Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-12 (2015) 203–208

Auxiliary Function-based Summation Inequalities for Quadratic Functions and their Application to Discrete-time Delay Systems ⋆

Won Il Lee * **PooGyeon Park** * **Seok Young Lee** ** **Robert W. Newcomb** ***

* *Department of Electronic and Electrical Engineering, Pohang University of Science and Technology, Pohang, Republic of Korea (e-mail: {wilee,ppg}@postech.ac.kr).*
** *Division of IT Convergence Engineering, Pohang University of Science and Technology, Pohang, Republic of Korea (e-mail: suk122@postech.ac.kr)*
*** *Department of Electrical and Computer Engineering, University of Maryland, College Park, MD 20742, USA (e-mail: newcomb@eng.umd.edu)*

Abstract: Jensen inequality has become a powerful tool of supporting summation inequalities for quadratic functions in order to obtain stability criteria for time-delayed systems since it achieves remarkable performance with a small number of decision variables. This paper suggests a new summation inequality for quadratic functions based on an auxiliary function, which is superior to the Jensen inequality. To demonstrate the superiority of the new inequality, its application to stability analysis for discrete-time delay system is provided with a simple numerical example.

Keywords: Summation inequality, Discrete-time delay systems, Stability analysis

1. INTRODUCTION

Time delays are easily encountered in many dynamic systems, and such time delays often cause poor performance or even system instabilities Richard (2003). Thus, stability analysis of discrete-time delay systems has become an important issue and considerable research efforts have been devoted in this field.
Most recent stability analyses for discrete-time delay systems have been based on the Jensen inequality Liu and Zhang (2012); Ramakrishnan and Ray (2013); Kwon et al. (2013); Xu et al. (2013) because such approaches require fewer decision variables than approaches based on the inequality developed by Moon et al. (2001); Kim (2012) or free-weighting matrix method Gao and Chen (2007); Zhang et al. (2008); Ma et al. (2010) while achieving identical or comparable performance behavior. Recently, however, there has been an effort to analyze the conservatism of the Jensen inequality itself Briat (2011). In addition, for continuous-time systems, there have been some studies for developing an alternative inequality reducing the gap of the Jensen inequality Seuret and Gouaisbaut (2013). However, to the author's best knowledge, there are no such efforts for discrete-time systems yet.
Motivated by above observation, in this paper, we investigate to develop a new summation inequality for quadratic functions to substitute Jensen inequality in the framework of discrete-time systems. By introducing some auxiliary functions, a new summation inequality for reducing the conservatism of the Jensen inequality is developed. With an appropriate choice of the auxiliary function, the proposed summation inequality becomes easily applicable to the stability analysis for discrete-time delay systems, which yields competitive results with those of conventional approaches containing Jensen inequality.

2. PRELIMINARIES

2.1 Jensen inequality

Lemma 2.1. (Gu et al., 2003) For a positive definite matrix $R > 0$ and a vector function $\{w_i \mid i \in [a, a+n-1]\}$, the following inequality holds:

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \left(\sum_{i=a}^{a+n-1} w_i \right)^T R \left(\sum_{i=a}^{a+n-1} w_i \right). \quad (1)$$

2.2 Lower bound lemma for reciprocal convexity

Lemma 2.2. (Park et al., 2011) Let $f_1, f_2, \ldots, f_N : \mathbf{R}^m \mapsto \mathbf{R}$ have positive values in an open subset $\mathbf{D}$ of $\mathbf{R}^m$. Then,

⋆ This research was supported by the MSIP(Ministry of Science, ICT and Future Planning), Korea, under the IT Consilience Creative Program (NIPA-2014-H0201-14-1001) supervised by the NIPA(National IT Industry Promotion Agency). This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2014R1A1A2055122).

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.09.378

the reciprocally convex combination of f_i over $\mathbf{D}$ satisfies

$$\min_{\{\alpha_i \mid \alpha_i>0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i \neq j} g_{i,j}(t)$$

subject to

$$\left\{ g_{i,j} : \mathbf{R}^m \mapsto \mathbf{R}, g_{j,i}(t) = g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \geq 0 \right\}.$$

Lemma 2.1 will be referred for comparison with a new summation inequality and Lemma 2.2 will be applied in deriving a stability condition in Section 3.

3. MAIN RESULTS

In this section, we provide new summation inequalities for quadratic functions via auxiliary functions and their application to the stability analysis for discrete-time delay systems.

