\omega_i$) for an LC structure. If there are n positive (including infinite) ω_i, we order these by

$$\omega_n \geqslant \omega_{n-1} \geqslant \ldots \geqslant \omega_2 \geqslant \omega_1 > 0 \quad . \quad . \quad . \quad . \quad (2)$$

For convenience we will call the set $\{\omega_i\}$ of ω_i of expr. 2 the ordered natural frequencies. The general natural-frequency theorem in question can then be stated as follows:

Theorem: Let an LC structure S' of ordered natural frequencies $\{\omega_i'\}$ be constructed by paralleling any LC structure S, of order natural frequencies $\{\omega_i\}$, with a C structure. Then

$$\omega_i \geqslant \omega_i'; \; i = 1, \ldots n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Proof: When Γ is positive definite, the fact that the ordered natural frequencies do not increase (expr. 3) is a consequence of a known result on quadratic forms.[3] If by ordered eigenvalues $\{\lambda_i\}$ of a positive semidefinite matrix A we mean that the eigenvalues λ_i of A satisfy expr. 2 with $\lambda_1 = 0$ allowed, the known result of interest is that the ordered eigenvalues $\{\lambda_i'\}$ for $A + A_0$ satisfy $\lambda_i' \geqslant \lambda_i$ when A_0 is positive semidefinite.[3]

Assuming that all structures have the same number m of node pairs, which can be obtained by adding open circuits if they do not, let C_0 be the capacitance matrix for the added C structure. Then for S and S', respectively,

$$Y_{nn} = (\Gamma/p) + pC \quad Y_{nn}' = (\Gamma/p) + p(C + C_0) \quad (4a)$$

Since Γ is positive semidefinite, there exists a real nonsingular matrix T, such that[5]

$$\tilde{T}\Gamma T = 1_\gamma \dot{+} 0_{m-\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

where γ is the rank of Γ, 1_γ is the identity matrix of order γ, $0_{m-\gamma}$ is the zero matrix of order $m - \gamma$, $\dot{+}$ denotes the direct sum, and $\sim$ denotes the transpose. Applying T to C and to $C' = C + C_0$ yields two new capacitance matrixes of interest:

$$C_t = \tilde{T}CT; \; C_t' = \tilde{T}C'T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4c)$$

First consider nonsingular Γ, i.e. $\gamma = m$. Then the natural frequencies of S are zeros of (p. 161 of Reference 4)

$$(\det T)^2 \det Y_{nn} = \det\{(1_\gamma + p^2 C_t)/p\} \quad . \quad . \quad . \quad (4d)$$

Consequently, the squares of the ordered natural frequencies are the reciprocals of the ordered eigenvalues of C_t, $\omega_i^2 = 1/\lambda_{n+1-i}$. The above mentioned known result applied to C_t and $C_t' = C_t + \tilde{T}C_0T$ then shows that $\lambda_i' \geqslant \lambda_i$, which is expr. 3.

When $\gamma \neq m$ in eqn. $4b$, we can work with the node admittance $\tilde{T}Y_{nn}T$, whose structure results from applying transformers to that for Y_{nn} (p. 307 of Reference 6). By replacing $\tilde{T}CT$ by its Norton equivalent seen at the first γ node pairs, and similarly for $\tilde{T}C_0T$, we can omit the final $m - \gamma$ node pairs from consideration (they add only natural frequencies at infinity). The singular Γ case is then reverted to the nonsingular one. The existence of the Norton equivalents derived from C_t and C_t' is almost physically obvious, but can be rigorously proved through simultaneous diagonalisation techniques.[7]

A simple consequence of the theorem, due to Lee,[1] is the following:

Corollary: If S is any LC structure and n_1 and n_2 are two nodes of S, connection of a passive capacitance between n_1 and n_2 tends to reduce all natural frequencies of S.

One of the most appealing applications of the result is the use of the corollary in proving the Fujisawa conditions.[1] However, consider a lossless 2-port designed for a resistive termination. If the termination has some inherent shunt capacitance, as is practically the case, then, by the theorem, the natural frequencies of the 'apparent' lossless 2-port (i.e. including the load capacitance) will be lower than those of the 'designed' 2-port.

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COUPLING IMPEDANCE FOR A RIPPLED ELECTRON BEAM IN A CYLINDRICAL GUIDE CARRYING THE E_{01} MODE

A hollow beam of electrons, injected into an axial magnetic field from a field-free region, has a diameter which varies periodically at the cyclotron frequency. An expression is derived for the coupling impedance between such a rippled beam and the fast E_{01} mode in a cylindrical waveguide.

In connection with work on a periodic-beam travelling-wave amplifier,[1] an attempt has been made to derive an expression for the coupling impedance of a thin hollow beam rippling at the cyclotron frequency in a cylindrical guide carrying the E_{01} mode. Here we are concerned with the electric field in the Z direction and with the periodic motion of the electrons in this field. In the following calculations space-charge effects have been ignored.

The radius r of the path of an electron injected from a magnetic-field-free region varies sinusoidally with distance Z:

$$r = r_0\{1 - \sin(\beta_h Z)\}$$

where $2r_0$ is the radial position of the electron at entry into the magnetic field and β_h is the cyclotron-frequency propagation coefficient:

$$\beta_h = \omega_h/u_0$$