3.1 Auxiliary function-based summation inequalities

Theorem 1. For a positive definite matrix $R > 0$, a vector function $\{w_i \mid i \in [a, a+n-1]\}$, and an auxiliary scalar function $\{\bar{p}_i \mid i \in [a, a+n-1]\}$, the following inequality holds:

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \left(\sum_{i=a}^{a+n-1} w_i \right)^T R \left(\sum_{i=a}^{a+n-1} w_i \right) + \left(\sum_{i=a}^{a+n-1} \bar{p}_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right)^T R \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right), \quad (2)$$

where

$$\sum_{i=a}^{a+n-1} \bar{p}_i = 0. \quad (3)$$

Proof. For any scalar function $\{p_i \mid i \in [a, a+n-1]\}$ and constant vector v, let us define $\{z_i \mid i \in [a, a+n-1]\}$ such as

$$z_i \triangleq w_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} w_j \right) - p_i v,$$

which satisfies

$$\begin{aligned} 0 &\leq \sum_{i=a}^{a+n-1} z_i^T R z_i \\ &= \sum_{i=a}^{a+n-1} \left\{ w_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} w_j \right) \right\}^T R \\ &\quad \times \left\{ w_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} w_j \right) \right\} + \left(\sum_{i=a}^{a+n-1} p_i^2 \right) v^T R v \\ &\quad -2 v^T R \left\{ \sum_{i=a}^{a+n-1} \left(p_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} p_j \right) \right) w_i \right\} \\ &= \sum_{i=a}^{a+n-1} w_i^T R w_i - \frac{1}{n} \left(\sum_{i=a}^{a+n-1} w_i \right)^T R \left(\sum_{i=a}^{a+n-1} w_i \right) \\ &\quad + \left(\sum_{i=a}^{a+n-1} p_i^2 \right) [v - \bar{v}]^T R [v - \bar{v}] \\ &\quad - \left(\sum_{i=a}^{a+n-1} p_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right)^T R \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right), \end{aligned}$$

where

$$\bar{p}_i = p_i - \frac{1}{n} \sum_{j=a}^{a+n-1} p_j,$$

$$\bar{v} = \left(\sum_{i=a}^{a+n-1} p_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right).$$

Let us choose $v = \bar{v}$, which yields

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \left(\sum_{i=a}^{a+n-1} w_i \right)^T R \left(\sum_{i=a}^{a+n-1} w_i \right) + \left(\sum_{i=a}^{a+n-1} p_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right)^T R \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right). \quad (4)$$

Since it holds that

$$\sum_{i=a}^{a+n-1} \bar{p}_i^2 = \sum_{i=a}^{a+n-1} p_i^2 - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} p_j \right)^2 \leq \sum_{i=a}^{a+n-1} p_i^2, \quad (5)$$

the choice of p_i as $\bar{p}_i$ will produce the tightest lower bound of (4) as the form of (2). Then, the resulting $\bar{p}_i$ should yield the kind of zero mean condition (3), which completes the proof. ∎

Remark 1. Since the last term in (2) is a positive quantity from the condition $R > 0$, the proposed inequality (2) is much tighter than Jensen inequality (1). Therefore, Theorem 1 can be applied to the stability analysis of discrete-time delay systems for reducing the conservatism of Jensen inequality.

Remark 2. For the application of Theorem 1, the design of auxiliary function $\bar{p}_i$ is essential. Let us determine α in the form of an auxiliary function $\bar{p}_i = (i-a+1)-\alpha$. From the condition (3),

$$\sum_{i=a}^{a+n-1} \{(i-a+1) - \alpha\} = \frac{n(n+1)}{2} - n\alpha = 0,$$

which implies that $\alpha = (n+1)/2$. Then we have that

$$\begin{aligned} \sum_{i=a}^{a+n-1} \bar{p}_i^2 &= \sum_{i=a}^{a+n-1} \left\{ (i-a+1) - \frac{n+1}{2} \right\}^2 \\ &= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{2} + \frac{n(n+1)^2}{4} \end{aligned}$$

$$= \frac{2n(n+1)(2n+1)-6n(n+1)^2+3n(n+1)^2}{12}$$
$$= \frac{n^3-n}{12}$$
$$= \frac{n(n+1)(n-1)}{12}, \quad (6)$$

$$\sum_{i=a}^{a+n-1} \bar{p}_i w_i = \sum_{i=a}^{a+n-1} \left\{ (i-a+1) - \frac{n+1}{2} \right\} w_i$$
$$= \sum_{i=a}^{a+n-1} \sum_{j=a}^{i} w_i - \frac{n+1}{2} \sum_{i=a}^{a+n-1} w_i$$
$$= \sum_{j=a}^{a+n-1} \sum_{i=j}^{a+n-1} w_i - \frac{n+1}{2} \sum_{i=a}^{a+n-1} w_i$$
$$= \sum_{i=a}^{a+n-1} \sum_{j=i}^{a+n-1} w_j - \frac{n+1}{2} \sum_{i=a}^{a+n-1} w_i. \quad (7)$$

By the results of (6)-(7), Theorem 1 can be rewritten as the following corollary.

Corollary 3.1. For a positive definite matrix $R > 0$ and a vector function $\{w_i \mid i \in [a, a+n-1]\}$, the following inequality holds:

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \left\{ \sum_{i=a}^{a+n-1} w_i \right\}^T R \left\{ \sum_{i=a}^{a+n-1} w_i \right\}$$
$$+ \frac{3(n+1)}{n(n-1)} \left\{ \sum_{i=a}^{a+n-1} w_i - \frac{2}{n+1} \sum_{i=a}^{a+n-1} \sum_{j=i}^{a+n-1} w_j \right\}^T R$$
$$\times \left\{ \sum_{i=a}^{a+n-1} w_i - \frac{2}{n+1} \sum_{i=a}^{a+n-1} \sum_{j=i}^{a+n-1} w_j \right\}. \quad (8)$$

Remark 3. In the field of stability analysis for discrete-time delay systems, summations of quadratic function appear in the forward difference of Lyapunov-Krasovskii functional as the form of

$$\sum_{i=a}^{a+n-1} w_i^T R w_i, \qquad w_i \triangleq x_{i+1} - x_i.$$

In this case, for directly applying the inequality proposed, Corollary 3.1 should be slightly modified as follows.

Corollary 3.2. For a positive definite matrix $R > 0$ and vector functions $\{w_i, x_i \mid i \in [a, a+n-1]\}$ such that $w_i = x_{i+1} - x_i$, the following inequality holds:

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \{x_{a+n} - x_a\}^T R \{x_{a+n} - x_a\}$$
$$+ \frac{3}{n} \left\{ \frac{2}{n+1} \sum_{i=a}^{a+n} x_i - x_{a+n} - x_a \right\}^T R$$
$$\times \left\{ \frac{2}{n+1} \sum_{i=a}^{a+n} x_i - x_{a+n} - x_a \right\}. \quad (9)$$

Proof. When $w_i = x_{i+1} - x_i$, we have that

$$\sum_{i=a}^{a+n-1} w_i = x_{a+n} - x_a, \quad (10)$$

$$\sum_{i=a}^{a+n-1} w_i - \frac{2}{n+1} \sum_{i=a}^{a+n-1} \sum_{j=i}^{a+n-1} w_j$$
$$= \{x_{a+n} - x_a\} - \frac{2}{n+1} \sum_{i=a}^{a+n-1} \{x_{a+n} - x_i\}$$
$$= \left\{ 1 - \frac{2n}{n+1} \right\} x_{a+n} - x_a + \frac{2}{n+1} \sum_{i=a}^{a+n-1} x_i$$
$$= \frac{2}{n+1} \sum_{i=a}^{a+n-1} x_i - \frac{n-1}{n+1} x_{a+n} - x_a$$
$$= \frac{2}{n+1} \left\{ \sum_{i=a}^{a+n-1} x_i + x_{a+n} - x_{a+n} \right\}$$
$$- \frac{n-1}{n+1} x_{a+n} - x_a$$
$$= \frac{2}{n+1} \sum_{i=a}^{a+n} x_i - x_{a+n} - x_a. \quad (11)$$

Then, substituting (10)-(11) into (8) provides that

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \{x_{a+n} - x_a\}^T R \{x_{a+n} - x_a\}$$
$$+ \frac{3(n+1)}{n(n-1)} \left\{ \frac{2}{n+1} \sum_{i=a}^{a+n} x_i - x_{a+n} - x_a \right\}^T R$$
$$\times \left\{ \frac{2}{n+1} \sum_{i=a}^{a+n} x_i - x_{a+n} - x_a \right\}$$
$$\geq \frac{1}{n} \{x_{a+n} - x_a\}^T R \{x_{a+n} - x_a\}$$
$$+ \frac{3}{n} \left\{ \frac{2}{n+1} \sum_{i=a}^{a+n} x_i - x_{a+n} - x_a \right\}^T R$$
$$\times \left\{ \frac{2}{n+1} \sum_{i=a}^{a+n} x_i - x_{a+n} - x_a \right\},$$

which concludes the proof. ∎

3.2 Application to discrete-time delay systems

Consider the following discrete-time delay system

$$x_{k+1} = Ax_k + A_d x_{k-d_k},$$
$$x_k = \phi_k, \quad k = -d_M, -d_M + 1, \ldots, 0, \quad (12)$$

where $x_k \in \mathbf{R}^{n_x}$ is the state vector, ϕ_k is an initial function and the time-varying delay d_k satisfying $0 \leq d_m \leq d_k \leq d_M$, $d_m < d_M$, and $d_\Delta = d_M - d_m$. We have the following theorem.

Theorem 2. For given scalars d_m and d_M, the system (12) is asymptotically stable if there exist matrices $P > 0$, $Q_i > 0, i = 1, \ldots, 4$ and $S_{ij}, i, j = 1, 2$ such that the following conditions hold for $d_k = d_m$ and $d_k = d_M$:

$$0 > \Omega+(\Pi_2-\Pi_1)^T PM(d_k)+M(d_k)^T P(\Pi_2-\Pi_1), \quad (13)$$

$$0 < \begin{bmatrix} Q_4 & 0 & S_{11} & S_{12} \\ 0 & 3Q_4 & S_{21} & S_{22} \\ S_{11}^T & S_{21}^T & Q_4 & 0 \\ S_{12}^T & S_{22}^T & 0 & 3Q_4 \end{bmatrix} \triangleq \Xi, \quad (14)$$

where

$$\begin{aligned}
\Omega \triangleq\ & \Pi_2^T P\Pi_2 - \Pi_1^T P\Pi_1 + e_1Q_1e_1^T - e_2Q_1e_2^T \\
& + e_2Q_2e_2^T - e_4Q_2e_4^T + d_m^2(e_0-e_1)Q_3(e_0-e_1)^T \\
& -(e_1-e_2)Q_3(e_1-e_2)^T \\
& -3(2e_5-e_1-e_2)Q_3(2e_5-e_1-e_2)^T \\
& + d_\Delta^2(e_0-e_1)Q_4(e_0-e_1)^T - \Pi_3^T\Xi\Pi_3,
\end{aligned}$$

$$\Pi_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & (d_m+1)I & 0 & 0 \\ 0 & -I & -I & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} A & 0 & A_d & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & (d_m+1)I & 0 & 0 \\ 0 & 0 & -I & -I & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} (e_2-e_3)^T \\ (2e_6-e_2-e_3)^T \\ (e_3-e_4)^T \\ (2e_7-e_3-e_4)^T \end{bmatrix},$$

$$M(d_k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (d_k-d_m+1)I & (d_M-d_k+1)I \end{bmatrix},$$

$e_i (i = 1, 2, \ldots, 7) \in \mathbf{R}^{7n\times n}$ is the i-th unit block matrix and $e_0 = e_1A^T + e_3A_d^T$.

Proof. Choose a candidate for a Lyapunov-Krasovskii functional as follows.

$$\begin{aligned}
V_k &= \sum_{i=0}^{4} V_{ik}, \\
V_{0k} &= \eta_k^T P\eta_k, \\
V_{1k} &= \sum_{i=k-d_m}^{k-1} x_i^T Q_1 x_i, \\
V_{2k} &= \sum_{i=k-d_M}^{k-d_m-1} x_i^T Q_2 x_i, \\
V_{3k} &= d_m \sum_{i=-d_m}^{-1}\sum_{j=k+i}^{k-1} y_j^T Q_3 y_j, \\
V_{4k} &= d_\Delta \sum_{i=-d_M}^{-d_m-1}\sum_{j=k+i}^{k-1} y_j^T Q_4 y_j,
\end{aligned}$$

where $y_j = x_{j+1} - x_j$ and

$$\eta_k \triangleq \mathrm{col}\left\{x_k, \sum_{i=k-d_m}^{k-1} x_i, \sum_{i=k-d_M}^{k-d_m-1} x_i\right\}.$$

The forward difference of V_k is computed as follows.

$$V_{k+1} - V_k = \sum_{i=0}^{4}\left\{V_{i(k+1)} - V_{ik}\right\}, \quad (15)$$

$$\begin{aligned}
V_{0(k+1)} - V_{0k} &= \eta_{k+1}^T P\eta_{k+1} - \eta_k^T P\eta_k \\
&= \zeta_k^T\{(\Pi_2+M(d_k))^T P(\Pi_2+M(d_k)) \\
&\quad -(\Pi_1+M(d_k))^T P(\Pi_1+M(d_k))\}\zeta_k \\
&= \zeta_k^T\{\Pi_2^T P\Pi_2 - \Pi_1^T P\Pi_1 \\
&\quad +(\Pi_2-\Pi_1)^T PM(d_k) \\
&\quad +M(d_k)^T P(\Pi_2-\Pi_1)\}\zeta_k, \quad (16)
\end{aligned}$$

$$\begin{aligned}
V_{1(k+1)} - V_{1k} &= x_k^T Q_1 x_k - x_{k-d_m}^T Q_1 x_{k-d_m} \\
&= \zeta_k^T\{e_1Q_1e_1^T - e_2Q_1e_2^T\}\zeta_k, \quad (17)
\end{aligned}$$

$$\begin{aligned}
V_{2(k+1)} - V_{2k} &= x_{k-d_m}^T Q_2 x_{k-d_m} - x_{k-d_M}^T Q_2 x_{k-d_M} \\
&= \zeta_k^T\{e_2Q_2e_2^T - e_4Q_2e_4^T\}\zeta_k, \quad (18)
\end{aligned}$$

$$\begin{aligned}
V_{3(k+1)} - V_{3k} &= d_m^2 y_k^T Q_3 y_k - d_m \sum_{i=k-d_m}^{k-1} y_i^T Q_3 y_i \\
&= \zeta_k^T\left\{d_m^2(e_0-e_1)Q_3(e_0-e_1)^T\right\}\zeta_k \\
&\quad - d_m \sum_{i=k-d_m}^{k-1} y_i^T Q_3 y_i, \quad (19)
\end{aligned}$$

$$\begin{aligned}
V_{4(k+1)} - V_{4k} &= d_\Delta^2 y_k^T Q_4 y_k - d_\Delta \sum_{i=k-d_M}^{k-d_m-1} y_i^T Q_4 y_i \\
&= \zeta_k^T\left\{d_\Delta^2(e_0-e_1)Q_4(e_0-e_1)^T\right\}\zeta_k \\
&\quad - d_\Delta \sum_{i=k-d_k}^{k-d_m-1} y_i^T Q_4 y_i \\
&\quad - d_\Delta \sum_{i=k-d_M}^{k-d_k-1} y_i^T Q_4 y_i, \quad (20)
\end{aligned}$$

where

$$\zeta_k = col\left\{x_k, x_{k-d_m}, x_{k-d_k}, x_{k-d_M}, \frac{1}{d_m+1}\sum_{i=k-d_m}^{k} x_i, \frac{1}{d_k-d_m+1}\sum_{i=k-d_k}^{k-d_m} x_i, \frac{1}{d_M-d_k+1}\sum_{i=k-d_M}^{k-d_k} x_i\right\}.$$

Applying Corollary 3.2, one can obtain

$$\begin{aligned}
& -d_m \sum_{i=k-d_m}^{k-1} y_i^T Q_3 y_i \\
&\le -\{x_k - x_{k-d_m}\}^T Q_3 \{x_k - x_{k-d_m}\} \\
&\quad -3\left\{\frac{2}{d_m+1}\sum_{i=k-d_m}^{k} x_i - x_k - x_{k-d_m}\right\}^T Q_3 \\
&\quad \times\left\{\frac{2}{d_m+1}\sum_{i=k-d_m}^{k} x_i - x_k - x_{k-d_m}\right\} \\
&= -\zeta_k^T\{(e_1-e_2)Q_3(e_1-e_2)^T \\
&\quad +3(2e_5-e_1-e_2)Q_3(2e_5-e_1-e_2)^T\}\zeta_k, \quad (21)
\end{aligned}$$

$$
\begin{aligned}
&- d_\Delta \sum_{i=k-d_k}^{k-d_m-1} y_i^T Q_4 y_i \\
&\quad \le -\frac{d_\Delta}{d_k - d_m} \left\{x_{k-d_m} - x_{k-d_k}\right\}^T Q_4 \left\{x_{k-d_m} - x_{k-d_k}\right\} \\
&\quad - 3\frac{d_\Delta}{d_k - d_m} \left\{ \frac{2}{d_k - d_m + 1} \sum_{i=k-d_k}^{k-d_m} x_i x_{k-d_m} - x_{k-d_k} \right\}^T \\
&\quad \times Q_4 \left\{ \frac{2}{d_k - d_m + 1} \sum_{i=k-d_k}^{k-d_m} x_i - x_{k-d_m} - x_{k-d_k} \right\} \\
&\quad = -\frac{d_\Delta}{d_k - d_m} \zeta_k^T \{(e_2 - e_3) Q_4 (e_2 - e_3)^T \\
&\quad + 3(2e_6 - e_2 - e_3) Q_4 (2e_6 - e_2 - e_3)^T\} \zeta_k, \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
&- d_\Delta \sum_{i=k-d_M}^{k-d_k-1} y_i^T Q_4 y_i \\
&\quad \le -\frac{d_\Delta}{d_M - d_k} \left\{x_{k-d_k} - x_{k-d_M}\right\}^T Q_4 \left\{x_{k-d_k} - x_{k-d_M}\right\} \\
&\quad - 3\frac{d_\Delta}{d_M - d_k} \left\{ \frac{2}{d_M - d_k + 1} \sum_{i=k-d_M}^{k-d_k} x_i - x_{k-d_k} - x_{k-d_M} \right\}^T \\
&\quad \times Q_4 \left\{ \frac{2}{d_M - d_k + 1} \sum_{i=k-d_M}^{k-d_k} x_i - x_{k-d_k} - x_{k-d_M} \right\} \\
&\quad = -\frac{d_\Delta}{d_M - d_k} \zeta_k^T \{(e_3 - e_4) Q_4 (e_3 - e_4)^T \\
&\quad + 3(2e_7 - e_3 - e_4) Q_4 (2e_7 - e_3 - e_4)^T\} \zeta_k. \qquad (23)
\end{aligned}
$$

From Lemma 2.2, we can infer that if the condition (14) holds, then the reciprocally convex combinations in (22)-(23) satisfy

$$
\begin{aligned}
&- \frac{d_\Delta}{d_k - d_m} \zeta_k^T \{(e_2 - e_3) Q_4 (e_2 - e_3)^T \\
&\quad + 3(2e_6 - e_2 - e_3) Q_4 (2e_6 - e_2 - e_3)^T\} \zeta_k \\
&- \frac{d_\Delta}{d_M - d_k} \zeta_k^T \{(e_3 - e_4) Q_4 (e_3 - e_4)^T \\
&\quad + 3(2e_7 - e_3 - e_4) Q_4 (2e_7 - e_3 - e_4)^T\} \zeta_k \\
&\quad \le -\zeta_k^T \Pi_3^T \Xi \Pi_3 \zeta_k. \qquad (24)
\end{aligned}
$$

By combining (15)-(24), we can obtain

$$
\begin{aligned}
V_{k+1} - V_k \le \zeta_k^T &\left\{ \Omega + (\Pi_2 - \Pi_1)^T P M(d_k) \right. \\
&\left. + M(d_k)^T P (\Pi_2 - \Pi_1) \right\} \zeta_k. \qquad (25)
\end{aligned}
$$

Since $\Omega + (\Pi_2 - \Pi_1)^T PM(d_k) + M(d_k)^T P(\Pi_2 - \Pi_1)$ is convex in $d_k \in [d_m, d_M]$, it's negativity condition is equivalent to (13) for $d_k = d_m$ and $d_k = d_M$. This ends the proof. ∎

4. EXAMPLE

Consider the system (12) with the parameters as follows:

$$
A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}.
$$

To show the effectiveness of our approach, the maximum allowable upper bound (MAUB) and the number of variables are compared with the existing results and are listed in Table 1 and Table 2, respectively. From Table 1, we can see that Theorem 2 outperforms the existing results in the literature except that the conditions provided in Kim (2012) and Kwon et al. (2013) are less conservative than Theorem 2 for some case of small d_m. However, Theorem 2 becomes identical or less conservative as d_m increases and given that the conditions in Kim (2012) and Kwon et al. (2013) are based on the Lyapunov-Krasovskii functional containing some terms with triple summation, it is obvious that the proposed approach is competitive with those conditions both in terms of conservatism and the number of decision variables.

Table 1. Maximum allowable upper bound (MAUB) d_M for given d_m.

d_m	6	12	15	20	25	30
Gao and Chen (2007)	14	17	18	22	26	30
Zhang et al. (2008)	14	18	20	24	29	33
Liu and Zhang (2012)	18	21	23	27	31	35
Kim (2012)	22	23	25	28	31	35
Ramakrishnan and Ray (2013)	18	21	23	27	31	35
Kwon et al. (2013)	22	23	25	28	32	36
Theorem 2	21	24	25	29	32	36

Table 2. The number of variables.

Methods	# of variables
Gao and Chen (2007)	$2.5n_x^2 + 1.5n_x$
Zhang et al. (2008)	$9n_x^2 + 3n_x$
Liu and Zhang (2012)	$4n_x^2 + 3n_x$
Kim (2012)	$90.5n_x^2 + 14.5n_x$
Ramakrishnan and Ray (2013)	$9n_x^2 + 3n_x + 2$
Kwon et al. (2013)	$27n_x^2 + 9n_x$
Theorem 2	$10n_x^2 + 3.5n_x$

5. CONCLUSION

This paper suggests a new auxiliary function-based summation inequality for quadratic functions which is stronger than Jensen inequality. Based on the simple Lyapunov-Krasovskii functional, we develop a delay-dependent stability criterion for discrete-time delay systems using the proposed inequality and the lower bound lemma for reciprocal convexity. Numerical example is provided to illustrate the effectiveness of the proposed approach.

ACKNOWLEDGEMENTS

This research was supported by the MSIP(Ministry of Science, ICT and Future Planning), Korea, under the IT Consilience Creative Program (NIPA-2014-H0201-14-1001) supervised by the NIPA(National IT Industry Promotion Agency). This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2014R1A1A2055122).

REFERENCES

Briat, C. (2011). Convergence and equivalence results for the jensen's inequality; application to time-delay and sampled-data systems. *IEEE Transactions on Automatic Control*, 56(7), 1660–1665.

Gao, H. and Chen, T. (2007). New results on stability of discrete-time systems with time-varying state delay. *IEEE Transactions on Automatic Control*, 52(2), 328–334.

Gu, K., Kharitonov, V.L., and Chen, J. (2003). *Stability of Time-Delay Systems*. Birkhäuser, Basel.

Kim, S. (2012). Relaxed inequality approach to robust $\mathcal{H}_\infty$ stability analysis of discrete-time systems with time-varying delay. *IET Control Theory & Applications*, 6(13), 2149–2156.

Kwon, O., Park, M., Park, J.H., Lee, S., and Cha, E. (2013). Stability and stabilization for discrete-time systems with time-varying delays via augmented lyapunovkrasovskii functional. *Journal of the Franklin Institute*, 350(3), 521–540.

Liu, J. and Zhang, J. (2012). Note on stability of discrete-time time-varying delay systems. *IET Control Theory & Applications*, 6(2), 335–339.

Ma, S., Boukas, E.K., and Chinniah, Y. (2010). Stability and stabilization of discrete-time singular markov jump systems with time-varying delay. *International Journal of Robust and Nonlinear Control*, 20(5), 531–543.

Moon, Y.S., Park, P., Kwon, W.H., and Lee, Y.S. (2001). Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, 74(14), 1447–1455.

Park, P., Ko, J.W., and Jeong, C. (2011). Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, 47(1), 235–238.

Ramakrishnan, K. and Ray, G. (2013). Robust stability criteria for a class of uncertain discrete-time systems with time-varying delay. *Applied Mathematical Modelling*, 37(3), 1468–1479.

Richard, J.P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10), 1667–1694.

Seuret, A. and Gouaisbaut, F. (2013). Wirtinger-based integral inequality: Application to time-delay systems. *Automatica*, 49(9), 2860–2866.

Xu, S., Lam, J., Zhang, B., and Zou, Y. (2013). A new result on the delay-dependent stability of discrete systems with time-varying delays. *International Journal of Robust and Nonlinear Control*, In press.

Zhang, B., Xu, S., and Zou, Y. (2008). Improved stability criterion and its applications in delayed controller design for discrete-time systems. *Automatica*, 44(11), 2963–2967.